A multispecies Calogero model

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Abstract

We study a multispecies one-dimensional Calogero model with two- and three-body interactions. Using an algebraic approach (Fock space analysis), we construct ladder operators and find infinitely many, but not all, exact eigenstates of the model Hamiltonian. Besides the ground state energy, we deduce energies of the excited states. It turns out that the spectrum is linear in quantum numbers and that the higher-energy levels are degenerate. The dynamical symmetry responsible for degeneracy is $SU(2)$. We also find the universal critical point at which the model exhibits singular behaviour. Finally, we make contact with some special cases mentioned in the literature.

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1 Introduction

The ordinary Calogero [1] model describes N indistinguishable particles on the line which interact through an inverse-square two-body interaction and are subjected to a common confining harmonic force. The model is completely integrable in both the classical and quantum case [2], the spectrum is known and the wave functions are given implicitly. The model and its various descendants (also known as Calogero-Sutherland-Moser systems [3]) are connected with a number of physical problems, ranging from condensed matter physics [4] to gravity and black hole physics [5]. The algebraic structure of the Calogero model, studied earlier using group theoretical methods [2,6], has recently been reconsidered by a number of authors in the framework of the S_N (permutational) algebra [7]. This operator approach is considerably simpler than the original one, yields an explicit expression for the wavefunctions and emphasizes the interpretation in terms of generalized statistics [8], especially Haldane's exclusion statistics [9]. In Haldane's formulation there is the possibility of having particles of different species with a mutual statistical coupling parameter depending on the i^{th} and j^{th} species coupled. On the level of the Calogero model, this corresponds to the generalization of the ordinary 1D Calogero model with identical particles to the 1D Calogero model with non-identical particles. This can be done by allowing particles to have different masses and different couplings to each other. In this way we obtain a 1D multispecies Calogero model. Very little is known about its spectra and wavefunctions [10-13].

In the present Letter, which is in a sense a continuation of our investigation of the ordinary Calogero model [14], we use an algebraic (operator) method to find some of the salient features of the multispecies Calogero model on the line with two-and three-body interactions. In Section 2 we present the S_N invariant Hamiltonian H of the model, together with its ground state wavefunction and ground state energy. After performing a certain similarity transformation, we get a much simpler Hamiltonian $\tilde{H}$, which we separate into parts describing the center-of-mass motion and the relative motion of particles. We express $\tilde{H}$ in terms of generators of the $SU(1,1)$ algebra. All analysis is made in Hilbert space. Section 3 contains our most important results. By applying Fock space analysis, we find some of the excited states of $\tilde{H}$, their energies and degeneracies. Closer inspection of the Fock space that corresponds to the relative motion of particles reveals the existence of the universal critical point at which the system exhibits singular behaviour. This result generalizes that mentioned in [14]. We also establish the necessary conditions for the equivalence of the two multispecies Calogero models. In Section 4 we briefly repeat the main points of the paper and make contact with the models studied in [10-13]. We particularly discuss the necessary conditions for vanishing of the three-body interaction in the starting Hamiltonian H.

2 A multispecies Calogero model with a three-body interaction

Let us consider the most general Calogero type ground state for the N-body quantum mechanical problem on the line ($\hbar = 1$):

$$\Psi_0(x_1, x_2, \cdots x_N) = \Delta e^{-\frac{\omega}{2}\sum_{i=1}^{N} m_i x_i^2}, \tag{1}$$

where the prefactor Δ is given by

$$\Delta = \prod_{i<j}(x_i - x_j)^{\nu_{ij}}, \qquad \nu_{ij} = \nu_{ji}, \qquad i,j = 1,2\cdots N$$

and ν_{ij} are symmetric statistical parameters between particles (i,j). The harmonic frequency ω is taken to be the same for all particles. Masses of the particles (m_i) are, in general, not equal.

