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Two-layered Plates with Assumed Shear Strain Fields

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Abstract. *We present a finite element model for a two-layered moderately thick plates based on the Mindlin plate theory. Starting from the pure displacement-based approach and the general expressions for the shear strains, selective constraints for the shear are assumed and a new layered plate element is introduced based on the mixed formulation. In the bending part of the formulation the starting transverse displacement interpolation has a cubic order and the rotation interpolations are quadratic while they are linked in both fields following the problem-dependent linked-interpolation expressions. In the membrane part the displacement interpolations are quadratic and also linked with the drilling nodal rotations by the constraints on constant stresses within the patch test.*

The element passes the general constant-bending patch test and has 36 degrees of freedom after the internal bubble parameters are statically condensed in the element stiffness matrix. The layers can have different material characteristics, but these are assumed to be linear for each layer.

The element is tested on a set of benchmark problems and compared with the results produced by the displacement-based layered elements without reduction in shear and which stiffness matrix is calculated by the reduced integration technique. The element is also compared with the other elements from the literature.

1 Introduction

A two-layered plate finite element is presented, developed in the research project entitled "Assumed strain method in finite elements for layered plates and shells with application on layer delamination problem – ASDEL", financially supported by the Croatian Science Foundation.

Two existing plane finite elements, one for the moderately thick plates and the other for the membrane effects are combined to model a space layer of the two-layered plate element (Fig. 1). Both incorporated elements are displacement based elements, involving only nodal displacements and nodal rotations as the unknown parameters linked by polynomial interpolations. The plate part of the element model is already presented in [1] and it involves cubic interpolation for transverse displacement and quadratic interpolation for the section rotations. The plate part of the element is problem dependent and it means that in the interpolations material parameters are involved. The membrane part has also only displacements as the unknown parameters and they include two components of a nodal displacement in the membrane plane and a nodal rotation around the normal to its plane.

Mindlin plate theory is adopted in the plate part of the layered element with the shear strains together with bending deformations included in the total strain energy. Equilibrium conditions are imposed by the minimization procedure on the total potential energy of the modelled problem.

Some constraints are introduced in the plate and the membrane part with the intention to avoid the locking effects present in the thin plate limit conditions and as well the membrane locking. This constraints are enforced on the shear strain expressions in the natural coordinate orientations for the plate part and on rotational degrees of freedom in membrane part as it will be shown further.

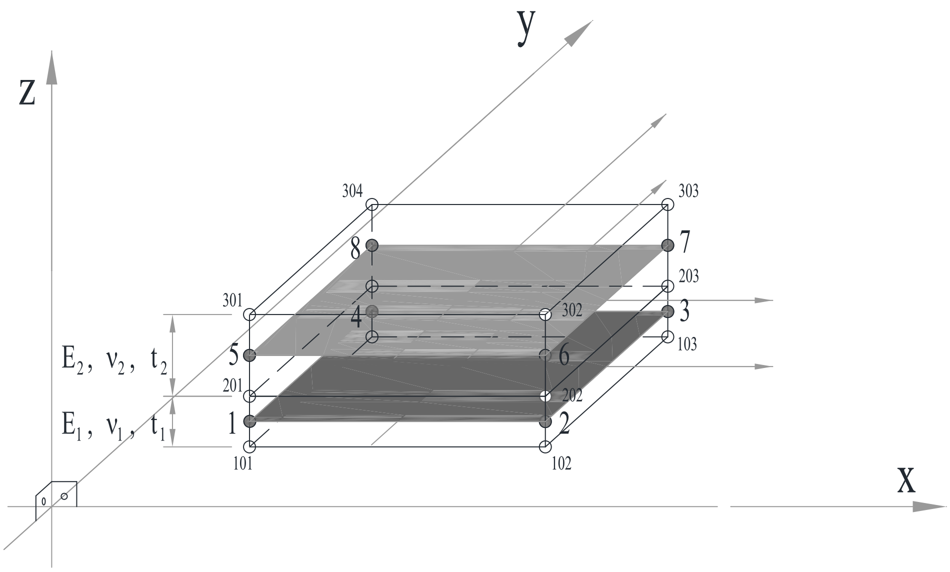

Figure1: Two layered plate model with four nodes per every plate, together with space nodes of the layered structure

