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Form-finding method of cable-net structures using Grassmann algebra and mass-points

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Abstract

In the paper, an application of Grassmann algebra and weighted points, called mass-points, to the Force Density Method (FDM) for cable-net structures is described. In classical mechanics, point represents location in space, and vector represents a force acting on that point with added mass. In this way, it is possible to study geometry using fundamental principles of mechanics. Procedure for determining the centre of mass-points is closely related to finding the position of a free node of cable-net structure based on FDM. The method presented in this paper is iterative, based on mass-points, where points represent fixed nodes, their masses represent force densities in cables of structure, and centre of the mass-points represents the position of a free node, whose coordinates are unknown. The procedure is presented through an example of a simple cable-net structure consisting of only one free node. Since the form-finding in the form diagram is studied using principles of mechanics, it is possible to construct the reciprocal force polygon using obtained centres of the given mass-points. Applying the method described in this paper, an example of a cable-net structure is shown in the form diagram whereas force polygons are assembled in the force diagram.

Keywords: graphic statics, reciprocal diagrams, form diagram, force diagram, mass-points, Grassmann algebra, Plücker coordinates, form-finding, equilibrium-finding

1 Introduction

Methods of graphic statics are used to evaluate the global equilibrium of structural systems and to determine the flow of internal axial forces in structural elements only by geometric operations. Since the actual trends in architecture give structures with complex and irregular geometry, there is an increasing need for an interactive tool that would enable direct and simultaneous control over the shape of the structure and internal forces caused by external loads, and thus finding an effective geometry of the structure at an early stage of structural design. Graphic statics methods are based on the construction of two reciprocal diagrams: the form diagram which shows the geometry of structure (location of internal nodes and supports, external loads and lengths of structural elements) and the force diagram where polygons of forces, assembled of vectors, represent equilibrium of internal forces in structural elements, forces in supports and external loads.

By attaching masses to points, it is possible to study geometry applying basic principles of mechanics (Goldman, [1]). On the other hand, using force densities and an iterative method node by node, the minimal shape of a cable-net structure can be obtained. In every step of the procedure, each free node is considered as an unknown, and its adjacent nodes are considered as supports. The similar procedure, based on an iterative method and force densities, has already been described by Enrique *et al.* [2]. The geometrical procedure of the form-finding, presented in this paper, is performed by adding corresponding masses as the given force densities to the adjacent, fixed nodes and by applying Grassmann algebra of mass-points to find the centre of the given mass-point system. In classical mechanics of particles, finding the centre of given weighted points coincides with the point at which the resultant weight of the given mass-point system acts.

2 Grassmann algebra of mass-points and vectors

2.1 Mass-points

In Grassmann algebra points are expressed as quadruples $P = (m, x, y, z)$, where $m \neq 0$ is a (generalized) mass of the point. Such a point is called the mass-point and we will denote it as a pair (m, mP). Division of the last three coordinates by the first one gives a normalized point $(1, mP/m) = (1, P) = (1, x/m, y/m, z/m)$, and P $= (x/m, y/m, z/m)$ represents the position of mass-point (m, mP) in Cartesian coordinates. The expression $\mathbf{v} = (0, \vec{v}) = (0, x, y, z)$ represents a point at infinity, where $\vec{v}$ is a vector directed from the origin to the point (m, mP).

