Level set topology optimization based on isogeometrical formulation of plane elasticity problems

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***Abstract*—In this paper a level-set approach to topology optimization is presented where level set function is parameterized using non-uniform rational basis splines (NURBS). Isogeometrical numerical method is used for calculating objective function, i.e. for performing structural analysis. Objective function is based on minimizing compliance to find the optimal distribution of material in the design domain under a specified volume constraint. Two benchmark examples are presented to illustrate the method.**

***Keywords*—topology optimization, level set, isogeometric analysis**

I. Introduction

Optimal design of structures is desirable in all fields of engineering. When comes to structural optimization, it is necessary to distinguish three categories of design optimization: sizing optimization, shape optimization and the most abstract, topology optimization. Topology optimization implies finding optimal distribution of material in a design domain, for a desired objective under specified constraints. Topology optimization has been most widely used in classical mechanics, however, not limited to [1], [2], [3]. Among various methods applied for topology optimization, most commonly used are SIMP (Solid Isotropic Material with Penalization) methods, ESO (Evolutionary Structural Optimization) methods and various implementations of density-based method. The level set method, initially proposed by Sethian and Osher [4] is used for the first time in structural optimization in work by Sethian and Wiegmann [5], and afterwards topology optimization using level set method has attracted many researchers and by now, many different formulations and implementations of level set method have been proposed [6], [7], [8]. Main differences and similarities of these methods, up to 2013, are greatly summarized in a review paper by van Dijk and co-workers [9].

The level set method describes the geometry of a structure via iso-contours of a level set function. In this paper implicit level set function is parameterized with NURBS surface. Sensitivity analysis is calculated based on the direct method and the level set function is updated in pseudo-time by solving Hamilton-Jacobi equation numerically, here in by using explicit forward Euler finite difference method. Method used for numerical analysis of structures is isogeometric analysis (IGA) [10], [11], developed in order to bridge the gap between the design and analysis tools. This means that the same mathematical description is used for describing geometry and approximating fields of interest in analysis.

The structure of the paper is as follows: first, brief introduction to isogeometric analysis is given. In the following text the basis of level set method are presented. Later, sensitivity analysis for minimum compliance under volume constraint is given, and all of procedure is summarized in simplified flowchart. Finally, benchmark examples are presented to test the developed MATLAB software for topology optimization using level set method and isogeometric analysis.

II. Isogeometric formulation of plane elasticity structural problems

The aim of isogeometric analysis is to unite Computer-Aided Design (CAD) representation and numerical analysis. As the CAD description is mostly obtained using spline geometry (B-Spline, NURBS, T-Spline etc.), the idea behind isogeometric analysis was to implement basis functions used to construct geometry as a basis functions to construct approximated fields of interests, e.g. displacement, as well. Apart from the good mathematical properties of these basis functions (in terms of analysis) [11], there is no geometry approximation since numerical mesh is directly embedded in the geometry via the knot vectors. Also, refinement procedures keep geometry intact.

By using NURBS basis functions, displacement field is defined as follows:

$$\mathbf{u}(\xi,\eta)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}R_{i,j}(\xi,\eta)\mathbf{u}_{i,j}^{cp} \tag{1}$$

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Geometry is also defined using the same basis functions:

$$\mathbf{x}(\xi,\eta)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}R_{i,j}(\xi,\eta)\mathbf{x}_{i,j}^{cp} \quad (2)$$

where ξ and η are parametric coordinates, $R_{i,j}(\xi,\eta)$ NURBS basis functions, $\mathbf{u}_{i,j}^{cp}$and $\mathbf{x}_{i,j}^{cp}$ control points defining displacement and geometry spline, respectively. Recursive definition of spline basis functions can be found in e.g. [11]

Following the standard procedure for deriving FEM stiffness matrix, the IGA global coefficient matrix is defined as follows:

$$\mathbf{K}=\int_V \mathbf{B}^T\mathbf{C}\mathbf{B}dV \quad (3)$$

where $\mathbf{B}$ is the strain-displacement matrix. In this paper numerical integration is carried out by using standard Gauss quadrature rule, using two Gauss quadrature points per each parametric direction. For the sake of brevity, full procedure is omitted, however, interested reader could refer to plentiful literature, e.g. [10], [11], [12].