2 Linked interpolation functions

2.1 The plate part

The plate displacement fields are interpolated as follows. The transverse displacement is interpolated with the cubic polynomial:

$$
\begin{aligned}
w = &\sum_{i=1}^{4} N_i w_i \\
&+\frac{1-\xi^2}{4}\frac{1-\eta}{2}\frac{1}{2}\left[(\theta_{y2}-\theta_{y1})(x_2-x_1)-(\theta_{x2}-\theta_{x1})(y_2-y_1)\right] \\
&+\frac{1-\xi^2}{4}\frac{1+\eta}{2}\frac{1}{2}\left[(\theta_{y3}-\theta_{y4})(x_3-x_4)-(\theta_{x3}-\theta_{x4})(y_3-y_4)\right]
\end{aligned}
\qquad (1)
$$

$$+\frac{1+\xi}{2}\frac{1-\eta^2}{4}\frac{1}{2}\left[\left(\theta_{y3}-\theta_{y2}\right)\left(x_3-x_2\right)-\left(\theta_{x3}-\theta_{x2}\right)\left(y_3-y_2\right)\right]$$

$$+\frac{1-\xi}{2}\frac{1-\eta^2}{4}\frac{1}{2}\left[\left(\theta_{y4}-\theta_{y1}\right)\left(x_4-x_1\right)-\left(\theta_{x4}-\theta_{x1}\right)\left(y_4-y_1\right)\right]$$

$$+\frac{1-\xi^2}{4}\frac{1-\eta^2}{4}w_{Bb0}$$

$$-\frac{\xi-\xi^3}{4}\frac{1-\eta}{2}\Delta\kappa_{12}\frac{L_{12}^2}{6}-\frac{1+\xi}{2}\frac{\eta-\eta^3}{4}\Delta\kappa_{23}\frac{L_{23}^2}{6}$$
$$+\frac{\xi-\xi^3}{4}\frac{1+\eta}{2}\Delta\kappa_{34}\frac{L_{34}^2}{6}+\frac{\xi-\xi^3}{4}\frac{1+\eta}{2}\Delta\kappa_{34}\frac{L_{34}^2}{6}$$

$$+\frac{\xi-\xi^3}{4}\frac{1-\eta^2}{4}\frac{1}{3}w_{Bb,3}+\frac{1-\xi^2}{4}\frac{\eta-\eta^3}{4}\frac{1}{3}w_{Bb,4}$$
$$+\frac{\xi-\xi^3}{4}\frac{\eta-\eta^3}{4}\frac{1}{3}w_{Bb,5},$$

The global rotations of the plate sections are interpolated with the quadratic polynomials:

$$\begin{aligned}\theta_x &= \sum_{i=1}^{4}N_i\theta_{xi}\\ &-\frac{1-\xi^2}{4}\frac{1-\eta}{2}\Delta\kappa_{12}\left(y_2-y_1\right)-\frac{1+\xi}{2}\frac{1-\eta^2}{4}\Delta\kappa_{23}\left(y_3-y_2\right)\\ &-\frac{1-\xi^2}{4}\frac{1+\eta}{2}\Delta\kappa_{34}\left(y_4-y_3\right)-\frac{1-\xi}{2}\frac{1-\eta^2}{4}\Delta\kappa_{41}\left(y_1-y_4\right)\\ &-\frac{1-\xi^2}{4}\frac{1-\eta^2}{4}\theta_{Bb,1},\end{aligned} \tag{2}$$

$$\begin{aligned}\theta_y &= \sum_{i=1}^{4}N_i\theta_{yi}\\ &+\frac{1-\xi^2}{4}\frac{1-\eta}{2}\Delta\kappa_{12}\left(x_2-x_1\right)+\frac{1+\xi}{2}\frac{1-\eta^2}{4}\Delta\kappa_{23}\left(x_3-x_2\right)\\ &+\frac{1-\xi^2}{4}\frac{1+\eta}{2}\Delta\kappa_{34}\left(x_4-x_3\right)+\frac{1-\xi}{2}\frac{1-\eta^2}{4}\Delta\kappa_{41}\left(x_1-x_4\right)\\ &+\frac{1-\xi^2}{4}\frac{1-\eta^2}{4}\theta_{Bb,2},\end{aligned} \tag{3}$$

