Quantum Computers, Discrete Space, and Entanglement

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We consider algebras underlying Hilbert spaces used by quantum information algorithms. We show how one can arrive at equations on such algebras which define n-dimensional Hilbert space subspaces which in turn can simulate quantum systems on a quantum system. In doing so we use MMP diagrams and linear algorithms. MMP diagrams are tractable since an n-block of an MMP diagram has n elements while an n block of a standard Greechie diagram has 2^n elements. An immediate test for such an approach is a generation of minimal and arbitrary Kochen-Specker vectors and we present a minimal $n \geq 5$ "state-independent" Kochen-Specker set of seven vectors.

Keywords: Quantum computer algebra, MMP diagrams, Hilbert lattices, Kochen-Specker theorem, Hilbert space

I. INTRODUCTION

In this paper we consider an algebra underlying Hilbert space used by quantum information algorithms and we explore whether one can use it for quantum computers in the same way one uses the Boolean algebra for classical computers. Since the answer is in the negative, we consider the possible modifications of the afore mentioned quantum algebra which allow general applications of the algebra in formulating algorithms and simulating quantum systems.

We will first present classical vs. quantum algebras in the next section stressing that the quantum one has to be infinite. Since this is in contrast with finite quantum algebras available on quantum computers we proceed with presenting possible new finite quantum algebras and algorithms in Sec. III. As a result we obtain a general algorithm for obtaining Kochen-Specker vectors and therefore an automated proof of the Kochen-Specker theorem in Sec. IV.

II. ALGEBRAS

Classical computers standardly manipulate two-valued (0 and 1; bits, *bi*nary dig*its*) elements of information using switches (physical devices) which are called gates. Their design is based on (a two-valued) Boolean algebra, also called *switching* algebra. We should stress here that a Boolean algebra based on n-valued elements is equivalent to the one based on two-valued elements.

A Boolean algebra is an algebraic structure consisting of a set of elements together with two binary operations *join*, $\cup$ and *meet*, $\cap$ and a unary operation *orthocomplement*, $^\perp$, such that the closure property holds, the law of distributivity, $a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$, associativity, and commutativity hold, and the identity and orthocomplement exist (so, it is a distributive lattice; a lattice is an ordered set in which all joins and meet exist). Each of the operations and their combinations can be implemented in the form of logic circuits by means of gates. Hence, one performs a classical task by first digitizing it, then manipulating bits, and in the end translating bits back to the original language of the task (no classical computer can directly mimic a classical physical process). In doing so, one can access the values of all bits at any stage of their manipulation.

Quantum computers manipulate *qubits* (*qu*antum *bits*)—elements of quantum information (which are actually not digits but vectors (states) from Hilbert space) by means of *quantum gates*.

Closed subspaces of a Hilbert space form an algebra called a Hilbert lattice. A Hilbert lattice is an orthomodular lattice which, is by definition a (relaxed) Boolean algebra in which the distributivity (see above) holds if $b\leq a$ and $c\perp a$. In any Hilbert lattice the operation *meet*, $a\cap b$, corresponds to set intersection, $\mathcal{H}_a\bigcap\mathcal{H}_b$, of subspaces $\mathcal{H}_a,\mathcal{H}_b$ of a Hilbert space $\mathcal{H}$, the ordering relation $a\leq b$ corresponds to $\mathcal{H}_a\subseteq\mathcal{H}_b$, the operation *join*, $a\cup b$, corresponds to the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{H}_a\bigcup\mathcal{H}_b$, and the *orthocomplement* $a^\perp$ corresponds to $\mathcal{H}_a^\perp$, the set of vectors orthogonal to all vectors in $\mathcal{H}_a$. One can define all the lattice operations on a Hilbert space itself following the above definitions: $\mathcal{H}_a\cap\mathcal{H}_b=\mathcal{H}_a\bigcap\mathcal{H}_b$, $\mathcal{H}_a\cup\mathcal{H}_b=(\mathcal{H}_a^\perp\bigcap\mathcal{H}_b^\perp)^\perp$. Also, the orthogonality (mentioned above) $\mathcal{H}_a\perp\mathcal{H}_b$ means $\mathcal{H}_a\leq\mathcal{H}_b^\perp$. [1, p. 175], [2, pp. 21-29], [3, pp. 66,67], [4, pp. 8-16]

Thus, using the properties of Hilbert space one arrives at a definition of the Hilbert lattice as an orthomodular lattice which satisfies:

Completeness: The meet and join of any subset of a Hilbert lattice always exist.