2.2 Lines and Forces

A line can be specified as a span of two points, namely by progressive product of two points in the form $l = (\mathbf{l_0}, \overline{\mathbf{l}}) = P_1 \wedge P_2 = (m_1, x_1, y_1, z_1) \wedge (m_2, x_2, y_2, z_2) = (l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})$, where $l_{ij} = x_i y_j - x_j y_i$ are homogeneous coordinates of a line, called Plücker line coordinates (Plücker [3], Pottmann and Wallner [4]). For such a line, the distance between two normalized points, that is the length of the line segment, can be determined as: $|l| = \sqrt{l_{01}^2 + l_{02}^2 + l_{03}^2}$.
A force can also be expressed in Plücker line coordinates $F = (\mathbf{f_0}, \overline{\mathbf{f}})$, where the first part represents vector $\mathbf{f_0} = (f_{01}, f_{02}, f_{03})$ of the force F, and the second part represents moment vector $\overline{\mathbf{f}}$ of $\mathbf{f_0}$ about the origin of the coordinate system. Also, the condition $\mathbf{f_0} \cdot \overline{\mathbf{f}} = 0$ must be fulfilled (the same holds for the lines). Force, as in the case of lines, can be expressed as a span of two normalized points $F = P_1 \wedge P_2$. Point P_1 can be regarded as the tail and P_2 as the head of the force F, and the length of the segment is equal to its intensity.

2.3 Operations on mass-points and vectors

Basic geometric operations of Grassmann algebra and their application to graphic statics and static equivalency can be found in Baniček *et al.* [5]. Here, Grassmann algebra of mass-points and vectors is used to present a graphical method for form-finding of cable-nets applying fundamental principles of mechanics. In this method, mass-points are expressed in Grassmann coordinates of points and vectors as described in sections 2.2 and 2.1.

2.3.1 Point and vector addition

The sum of a given mass-point (m, mP) and a vector $\mathbf{f_0}$, given as a force $F = (\mathbf{f_0}, \bar{\mathbf{f}})$ acting at the point (m, mP), can be expressed as (see Figure 1):

$$(m, mP) + (0, \mathbf{f_0}) = (m, mP + \mathbf{f_0}) = \left(1, \frac{mP + \mathbf{f_0}}{m}\right) = \left(1, P_1 + \frac{\mathbf{f_0}}{m}\right) \tag{1}$$

In mechanical interpretation, the given vector $\mathbf{f_0}$ represents a force $F = (\mathbf{f_0}, \bar{\mathbf{f}})$ acting on the mass m and pulling the point $(1, P)$ to the new position $(1, P+\mathbf{f_0}/m)$. Physically, the expression $\mathbf{f_0}/m$ represents velocity and varies inversely with the mass, while the vector $\mathbf{f_0}$ represents momentum vector (Goldman, [1]).

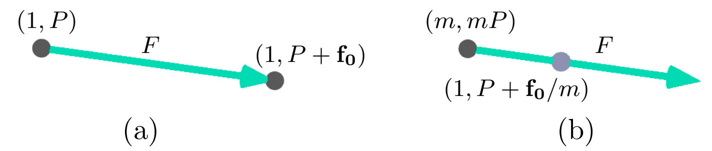

Figure 1: Addition of the mass-point (m, mP) and vector $F = (\mathbf{f_0}, \bar{\mathbf{f}})$, (a) $m = 1$, (b) $m \neq 0, 1$

2.3.2 Centre of mass-point system

According to well-known Archimedes' law of the lever of two mass points (Grassmann [6]), we can find the centre $(m_c, m_cP_c) = (m_1 + m_2, m_1P_1 + m_2P_2)$ of two given mass-points as a sum of their coordinates and masses. The mass-centre (m_c, m_cP_c) is also the balancing point of the given mass system, so the sum of moments of the two masses, called mass moments (Browne [7]), about the mass-centre must vanish:

$$m_1 \cdot d_1 - m_2 \cdot d_2 = 0 \tag{2}$$

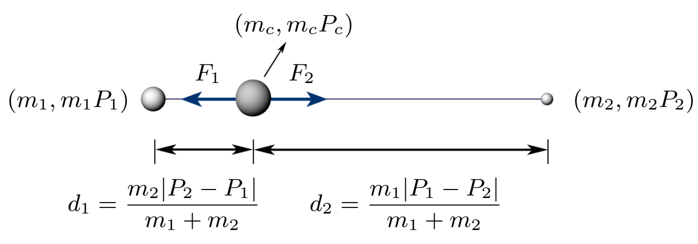

Figure 2: Archimedes' law of the lever

Consider the lever shown in Figure 2 as a cable-net structure consisting of two tensioned cables connected in the node (m_c, m_cP_c). Since forces F_1 and F_2 in the node (m_c, m_cP_c) must be in equilibrium, the condition (2) can be written in the form:

$$F_1 - F_2 = 0 \tag{3}$$

In mechanics of particles, mass-point has inertia. Thus, the effect of a force acting on two points with different masses will be less in the case of the larger mass (larger inertia), and conversely. In other words, translations d_1 and d_2 from the point of the same mass $m_c = m_1 + m_2$ will depend on the force magnitude pulling the mass m_c to the new position (Figure 3a and 3b). Since the two given mass-points uniquely determine the mass-centre, two forces S_1 and S_2 will translate the mass m_c from the point $(1, P_1)$ or $(1, P_2)$ to the point $(1, P_c)$ by the distance d_1 or d_2 accordingly. Therefore, the position of the point $(1, P_c)$ is determined in the form:

$$(1, P_c) = (1, P_1 + d_1) = \left(1, P_1 + \frac{\mathbf{f_{01}} \cdot m_2}{m_1 + m_2}\right),$$

$$(1, P_c) = (1, P_2 + d_2) = \left(1, P_2 + \frac{\mathbf{f_{02}} \cdot m_1}{m_1 + m_2}\right).$$

From equations (1) and (2) it can be seen that the force F_1 in the left cable is equal to $m_1 \cdot d_1$ and the force F_2 in the right cable is equal to $m_2 \cdot d_2$, as shown in the force diagram in Figure 3c. Notice that the expression of the two forces are the same as in the case of force densities with cable force $F = q \cdot l$, where q is force density and l is length of the cable.

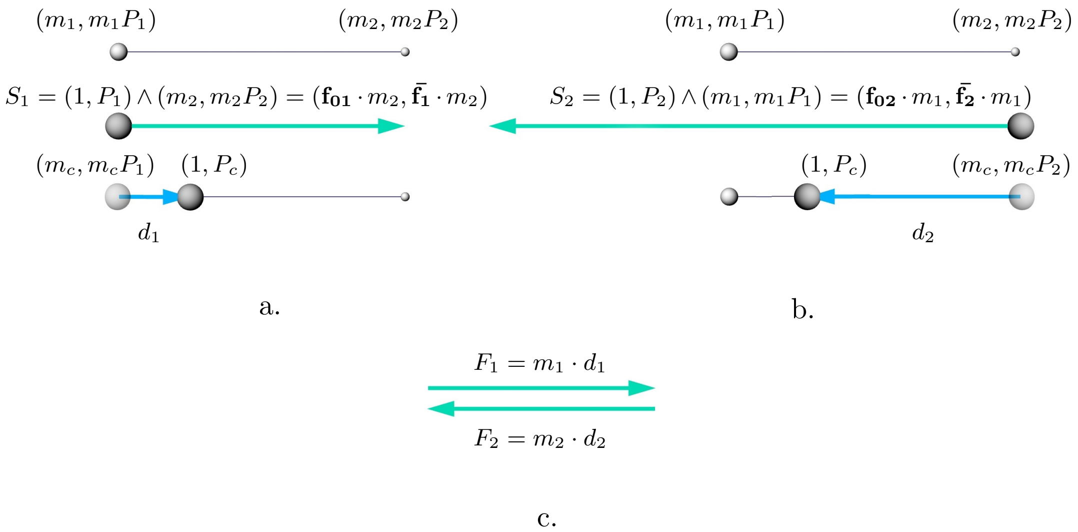

Figure 3: Geometric interpretation of Archimedes' law of the lever

For a mass-point system consisting of an arbitrary number of n mass-points, the mass-centre can be found iteratively, starting with the first two points and by adding in each step of the iteration one of the remaining masses. The equilibrium-finding procedure of the mass-centre, as shown in Figure 3, can be graphically followed in the force diagram. In the next section, we will show the construction of the force diagram for an arbitrary number of mass-points.
The sum of the coordinates of the given mass-points gives us the centre of the mass-points system. To find the position of the centre of the mass-points we can also use the following equation (Möbius [8]):

$$(m_c, m_c\mathbf{P}_c) = \left(\sum_{n=1}^{n} m_i, \sum_{n=1}^{n} m_i\mathbf{P}_i\right) \tag{4}$$