III. Level set method

In the level set approach, zero contours of a level set function $\Phi(\mathbf{x})$ define boundaries of a structure in a fixed domain. The level set function is defined over the whole domain *D* , and depending on the level set value, the topology of a structure is defined as follows:

$$\begin{aligned}&\Phi(\mathbf{x})>0,\mathbf{x}\in\Omega\\&\Phi(\mathbf{x})=0,\mathbf{x}\in\delta\Omega\\&\Phi(\mathbf{x})<0,\mathbf{x}\in D\backslash\Omega\end{aligned} \quad (4)$$

where Ω is a part of the fixed domain *D* where material exists. By letting zero-level of a level set function change in pseudo-time t , boundary of a structure can be defined as:

$$S(t)=\mathbf{x}(t):\Phi(\mathbf{x},t)=0 \quad (5)$$

The equation that describes evolution of a structure boundaries is Hamilton-Jacobi partial differential equation, and is derived by total differentiation of the previous equation (5):

$$\frac{\partial\Phi}{\partial t}+\nabla\Phi(\mathbf{x},t)\frac{d\mathbf{x}}{dt}=0 \quad (6)$$

where $\frac{d\mathbf{x}}{dt}$ is the velocity of the boundary.

IV. Isogeometrical approach to structural level set method

Typically, level set method is based on finite element method, and the level set function is either implicitly defined or approximated over finite elements using their shape functions. Herein, level set function is approximated by using the same basis functions used for geometry and displacement field, i.e. NURBS,

$$\Phi(\xi,\eta)=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}R_{i,j}(\xi,\eta)\varphi_{i,j} \quad (7)$$

where $\varphi_{i,j}$ are level set function control point coordinates. The control points of level set function are also used as optimization variables. Since the control points of a level set function mostly do not interpolate the function itself, enough number of control points is required for discretization. However, this approach requires less computational time since no additional determination of level set function values and control points after updating the function has to be performed.

V. Formulation of optimization problem

A. Sensitivity analysis

To update the level set function towards optimal topology, i.e. solve Hamilton-Jacobi equation, it is necessary to properly define velocity. To minimize the Lagrangian function of the optimization problem, changes in Lagrangian with respect to time have to be negative:

$$\frac{\partial L(\Phi,t)}{\partial t}=\frac{\partial L(\Phi,t)}{\partial\Phi}\nabla\Phi(\mathbf{x},t)\frac{d\mathbf{x}}{dt}<0 \quad (8)$$

Based on the steepest descent optimization algorithm, the velocity is chosen as [7], [13]:

$$\nabla\Phi(\mathbf{x},t)\frac{d\mathbf{x}}{dt}=-\frac{\partial L(\Phi,t)}{\partial\Phi} \quad (9)$$

The derivation of Lagrangian with respect to the level set function is obtained by sensitivity analysis based on the formulation of the optimization problem: in this paper, the objective of the optimization problem is to minimize the compliance of the structure, for a certain volume constraint, while satisfying equilibrium in each iteration:

$$\begin{aligned}&Minimize: J(\mathbf{u},\Phi(\mathbf{x}))=\int_\Omega F(\mathbf{u},\Phi(\mathbf{x}))H(\Phi(\mathbf{x})d\Omega\\&Subject\,to: \mathbf{k}_e\mathbf{u}=\mathbf{f}\\&\qquad V\leq V_{max}\end{aligned} \quad (10)$$

where $F(\mathbf{u},\Phi(\mathbf{x}))$ is the strain energy of an element, as follows:

$$F(\mathbf{u},\Phi(\mathbf{x}))=\frac{1}{2}\mathbf{u}^T\mathbf{k}^e\mathbf{u} \quad (11)$$

This formulation is converted to the nested formulation of optimization problem by introducing equilibrium constraint implicitly into formulation [14]. After conducting sensitivity analysis the final expression for velocity of the boundary is [13]:

$$v_n=-\frac{1}{2}\int_\Omega \mathbf{u}^T\mathbf{k}_e\mathbf{u}H(\Phi(\mathbf{x}))d\Omega+\lambda \quad (12)$$

where λ is Lagrange multiplier for volume constraint that is updated in each iteration based on KKT conditions, using the following expression:

$$\lambda=\frac{\int_\Omega(\frac{1}{2}\mathbf{u}^T\mathbf{k}_e\mathbf{u}\delta(\Phi)|\nabla\Phi|d\Omega)}{\int_\Omega\delta(\Phi)|\nabla\Phi|d\Omega} \quad (13)$$