where w_i, $\theta_{x,i}$ and $\theta_{y,i}$ are the nodal unknown parameters: the transversal displacement and the two rotations of the plate sections around the local in plane x and y coordinates respectfully, for every element node ($i= 1,\ldots 4$). $w_{Bb,0}$, $w_{Bb,3}$, $w_{Bb,4}$ and $w_{Bb,5}$ are four internal bubble parameters that do not affect the displacement field on the element sides and also $\theta_{Bb,2}$ and $\theta_{Bb,3}$ are internal bubble parameters in the rotation fields, which compete the cubic and the quadratic interpolation polynomials respectively. This bubble parameters satisfy the conformity of the element and they can be statically condensed on the element stiffness matrix level. N_i are the bi-linear Lagrangean interpolations, and the $\Delta\kappa_{ij}$ are the element side curvature increments between nodes *i* and *j* (hierarchical rotation vectors in Fig. 2) that can be expressed in terms of the shear strains along that element side with the aim to pass the constant bending stress condition on the standard patch test of five elements.

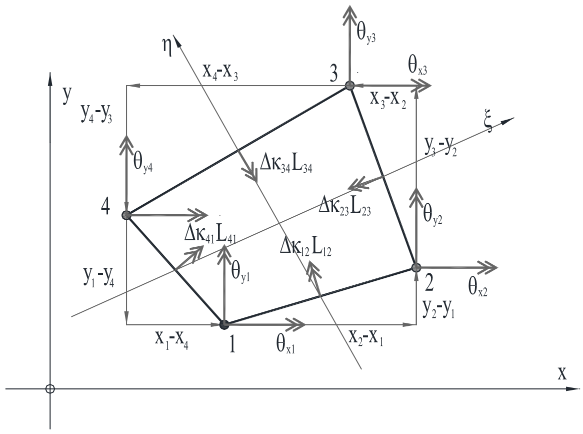

Figure2: Four node plate element, nodal rotation vector and hierarchical rotation vectors perpendicular to the element's sides

This higher order parameters are expressed with the nodal displacements and rotations of that element side and the plate material parameters:

$$\Delta\kappa_{ij} = -\left(w_j - w_i - \frac{\theta_{xi} + \theta_{xj}}{2} \cdot \left(y_j - y_i\right) + \frac{\theta_{yi} + \theta_{yj}}{2} \cdot \left(x_j - x_i\right) \right) \frac{1}{1 + \dfrac{12D}{L_{ij}^2 (Gtk)}} \frac{6}{L_{ij}^2} \tag{4}$$

In the expression (4) x_i i y_i are the coordinates of the four element nodes i=*1,..,4*, L_{ij} are the lengths of the element sides between neighboring nodes *i* and *j*, *D* is the plate bending rigidity $D=Et^3/12(1-v^2)$, *E* and *G* are the elasticity and the shear moduli, *v* the Poisson's coefficient, *t* the plate thickness and *k* the shear coefficient, usually taken as *k=5/6.*

2.2 The membrane part

The membrane fields are interpolated as follows. The two in-planar components of the displacement for the membrane deformations are interpolated in terms of the nodal displacements and the nodal rotations of element with four nodes.

$$u = \sum_{i=1}^{4} N_i u_i$$

$$-\frac{1-\xi^2}{4}\frac{1-\eta}{2}\frac{1}{2}(y_2-y_1)(\theta_{z1}-\theta_{z2})-\frac{1+\xi}{2}\frac{1-\eta^2}{4}\frac{1}{2}(y_3-y_2)(\theta_{z2}-\theta_{z3})$$
$$-\frac{1-\xi^2}{4}\frac{1+\eta}{2}\frac{1}{2}(y_4-y_3)(\theta_{z3}-\theta_{z4})-\frac{1-\xi}{2}\frac{1-\eta^2}{4}\frac{1}{2}(y_1-y_4)(\theta_{z4}-\theta_{z1}) \quad (5)$$
$$+\frac{1-\xi^2}{4}\frac{1-\eta^2}{4}u_{Bb,0},$$

$$v = \sum_{i=1}^{4} N_i v_i$$

$$-\frac{1-\xi^2}{4}\frac{1-\eta}{2}\frac{1}{2}(x_2-x_1)(\theta_{z1}-\theta_{z2})-\frac{1+\xi}{2}\frac{1-\eta^2}{4}\frac{1}{2}(x_3-x_2)(\theta_{z2}-\theta_{z3})$$
$$-\frac{1-\xi^2}{4}\frac{1+\eta}{2}\frac{1}{2}(x_4-x_3)(\theta_{z3}-\theta_{z4})-\frac{1-\xi}{2}\frac{1-\eta^2}{4}\frac{1}{2}(x_1-x_4)(\theta_{z4}-\theta_{z1}) \quad (6)$$
$$+\frac{1-\xi^2}{4}\frac{1-\eta^2}{4}v_{Bb,0},.$$