Atomicity: Every non-zero element in an *HL* is greater than or equal to an atom. (An atom a is a non-zero lattice element with $0<b\leq a$ only if $b=a$.)

Superposition Principle: (The atom c is a superposition of the atoms a and b if $c\neq a$, $c\neq b$, and $c\leq a\cup b$.) 1. Given two different atoms a and b, there is at least one

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other atom c, $c \neq a$ and $c \neq b$, that is a superposition of a and b; 2. If the atom c is a superposition of distinct atoms a and b, then atom a is a superposition of atoms b and c.

Minimal length: The lattice contains at least three elements a, b, c satisfying: $0 < a < b < c < 1$.

One can also prove the other direction and therefore a Hilbert lattice is isomorphic to the set of closed subspaces of a Hilbert space. [5] Here comes a result we want to stress: It can be proved that a Hilbert lattice *must* contain infinite number of atoms. [6] Moreover, if we wanted a Hilbert lattice to provide us with a complex field over which Hilbert space can be defined, we should assume that the Hilbert lattice contains a countable infinite sequence of orthogonal elements.

Infinite dimensionality of a Hilbert space corresponds to the space continuity, to the integrals instead of sums, to radial functions and spherical harmonics, etc.; in a word, to all solutions of the Schrödinger equation we are used to. Therefore, the usual space distribution of, e.g., a wave function of electrons within, e.g., a molecule requires an infinite dimensional Hilbert space. Since we cannot have infinite dimensionality on a quantum computer we cannot directly simulate quantum mechanics on a quantum computer. But since both systems are quantum systems, a simulation—as opposed to the classical case—is nevertheless possible.

III. ALGORITHMS

In the literature simulation of quantum mechanics on a quantum computer has been approached in basically two ways. The first approach is to simulate one quantum system by another which resides in a quantum computer and might be simpler, e.g., proton spins by electrons in quantum dots. [7, 8] Or even "universal quantum computation over continuous variables for transformations that are polynomial in those variables." [9] This approach does not help us, though, since we have to find the algebra of the quantum computer system itself. The second approach is to simulate quantum mechanics by means of quantum gas model. [10, 11] Basically this boils down to application of the corresponding Schrödinger equation on points in a grid. As a result the points sit in the grid so as to fit the continuous wave function. Hence, a discrete set of points approximates a continuous wave function but we still do not have a genuine discrete algebra and discrete Hilbert space. Our aim is to investigate whether such discretization is possible.

We consider finite orthomodular lattices and filter them through the conditions stated in Sec. II and investigate properties which hold and which fail in the lattices. We want to find classes of such lattices which would approximate lattices with infinite number of atoms and in the end we want to compare them with lattices we derive from a finite dimensional Hilbert space.

The most attractive feature of such a procedure is that one can define finite lattices by algorithms which are not simply read off from the standard Hilbert space properties but are derived form highly nontrivial theorems derived in the theory of Hilbert lattices in the last 20 years. These algorithms also speed up calculations for several orders of magnitude. It can be shown that finite orthomodular lattices can be obtained from MMP diagrams which are organized as connected blocks of mutually orthogonal atoms. MMP diagrams are diagrams that are defined as follows:

1. Every vertex (i.e., atom when a diagram corresponds to a lattice) belongs to at least one block;
2. If there are at least two vertices then every block is at least 2-element;
3. Every block which intersects with another block is at least 3-element;

and then generated by the the isomorph-free generation procedure according to the following algorithm [12]:

procedure scan (D: diagram; β: integer)

  if D has exactly β blocks **then**
    output D
  else
    for each equivalence class of extensions $D + e$ **do**
      if $e \in m(D + e)$ **then** scan($D + e$,β)

end procedure

Without the latter algorithm MMP diagrams would be nothing but Greechie diagrams [13] with one of the conditions dropped. The isomorph-free generation procedure is what make them very different. Greechie diagrams are a handy way to draw Hasse diagrams but Hasse diagrams get more and more intrinsically complicated when we enlarge the number of atoms. E.g., a four-atom Greechie block has 16 elements, a five-atom Greechie block has 32 elements, and an n-atom Greechie block has 2^n elements, so they soon become intractable. MMP diagrams are however just strings. A five vertex block has 5 elements, an n vertex block has n elements.

Depending on parameters we use in their generation (parameters appear as options in our programs) MMP diagrams can be represented as lattices, but also as partially ordered sets, or as vectors from a Hilbert space which do not form a lattice; they can even be used for representing relations between vectors, planes, and subspaces of any n-dim space in classical physics. Which diagram will be appropriate for which purpose is determined by a selection procedure we use once they are generated.

So, the 3 simple aforementioned conditions imposed on diagrams gives us all we need to get all finite lattices of arbitrary complexity: we just eliminate diagrams in which Hilbert lattice properties do not hold. We currently use programs which generate and use lattices with up to 100 atoms but for all results we have obtained so far, 15 to 28 atoms suffice.

We were also able to reformulate Hilbert lattice properties and substitute 3 (conjectured all) classes of polynomial equations of the n-th order for the afore stated conditions. One such class was known before. [14] And the other two, the orthoarguesian class of n-th order and *quantum state equation* class, we found only recently. [15–17] As the name of the latter class tells us, it is determined by the kind of states imposed on any Hilbert lattice of Hilbert space, i.e., by possible evaluation of Hilbert lattice elements.

Let us see what makes a difference between a classical and a quantum state.

A state on a lattice L is a function $m : \mathrm{L} \longrightarrow [0,1]$ such that $m(1) = 1$ and $a \perp b \ \Rightarrow\ m(a \cup b) = m(a) + m(b)$. This yields $m(a) + m(a') = 1$ and $a \le b \ \Rightarrow\ ((m(a) = 1 \ \Rightarrow\ m(b) = 1)$.

A nonempty set S of states on L is *classical* if
$(\exists m \in S)(\forall a, b \in \mathrm{L})((m(a) = 1 \ \Rightarrow\ m(b) = 1) \ \Rightarrow\ a \le b)$
and *quantum* if
$(\forall a, b \in \mathrm{L})(\exists m \in S)((m(a) = 1 \ \Rightarrow\ m(b) = 1) \ \Rightarrow\ a \le b)$

Now we are able to prove the following

Theorem. Any orthomodular lattice that admits classical states is a Boolean algebra.

Theorem. Any Boolean algebra admits classical states and any Hilbert lattice admits quantum states.

Theorem. An orthomodular lattice that admits quantum states is still not necessarily a Hilbert algebra (lattice).

The proof of the latter theorem is simple: many of the orthoarguesian equations (characteristic of any 3 and more dimensional Hilbert space) fail in many MMP diagrams with loops of at least 5 blocks and interpreted as Hasse diagrams which allow quantum states.

Taken together, we conjecture that an infinite dimensional Hilbert space can be represented by a polynomial quantum algebra (Hilbert lattice reformulated by means of orthoarguesian and state equations) of the n-th order with $n \to \infty$ and for finite n such an algebra can be implemented on a would-be quantum computer. Qubits as the elements of the algebra obey superposition principle but do not allow a fixed evaluation (see above: there is no state for *every* element of the algebra). This is due to particular way in which the orthogonality can be defined in MMP diagrams, i.e., in Hilbert lattices and Hilbert space, and this orthogonality turns out to be very promising in solving problems because it can be reduced to linear equations as opposed to the standard approach to the orthogonality which is nonlinear. The details are presented in the next section.