If emerges negative from the equation above, λ is set to zero. It should be emphasized that, in order to avoid numerical burden in the form of remeshing in each iteration, smeared Heaviside function is introduced:

$$H(\Phi) = \begin{cases} \alpha, & \Phi \leq \Delta \\ \frac{1}{2}(1 + \sin \frac{\pi\Phi}{2\Delta}), & \delta < \Phi < \Delta \\ 1, & \Phi \geq \Delta \end{cases} \tag{14}$$

and its derivative, the smeared Dirac delta function is expressed as:

$$\delta(\Phi) = \begin{cases} \frac{\pi}{4\Delta} \cos \frac{\pi\Phi}{2\Delta}, & |\Phi| \leq \Delta \\ 0, & |\Phi| > \Delta \end{cases} \tag{15}$$

where Δ is the approximation width of the smeared Heaviside function. In this paper, approximation width is calculated from the following relation:

$$\Delta = \sqrt{\Delta x^2 + \Delta y^2} \tag{16}$$

where Δx and Δy are control points distances in x and y directions respectively.

B. Hamilton Jacobi equation updating scheme

After appropriately finding the velocity, level set function can be modified in pseudo time by numerically solving discretized Hamilton Jacobi equation (using explicit forward Euler finite difference method):

$$\Phi_{i,j}^{k+1} = \Phi_{i,j}^{k} - \Delta t v_{i,j}^{k} |\nabla\Phi|_{i,j}^{k} \tag{17}$$

where $\Phi_{i,j}^{k}$ is the value of control point in i-th row and j-th column of the control mesh, $v_{i,j}^{k}$ is the velocity value at the respective point, calculated from equation (12), corresponding to the iteration k . Time step is calculated with respect to CFL condition in order to maintain stability:

$$\Delta t = \frac{\beta \max(\Delta x, \Delta y)}{|v_{max}|} \tag{18}$$

where Δx and Δy are control points distances in x and y directions respectively. β is the move limit factor, which additionally controls the time step size i.e. the rate of convergence.

C. Flowchart of isogeometrical approach to level set topology optimization

To summarize the above stated and to complete the full image of the overall process, the simplified flowchart is given, Fig.1.
One segment of the full procedure has not been mentioned yet, however plays important role in the overall convergence: regularization of the level set function to the signed distance function, to avoid level set function becoming too steep or too flat [7], [9], [15]. The regularization is performed after the initialization of the level set function, and after each update of the level set function, as can be seen from the flowchart, Fig.1.