In the expressions (5) and (6) u_i and v_i are the nodal displacement in the local coordinate orientations, $\theta_{z,i}$ are the nodal rotations around the axis perpendicular to the plane of the element and $u_{Bb,0}$ and $v_{Bb,0}$ are again the internal bubble parameters that do not affect the displacement conformity on the element sides and are statically condensed on the element stiffness level. N_i are again the bi-linear Lagrangean interpolations, and the rotational differences are controlling the hierarchical (quadratic) polynomial part of the interpolations (Fig. 3). This kind of interpolations most membrane elements from literature use if the rotational degrees of freedom are involved.

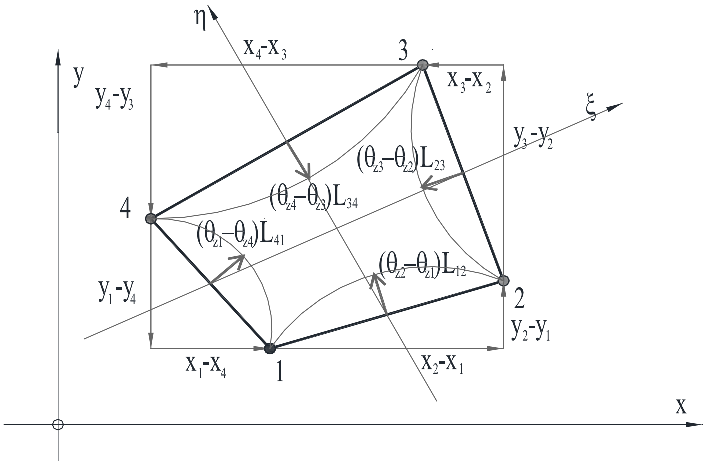

Figure3: Hierarchical perpendicular side displacements

Since the rotation field is a kinematical field that can be expressed from the skew stress tensor part by

$$\theta_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{7}$$

the nodal rotation differences can be derived to satisfy the patch test for the constant strain condition, where the Lagrangean interpolation part suffices to pass it and the quadratic part must vanish. From that criteria the nodal rotation differences can be expressed like in (8) for the element side between nodes 1 and 2 and where the rotation difference between these nodes is expressed:

$$\begin{aligned}\theta_{1z} - \theta_{2z} = {} & \frac{v_1(y_2 - y_4) + v_2(y_4 - y_1) + v_4(y_1 - y_2)}{8 \det J_1} + \frac{u_1(x_2 - x_4) + u_2(x_4 - x_1) + u_4(x_1 - x_2)}{8 \det J_1} \\ & - \frac{v_1(y_2 - y_3) + v_2(y_3 - y_1) + v_3(y_1 - y_2)}{8 \det J_2} + \frac{u_1(x_2 - x_3) + u_2(x_3 - x_1) + u_3(x_1 - x_2)}{8 \det J_2}.\end{aligned} \tag{8}$$

3 Assembled layer stiffness matrix

From plate and membrane element stiffnesses, generated with presented interpolations by standard finite element procedures, the assembled layer stiffness is calculated involving only nodal displacements and eliminating all rotation degrees of freedom.

4 Conclusion

The element is tested on a set of benchmark problems and compared with the results produced by the layered elements without the constraints in shear and the membrane constraints. The element is also compared with the other elements from the literature and with the layered beam element models [2].

The results of the presented layered model will be published in the research paper with more details and comparative numerical examples.

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References

[1] D. Ribarić, "Problem-dependent cubic linked interpolation for Mindlin plate four-node quadrilateral finite elements," *Structural Engineering and Mechanics,* 59.6, 1071-1094, 2016.

[2] L. Škec, G. Jelenić, Analysis of a geometrically exact multi-layer beam with a rigid interlayer connection. *Acta mechanica* 225**:** 523-541, 2014.