As seen from the expression (4), only operations of addition and multiplication by scalar are used. Therefore, it could be computationally more efficient for preforming of the form-finding because it lets us avoid, or at least postpone, operations of the division until the end of the procedure, thus achieving results with better accuracy.

3 Application of the mass-points algebra to the form-finding

3.1 Equilibrium of a free node and force diagram construction

Using Archimedes' law and the procedure of finding the mass-centre, described in the previous section 2.3.2, it is possible to graphically obtain the equilibrium position of a free node of a cable-net structure by using the iterative method for finding the centre of mass system. The procedure will be applied to the example of a free node in the form diagram, shown in Figure 4a, and will be simultaneously followed by polygons of internal forces in the force diagram.

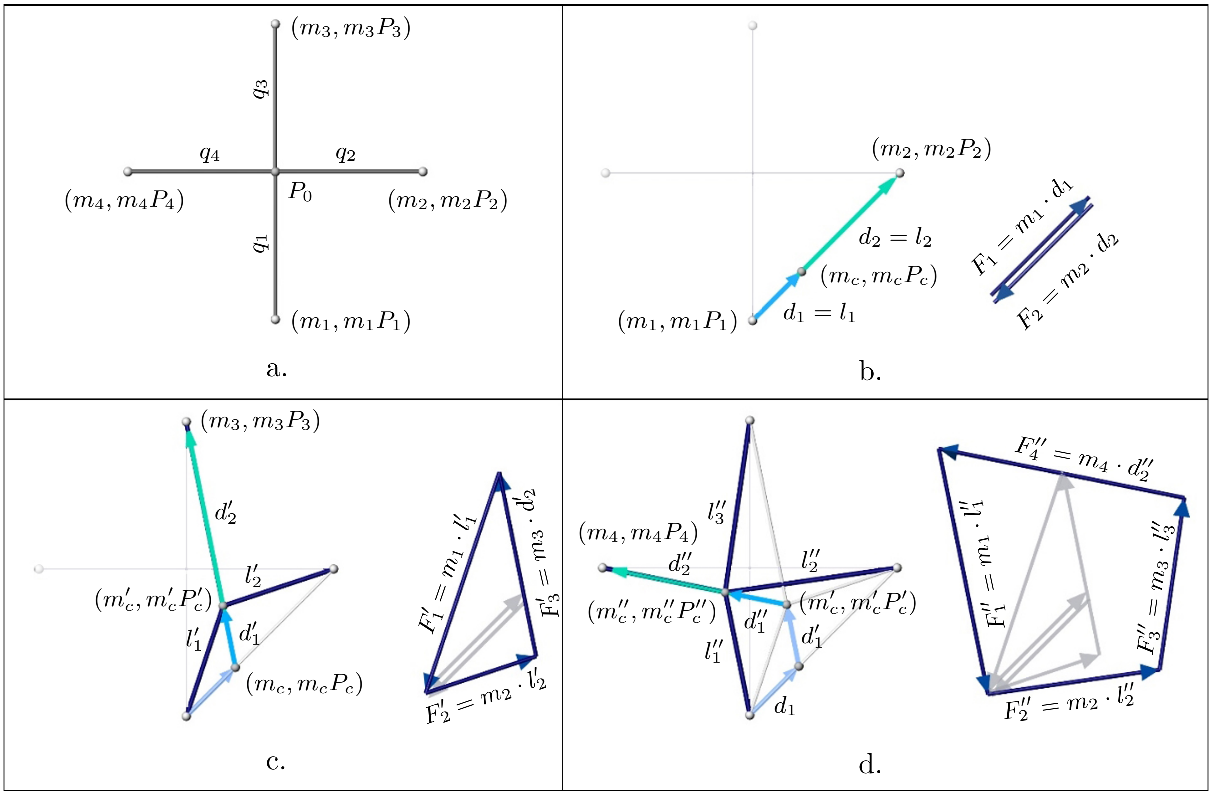

Figure 4: Iterative graphical procedure for form-finding of a free node

Let us consider a single node cable-net system shown in Figure 4. For the free node P_0, connected to fixed points $P_i, i = 1, 2, 3, 4$ with four pre-stressed cables, we can determine its equilibrium position using procedure for finding the mass-centre. Point of the free node is given as a normalized point $P_0 = (1, x_0, y_0, z_0)$ and points of the fixed nodes are given in the normalized form $P_i = (1, x_i, y_i, z_i), i = 1, 2, 3, 4$. The values q_i in Figure 4a present force densities or tension coefficients in corresponding cables.
Before we start the procedure, the given force densities are removed from cables and attached to the corresponding fixed nodes in terms of masses $m_i = q_i$. Therefore, the coordinates of the fixed

nodes becomes Grassmann coordinates of the mass-points $(m_i, m_iP_i) = (m_i, m_i{\cdot}x_i, m_i{\cdot}y_i, m_i{\cdot}z_i)$. Such points form a mass system; thus, the equilibrium-finding for the free node P_0 can be carried out.