IV. ORTHOGONALITY, ENTANGLEMENT, AND KOCHEN-SPECKER THEOREM

In 1993 "[w]e propos[ed] a new experiment employing two independent sources of spin correlated photon pairs. Two photons from different unpolarized sources each pass through a polarizer to a detector. Although their trajectories never mix or cross they exhibit 4th–order–interference–like correlations when the other two photons interfere on a beam splitter even when the latter two do not pass any polarizers at all" [18, 19], independently of [20, 21] and simultaneously with [21]. Later the obtained results have been verified "experimentally ... [by] two pairs of polarization entangled photons and subject[ing] one photon from each pair to a Bell-state measurement. This results in projecting the other two outgoing photons into an entangled state." [22] The very same scheme was also used for teleportation. [18–21, 23, 24]

That "entangle[ment] and correlat[ion] in polarization [of the other two photons] even when we do not measure polarization on the first two at all" [19] and also our discovery of a 100% polarization correlation between unpolarized photons [25] we arrived at by investigating creation and annihilation operators when acting on orthogonal states in the second quantization formalism. We realized that this orthogonality is crucially different from the classical orthogonality. In entanglement we make a tensor product state and then extract just a part of the state. Since in the obtained Hilbert subspace the orthogonality of the one dimensional subspaces containing relevant vectors means that they are included in the span of the other one dimensional subspaces of the subspace, i.e., that the vectors are orthogonal to each other, span the considered subspace, and make it a Hilbert space.

Exactly this property of the quantum orthogonality enables us to use linear instead of classical nonlinear equations. To see the meaning of the difference we will consider the old problem of finding finite Kochen-Specker vectors which prove the Kochen-Specker theorem.

Recently proposed experimental tests of Kochen-Specker theorem [26, 27] and disputes on feasibility of such experiments [28–32] prompted a renewed interest in the theorem and this an additional reason for reconsidering the theorem.

The original Kochen-Specker theorem [33] produced a set of 117 3-dimensional Hilbert space vectors for which there is no way to assign 1's and 0's to their states and therefore no way to provide quantum space with a classical Boolean model. The proof was tedious and subsequent attempts to reduce the number of vectors gave the following minimal results: 33 [34] and 31 [35, p. 114] 3-dim vectors, 18 [36] and 14 [37] 4-dim vectors, 29, 31, and 34 5-dim, 6-dim, and 7-dim vectors, respectively [38], 36 8-dim vectors [39], etc. Reducing the number of vectors turn out to be important because a direct connection between such vectors and an experimental setup can be established. [38] However, no general method for constructing sets of Kochen-Specker vectors has been proposed so far and here we give one.

The main idea of our approach is to first show that for particular set of orthogonal Hilbert space vectors one can impose no 0-1 state on the vectors. However, we do that using the Hilbert space orthogonality: $a \le b \cup c \cup \dots$, not the standard one: $(a, b) = 0$, $(a, c) = 0$, $\dots$, which boils down to a non-linear system: $a_1b_1+a_2b_2+a_3b_3+\cdots=0$, $a_1c_1+a_2c_2+a_3c_3+\cdots=0$, ... But even this Hilbert orthogonality we do not "calculate"—it is "built in" in the MMP diagrams by its generations algorithm. We only check whether one can or cannot impose classical 0-1 state on the diagrams. We then only have to find the one which does not allow such a state and this is done by a simple program which follows the definition of the classical state.

In order to convince the reader we will find a minimal set of vectors from a 5 dim space as an example. The smallest MMP diagram we find not to allow a classical state 0-1 is the following one: abc, cde, efa, egb, dgf (where, e.g, abc means an orthogonal triple: $a\perp b$, $a\perp c$, and $b\perp c$). Then we form equations corresponding to the inner products of 5 dim orthogonal vectors being equal to 0 and solve the system. We deal with triples and not with quintuples since we only have to find a set which does not allow 0-1 valuation. I.e., we follow the two Kochen-Specker (actually, Gleason's [40]) conditions:

1. No two orthogonal rays are both assigned the value 1;
2. In any group of n mutually orthogonal rays, not all of the rays are assigned the value 0.