The Hamiltonian which possesses the above state (1) as the ground state is ($p_i = -i\frac{\partial}{\partial x_i}$)

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \frac{\omega^2}{2} \sum_{i=1}^{N} m_i x_i^2 + \frac{1}{4} \sum_{i \neq j} \frac{\nu_{ij}(\nu_{ij}-1)}{(x_i - x_j)^2} (\frac{1}{m_i} + \frac{1}{m_j})$$

$$+\frac{1}{2} \sum_{i,j,k\neq} \frac{\nu_{ij}\nu_{jk}}{m_j(x_j - x_i)(x_j - x_k)}, \qquad (2)$$

$$H\Psi_0(x_1, x_2, \cdots x_N) = E_0 \Psi_0(x_1, x_2, \cdots x_N),$$

$$E_0 = \omega(\frac{N}{2} + \frac{1}{2} \sum_{i\neq j} \nu_{ij}).$$

The symbol $(i,j,k\neq)$ in the last term denotes the summation over all triples of mutually distinct indices.

The Hamiltonian in Eq.(2) is invariant under the group of permutation of N elements, S_N. The elementary generators K_{ij} of the symmetry group S_N exchange labels i and j in *all quantities*, according to the rules:

$$K_{ij}x_j = x_i K_{ij},\ K_{ij}m_j = m_i K_{ij},\ K_{ij}\nu_{jl} = \nu_{il}K_{ij},$$

$$K_{ij} = K_{ji},\ (K_{ij})^2 = 1,$$

$$K_{ij}K_{jl} = K_{jl}K_{il} = K_{il}K_{ij},\ \text{for } i \neq j,\ i \neq l,\ j \neq l.$$

It follows that $K_{ij}HK_{ij} = H$, i.e. $[H, K_{ij}] = 0$, $\forall i, j$.
In a sense, one may think of (2) as a smooth, sufficiently small deformation of the ordinary Calogero model.

A few additional remarks concerning the Hamiltonian (2) are in order.
(1) It describes distinguishable particles on the line, interacting with harmonic, two-body and three-body potentials. As far as we know, it is the first time that such Hamiltonian is considered in full generality. An earlier attempt to solve the similar, but less general Hamiltonian (with all masses m_i equal) can be find in [11] (see also Ref.[13]).
(2) The asymptotic behaviour of its eigenstates should be $\Psi \propto (x_i - x_j)^{\nu_{ij}}$ as $(x_i - x_j) \to 0$.
(3) A well-known stability condition demands that the two-body couplings $\nu_{ij}(\nu_{ij} - 1)$ should be greater than $-\frac{1}{4}$, $\forall i, j$.
(4) Setting $\nu_{ij} = \nu$, $\forall i, j$ and $m_i = m$, $\forall i$, we recover the ordinary N-body Calogero model [1]. In that case, owing to the identity $\sum_{cycl.} \frac{1}{(x_i - x_j)(x_i - x_k)} = 0$ which holds in 1D, the three-body term in Eq.(2) trivially vanishes. When ν_{ij} are mutually different but $m_i = m$, $\forall i$, we recover the model treated in [11,13]. Finally, when $\nu_{ij} = \alpha m_i m_j$, α being constant, we obtain the model mentioned in [10,12]. We comment on these cases in Section 4.
In the following we analyse the most general case, namely the Hamiltonian of Eq.(2) without restrictions (4), given above.

Let us perform the non-unitary transformation on $\Psi_0(x_1, x_2, \cdots x_N)$:

$$\tilde{\Psi}_0(x_1, x_2, \cdots x_N) = \Delta^{-1}\Psi_0(x_1, x_2, \cdots x_N) = e^{-\frac{\omega}{2}\sum_{i=1}^{N} m_i x_i^2}. \tag{3}$$

It generates a similarity transformation which leads to another S_N invariant (but non-Hermitean) Hamiltonian $\tilde{H}$:

$$\tilde{H} = \Delta^{-1} H \Delta,$$

$$\tilde{H}\tilde{\Psi}_0(x_1, x_2, \cdots x_N) = E_0 \tilde{\Psi}_0(x_1, x_2, \cdots x_N),$$

with E_0 given in Eq.(2).

We find $\tilde{H}$ as

$$\tilde{H} = -\frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\frac{\partial^2}{\partial x_i^2} + \frac{\omega^2}{2}\sum_{i=1}^{N} m_i x_i^2 - \frac{1}{2}\sum_{i\neq j}\frac{\nu_{ij}}{(x_i - x_j)}\left(\frac{1}{m_i}\frac{\partial}{\partial x_i} - \frac{1}{m_j}\frac{\partial}{\partial x_j}\right). \qquad (4)$$

Notice that in Eq.(4) two-body and three-body interactions apparently disappeared but they are hidden in the last term of Eq. (4).