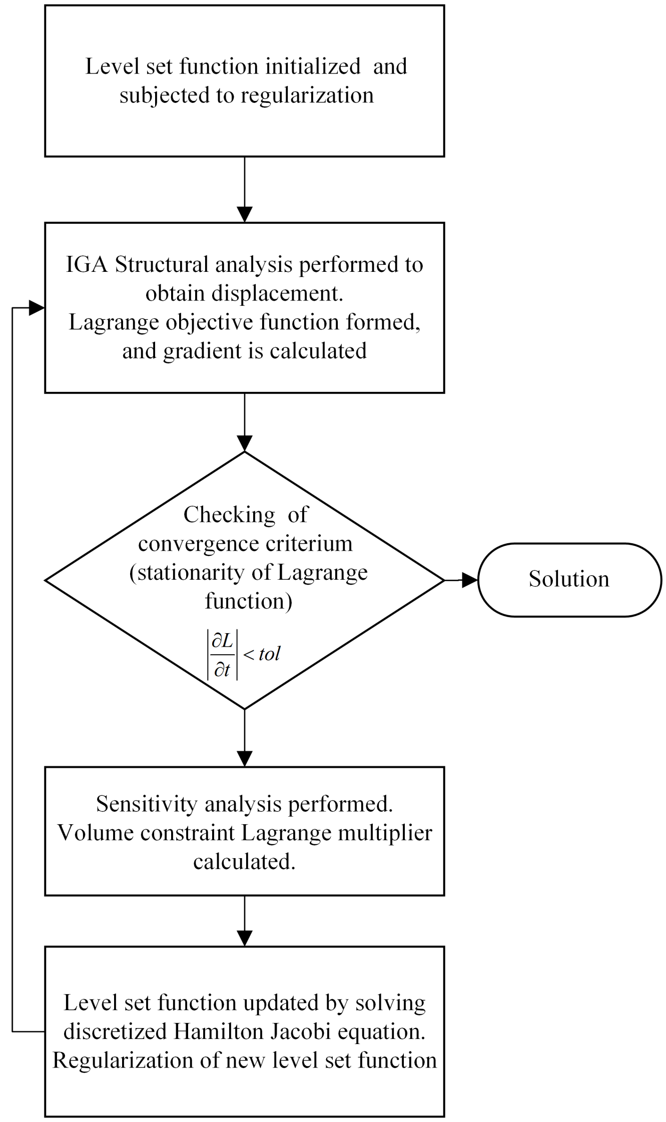

Fig. 1. Flowchart of topology optimization based on level set method and isogeometric analysis

VI. Numerical examples

In the following text, two benchmark examples ([13], [8], [16], [17]) will be presented, with focus on implementation details and their effect on the overall convergence of the process. It should be noted that all units are consistent, hence unit symbols are omitted. In both examples the Poissons ratio is set to $\nu = 0.3$, and Youngs modulus of elasticity is set to $E = 1$. Heaviside function is approximated with $\alpha = 10^{-4}$ and $\Delta = 0.2\sqrt{\Delta x^2 + \Delta y^2}$ where Δx and Δy are control point distances in x and y direction respectively. Move limit factor is set to $\beta = 0.3$.

A. Example 1

Problem setup is shown on Fig. 2: Cantilever beam with dimension of $L = 5$ and $H = 2.5$ is loaded with point load $P = 1$ at the end (right edge, lower corner). Left edge is fixed

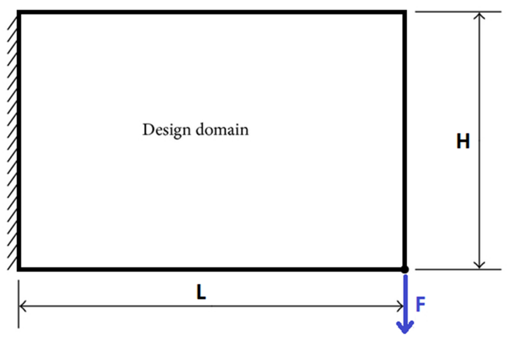

Fig. 2. Example 1: Problem setup

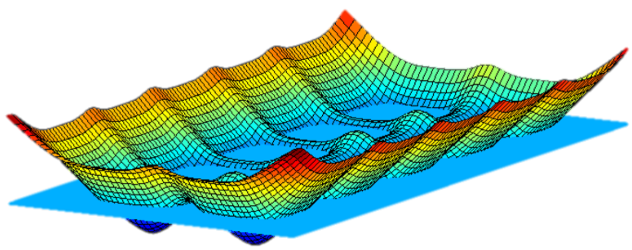

Fig. 3. Initial level set function

to support structure. The volume constraint is set to 40% of the maximum design domain volume.

Fig. 3 shows the initial level set function and xy plane whose intersection defines initial topology, Fig. 4. Initial level set function is of degree 3 and defined with 51×26 control points (as in [13]), meaning that the optimization problem is defined with 1326 optimization variables. The resulted topology and convergence of the process are presented on Fig. 5 and Fig. 6, respectively. It can be seen from Fig. 6 that a high amount of noise is introduced, however topology converges to what is expected (Fig. 5), with significant reduction in the objective function, i.e. compliance. The process is terminated at iteration 250 since the termination criterion threshold (stationarity of Lagrange function) is not reached due to high amount of noise which is consequence of constant „skipping" from feasible space to unfeasible and vice versa, i.e. the process is too sensitive to Lagrange multiplier.