First iteration step of the equilibrium-finding is to determine centre of the first two masses (m_1, m_1P_1) and (m_2, m_2P_2) (Figure 4b). The system consisting of two mass-points can be regarded as a two-cable structure, and the centre of masses can be obtained as described in section 2.3.2. That is,

$$(1, P_c) = (1, P_1 + d_1) = \left(1, P_1 + \frac{\mathbf{f_{01}} \cdot m_2}{m_1 + m_2}\right)$$

The distances d_1 and d_2 are also the lengths of the two cables. Thus, magnitudes of the forces in the cables are equal and opposite orientated. In order to construct the force diagram, magnitudes of forces F_1 and F_2 can be calculated as products of the corresponding masses and lengths of cables (Figure 4b right).

As mentioned before, for a mass-point system consisting of n arbitrary number of mass-points the mass-centre can be found iteratively, starting with the first two points and by adding in each step of the iteration one of the remaining masses. In the similar way, the equilibrium position of a single node, connected to the fixed nodes with n cables, can be obtained using mass-points. So, the next step is to attach the third mass to existing two-mass system (m_c, m_cP_c). In this way, a new two-mass system $(m'_c, m'_cP'_c)$ (Figure 4c), where the new mass is the sum of the three masses $m'_c = m_1 + m_2 + m_3 = m_c + m_3$, and its location are determined by pulling the mass m'_c of the new system with the force $S_3 = (1, P_c) \wedge (m_3, m_3P_3) = \mathbf{f_{02}} \cdot m_3$ from the point $(1, P_c)$ for the distance d'_1:

$$(1, P'_c) = (1, P_c + d'_1) = \left(1, P_c + \frac{\mathbf{f_{02}} \cdot m_3}{m_1 + m_2 + m_3}\right) \tag{5}$$

For the example shown in figure 4a, the last iteration step is attaching the fourth mass to the existing two-mass system $(m'_c, m'_cP'_c)$. The new mass-system $(m''_c, m''_cP''_c)$ has the mass $m''_c = m_1 + m_2 + m_3 + m_4 = m'_c + m_4$ and it is translated from the point $(1, P'_c)$ by the force $S_4 = (1, P'_c) \wedge (m_4, m_4P_4) = \mathbf{f_{03}} \cdot m_4$ to the new position:

$$(1, P''_c) = (1, P'_c + d''_1) = \left(1, P'_c + \frac{\mathbf{f_{03}} \cdot m_4}{m_1 + m_2 + m_3 + m_4}\right) \tag{6}$$