Such triples are in principle just a part of a possible experiment. What is important is that for particular orthogonalities between chosen vectors we cannot ascribe 0-1 values to them.

Since only the directions of the vectors ("rays" [34]) are relevant they must be real. Since we did not care to find "nice looking" vectors some vectors are "big" due to a recursion procedure. This however do not affect the main aim of finding the vectors and it is to show that our approach works. a={608683911, 17315878, -22061625, -111556858, 20961326}, b={3, 68, -123, 52, 4}, c={1, 3, 5, 7, 11}, d={11, -11, 11, -11, 4}, e={1788, -8663, -1348, 8223, -2420}, f={5791304343, -304905182408, -1387655556967, 1686769435032, 7600253389432}, g={1, 1, 1, 1, 0}. The reader can introduce the vectors into, e.g., Wolfram's *Mathematica* and convince her- or himself that they really are orthogonal and, by simple combinatorics, that one cannot ascribe 0 or 1 to all of them (in each triple one element must be 1 and the other two zero and this is impossible).

Cabello [38] related his Kochen-Specker set of 18 vectors in 9 blocks with his experimental proposal in a four-dimensional Hilbert space [26] and he deals with 4-tuples. We deal with triples and leave a problem of finding a related experiment open. This is because we are first of all interested in finding a general algorithm for find all orthogonalities that do not allow 0-1 states. So, e.g., Cabello's 18 vectors in 9 blocks can form about cca 7000 MMP diagrams that do not allow 0-1 states. but do allow quantum states. Still, none of these examples (therefore not even the one elaborated in Ref. [38]) by itself correspond to a Hilbert space because their MMP diagrams do not correspond to lattices: the smallest triple lattice which do not allow states are two lattices with 19 atoms and 13 blocks and a quadruple lattice can only have more atoms and/or blocks. Other smaller cases with 18 vectors which do not allow 0-1 states are: 4 diagrams with 8 (quadruple blocks) blocks. The lowest number of quadruple blocks and vectors are: 1 4-block case with 10 vectors.

Let us be more specific: one of the obtained 18-9 MMP diagrams is: $abcd$, $defg$, $ghij$, $jklm$, $mnop$, $pqra$, $bikr$, $celn$, $fhoq$. And with a=$100\bar{1}$, b=0110, c=$11\bar{1}1$, d=$1\bar{1}11$, e=$111\bar{1}$, f=0101, g=$10\bar{1}0$, h=$010\bar{1}$, i=$1\bar{1}1\bar{1}$, j=1111, k=$11\bar{1}\bar{1}$, l=$1\bar{1}00$, m=$001\bar{1}$, n=0011, o=1000, p=0100, q=0010, r=1001, this is nothing but Cabello's 18-9 case. Graphically it means a hexagram with 3 ellipses contained in it. The smallest 4-block 10-5 case is: $abcd$, $defg$, $ghia$, $bfij$, $cehj$. Graphically it means a triangle with 2 ellipses contained in it (with one common vertex not contained in the triangle). However, it might turn out (we still have not checkedt) that so small a diagram cannot be ascribed real vectors in a 4-dim space and that we should go to higher dimensions to find real vector sets.

V. CONCLUSION

We have shown that one can build an algebra underlying Hilbert space which could be a universal algebra for quantum computers in the same way the Boolean algebra is for classical computers. In our approach the algebra is based on polynomial series of relations between one dimensional subspaces of a Hilbert space and linearly defined orthogonality relations between either subspaces or vectors of a Hilbert space.

Linear orthogonality defined through MMP diagrams possibly opens a way to substitute a linear for nonlinear coupling between qubits presently required for universal quantum computation. On the other hand such linear orthogonality defined through MMP diagrams already on our classical computers enabled speeding up calculations for more than 5 order of magnitude on the CPU time scale and enabled us to find polynomial expressions of the n-th order representing any Hilbert space and unknown so far. It also enabled us to find a general approach to finding Kochen-Specker vectors, some of which we presented above.

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