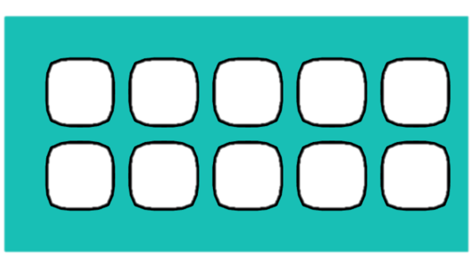

Fig. 4. Initial topology

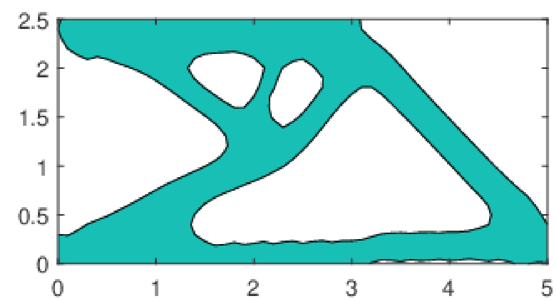

Fig. 5. Example 1: Resulted topology, after 250 iterations

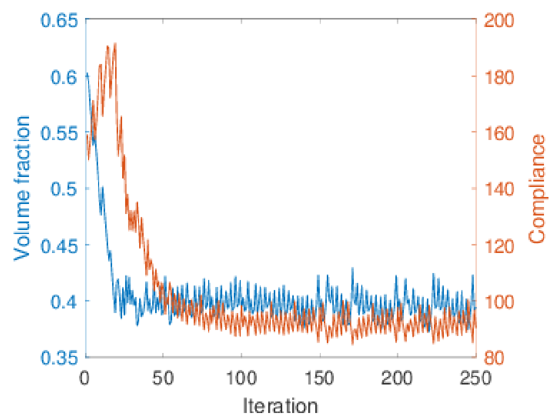

Fig. 6. Example 1: Convergence graph

B. Example 2

Problem setup is shown on Fig. 7: Cantilever of the same properties as in previous example is loaded with point load of amplitude $P = 1$ at the end center point. The same initial level set function (Fig. 3) is used for initial solution as in previous example, Fig. 4. Maximum volume fraction is set to 40% of the design domain. The topology of the structure after 250 iterations is presented in Fig. 8. It can be seen from Fig. 9 that this example is even more prone to convergence oscillations, and the stationarity of the Lagrange function is not reached as well. However, as in previous example, topology of the structure visually converges to what is expected.

VII. Conclusion

In this paper, topology optimization is performed using NURBS for geometry description, level set function approximation and structural analysis (in the isogeometrical analysis form). The nested formulation of minimum compliance that satisfies static equilibrium in each iteration subjected to specified volume constraint is presented. Based on that formulation,

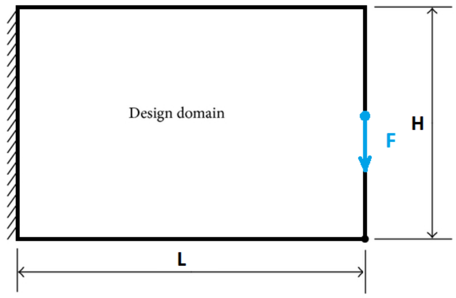

Fig. 7. Example 2: Problem setup

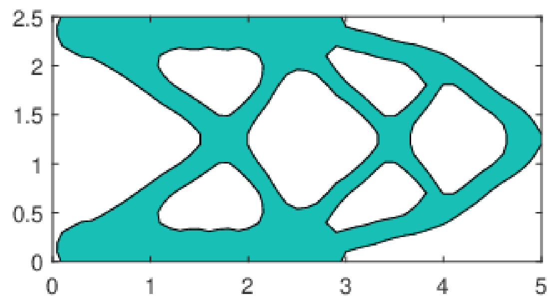

Fig. 8. Example 2: Resulted topology after 250 iterations.

sensitivity analysis based on direct method is presented and velocity field that updates Hamilton-Jacobi equation is obtained. Hamilton-Jacobi equation is updated based on explicit upwind procedure. The optimization variables are control points of the level set function.

Two benchmark examples are presented, and although topology converges to expected, high amount of noise can be observed from the convergence graphs of the procedure, due to high sensitivity to Lagrange volume multiplier. This numerical difficulty could be avoided by extending the Lagrange function with penalty term, i.e. by using augmented Lagrangian formulation.