3.2 Application to a cable-net structure

Applying the procedure for a single node, described in section 3.1, finding the shape of the cable-net structure (Figure 5) is carried out by using an iterative method node by node until equilibrium of the given cable net structure is established. For this example, in all cables the force density values are set to 1 except for the boundary cables, where the values are set to 2, and for the inner cable loop, with the force densities set to 3.

To construct the force diagram from the obtained form diagram, we have developed a computer program based on Bow's notation [9] of nodes and forces and the approach by Micheletti [10]. The code of the program is written in GhPython [11] (Python interpreter component and plug-in for Grasshopper [12] and Rhinoceros [13]). After the equilibrium of each node is found, the code calculates forces F_i of each cable by multiplying obtained lengths l_i by their corresponding masses m_i, where $i = 1...n$ number of cables connected at the considered node. To determine the lengths of the cables, we can apply the expression of line segment as a span of two normalized points, that is: $|l_i| = \sqrt{l_{i01}^2 + l_{i02}^2 + l_{i03}^2}$.

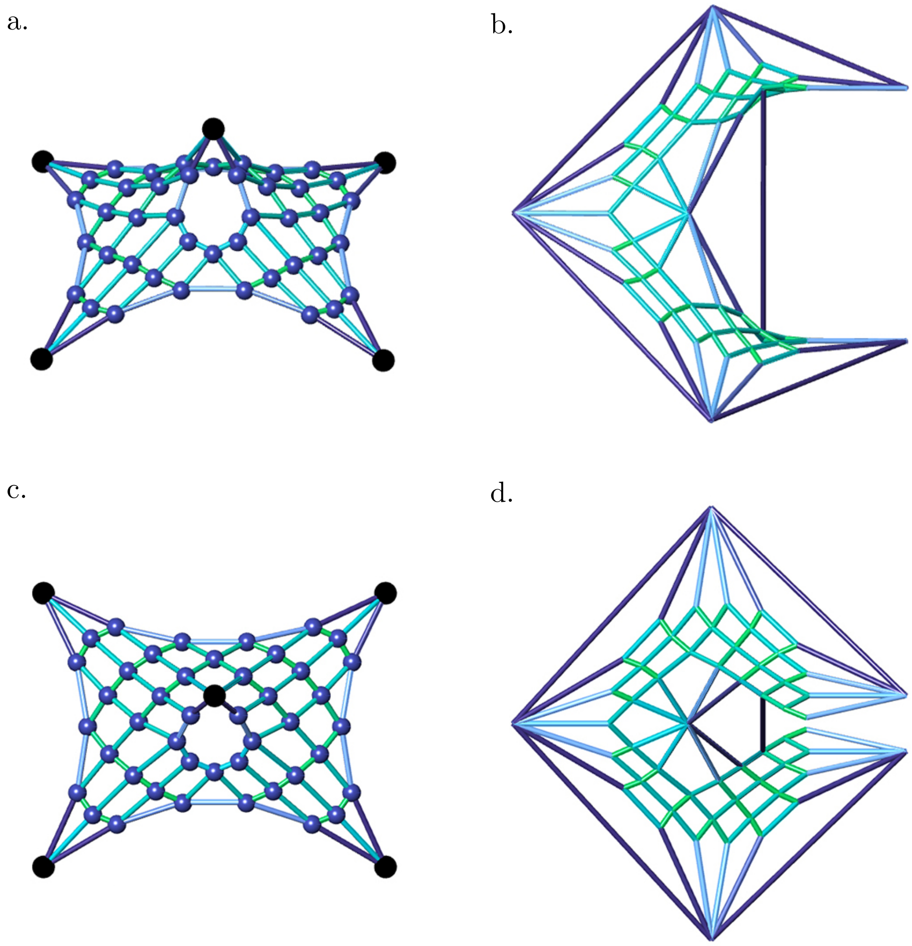

Figure 5: Form-finding of the given cable-net structure using presented graphical iterative method (a. and c. form diagram; b. and d. force diagram; a. and b. 3D view; c. and d. top view)

4 Conclusion and future work

By adding masses to points, it is possible to study geometry by using fundamental mechanical principles. This was applied to a single node problem using described Archimedes' law and thereafter to an example of a cable-net structure using the method node by node. The method presented in the paper allows user to simultaneously explore structural shape in the form diagram and internal forces in the force diagram, and if one applies the tension-compression analogy, the method is convenient for both cable-net and self-supported structures.
The method could also be graphically described with Hooke's law of the spring system, where the position of a free node can be obtained iteratively using the equation $F = k\mathbf{x}$, where F is

the force acting on the spring, $\mathbf{x}$ represents elongation of the spring, and k is a positive constant that characterizes spring stiffness. If one compares Hooke's law and Archimedes' law approach, it is clear that the elongation $\mathbf{x}$ is equal to $d = \mathbf{f_0}/\sum m_i, i = 1, ..., n$, while the spring stiffness k is equal to $m_c = \sum m_i, i = 1, ..., n$.
Similarly, the form-finding of a free node could be described by Newton's second law of motion used in dynamics of particles. When a particle of mass m is acted upon by force F, acceleration of the particle and the force must satisfy the relation $F = m\mathbf{a}$.
Since the graphical method can be described using basic principles of mechanics, they are convenient in engineering education for geometrical study and understanding of iterative methods, evaluation of global equilibrium as well as understanding internal equilibrium of structures.

References

[1] R. Goldman, *Rethinking Quaternions: Theory and Computation*, Morgan and Claypool Publishers, 2010.

[2] L. Enrique, P. Block. and J. Schwartz, "Form-finding method for prestress-dominant surface structures using graphic statics", *Proceedings of the International Association for Shell and Spatial Structures (IASS) Symposium 2015*, August, 2015, Amsterdam, Netherlands.

[3] J. Plücker, "On a new geometry of space", *Philosophical Transactions of the Royal Society of London*, vol. 155, pp. 725-791, 1865.

[4] H. Pottmann, J. Wallner, *Computational Line Geometry*, Springer, Berlin, Heidelberg, 2001.

[5] M. Baniček, K. Fresl and I. Kodrnja, "Some examples of static equivalency in space using descriptive geometry and Grassmann algebra", *Proceedings of the International Association for Shell and Spatial Structures (IASS) Symposium 2018*, July, 2018, MIT, Boston, USA.

[6] H. Grassmann, *A New Branch of Mathematics: The "Ausdehnungslehre" of 1844 and Other Works*, (Transl.: C. L. Kannenberg), Open Court Publishing, 1995.

[7] J. Browne, *Grassmann Algebra Volume 1: Foundations: Exploring extended vector algebra with Mathematica*, Melbourne: Birkhauser, Quantica Publishing, 2012.

[8] A. F. Möbius, *Der barycentrische Calcul*, Verlag von J.A. Barth, Leipzig, 1827.

[9] R. H. Bow, *Economics of construction in relation to framed structures*, London, 1873.

[10] A. Micheletti, "On generalized reciprocal diagrams for self-stressed frameworks", *International Journal of Space Structures*, vol. 23(3), pp. 153-166, 2008.

[11] R.McNeel & Assoc., *GhPython 3D*, http://www.food4rhino.com/app/ghpython, (accessed: 28.5. 2019.)

[12] R.McNeel & Assoc., *Grasshopper 3D*, http://www.grasshopper3D.com, (accessed: 28.5. 2019.)

[13] R.McNeel & Assoc., *Rhinoceros 3D*, http://www.rhino3d.com, (accessed: 28.5. 2019.)