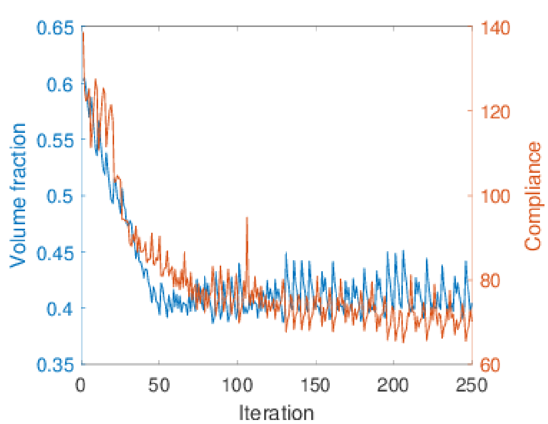

Fig. 9. Example 2: Convergence graph

References

[1] T. Borrvall and J. Petersson, "Topology optimization of fluids in Stokes flow," *International Journal for Numerical Methods in Fluids*, vol. 41, pp. 77–107, jan 2003.

[2] O. Sigmund, "Design of multiphysics actuators using topology optimization Part I: One-material structures," *Computer Methods in Applied Mechanics and Engineering*, vol. 190, pp. 6577–6604, oct 2001.

[3] S. Zhou, W. Li, and Q. Li, "Level-set based topology optimization for electromagnetic dipole antenna design," *Journal of Computational Physics*, vol. 229, pp. 6915–6930, sep 2010.

[4] S. Osher and J. A. Sethian, "Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations," *Journal of Computational Physics*, vol. 79, pp. 12–49, nov 1988.

[5] J. Sethian and A. Wiegmann, "Structural Boundary Design via Level Set and Immersed Interface Methods," *Journal of Computational Physics*, vol. 163, pp. 489–528, sep 2000.

[6] T. Belytschko, S. P. Xiao, and C. Parimi, "Topology optimization with implicit functions and regularization," *International Journal for Numerical Methods in Engineering*, vol. 57, pp. 1177–1196, jun 2003.

[7] G. Allaire, F. Jouve, and A.-M. Toader, "Structural optimization using sensitivity analysis and a level-set method," *Journal of Computational Physics*, vol. 194, pp. 363–393, feb 2004.

[8] M. Y. Wang, X. Wang, and D. Guo, "A level set method for structural topology optimization," *Computer Methods in Applied Mechanics and Engineering*, vol. 192, pp. 227–246, jan 2003.

[9] N. P. van Dijk, K. Maute, M. Langelaar, and F. van Keulen, "Level-set methods for structural topology optimization: a review," *Structural and Multidisciplinary Optimization*, vol. 48, pp. 437–472, sep 2013.

[10] T. Hughes, J. Cottrell, and Y. Bazilevs, "Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement," *Computer Methods in Applied Mechanics and Engineering*, vol. 194, pp. 4135–4195, oct 2005.

[11] J. A. Cottrell, T. J. R. Hughes, and Y. Bazilevs, *Isogeometric Analysis: Toward Integration of CAD and FEA*. Wiley, 1st editio ed., 2009.

[12] G. Beer and S. Bordas, eds., *Isogeometric Methods for Numerical Simulation*, vol. 561 of *CISM International Centre for Mechanical Sciences*. Vienna: Springer Vienna, 2015.

[13] H. A. Jahangiry and S. M. Tavakkoli, "An isogeometrical approach to structural level set topology optimization," *Computer Methods in Applied Mechanics and Engineering*, vol. 319, pp. 240–257, jun 2017.

[14] P. Christensen and A. Klarbring, *An Introduction to Structural Optimization*. Springer, 2009.

[15] Y. Wang and D. J. Benson, "Isogeometric analysis for parameterized LSM-based structural topology optimization," *Computational Mechanics*, vol. 57, pp. 19–35, jan 2016.

[16] L. Dedè, M. J. Borden, and T. J. R. Hughes, "Isogeometric Analysis for Topology Optimization with a Phase Field Model," *Archives of Computational Methods in Engineering*, vol. 19, pp. 427–465, sep 2012.

[17] B. Hassani, M. Khanzadi, and S. M. Tavakkoli, "An isogeometrical approach to structural topology optimization by optimality criteria," *Structural and Multidisciplinary Optimization*, vol. 45, pp. 223–233, feb 2012.