It is convenient to introduce the variables (X, ξ_i)

$$X = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} \equiv \frac{1}{M}\sum_{i=1}^{N} m_i x_i \quad , \qquad \xi_i = x_i - X, \quad i = 1, 2, \cdots N, \qquad (5)$$

and the linear combinations of derivatives $(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial X})$

$$\frac{\partial}{\partial \xi_i} = \frac{\partial}{\partial x_i} - \frac{m_i}{M}\frac{\partial}{\partial X} \quad , \qquad \frac{\partial}{\partial X} = \sum_{i=1}^{N}\frac{\partial}{\partial x_i}, \quad i = 1, 2, \cdots N. \qquad (6)$$

Note that the variables ξ_i (as well as $\frac{\partial}{\partial \xi_i}$) are not linearly independent, i.e. $\sum_{i=1}^{N} m_i \xi_i = \sum_{i=1}^{N} \frac{\partial}{\partial \xi_i} = 0$.

In terms of the variables just introduced, the Hamiltonian $\tilde{H}$, Eq.(4), separates into parts which describe its center-of-mass motion (CM) and its relative motion (R), namely $\tilde{H} = \tilde{H}_{CM} + \tilde{H}_R$, with

$$\tilde{H}_{CM} = -\frac{1}{2M}\frac{\partial^2}{\partial X^2} + \frac{1}{2}M\omega^2 X^2,$$

$$\tilde{H}_R = -\frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\frac{\partial^2}{\partial \xi_i^2} + \frac{1}{2}\omega^2\sum_{i=1}^{N} m_i \xi_i^2 - \frac{1}{2}\sum_{i\neq j}\frac{\nu_{ij}}{(\xi_i - \xi_j)}\left(\frac{1}{m_i}\frac{\partial}{\partial \xi_i} - \frac{1}{m_j}\frac{\partial}{\partial \xi_j}\right). \qquad (7)$$

The Hamiltonian H, Eq.(2), can also be decomposed into H_{CM} and H_R.

The wave function (3) separates as

$$\tilde{\Psi}_0(x_1, x_2, \cdots x_N) = \tilde{\Psi}_0(X)\tilde{\Psi}_0(\xi_1, \xi_2 \cdots \xi_N) = e^{-\frac{M\omega}{2}X^2} e^{-\frac{\omega}{2}\sum_{i=1}^N m_i \xi_i^2}.$$

The ground state energy E_0 splits into the energy of CM ($E_{0CM} = \frac{1}{2}\omega$) and the energy of relative motion ($E_{0R} = \frac{N-1}{2}\omega + \frac{1}{2}\omega \sum_{i \neq j} \nu_{ij}$).

We define the set of operators $\{T_+, T_-, T_0\}$ as

$$T_+ = \frac{1}{2}\sum_{i=1}^N m_i x_i^2,$$

$$T_- = \frac{1}{2}\sum_{i=1}^N \frac{1}{m_i}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i \neq j} \frac{\nu_{ij}}{(x_i - x_j)}(\frac{1}{m_i}\frac{\partial}{\partial x_i} - \frac{1}{m_j}\frac{\partial}{\partial x_j}),$$

$$T_0 = \frac{1}{2}(\sum_{i=1}^N x_i \frac{\partial}{\partial x_i} + \frac{E_0}{\omega}). \tag{8}$$

Using Eqs.(5) and (6) one can easily show that these operators also split as $T_{\pm,0} = T_{\pm,0_{(CM)}} + T_{\pm,0_{(R)}}$.

Operators $\{T_+, T_-, T_0\}$ satisfy the $SU(1,1)$ algebra:

$$[T_-, T_+] = 2T_0 \quad , \qquad [T_0, T_\pm] = \pm T_\pm.$$

In terms of these operators, the Hamiltonian (4) reads $\tilde{H} = \omega^2 T_+ - T_-$.

3 Ladder operators and Fock space representation

Now we introduce pairs of creation and annihilation operators $\{A_1^+, A_1^-\}$ and $\{A_2^+, A_2^-\}$:

$$A_1^\pm = \frac{1}{\sqrt{2}}(\sqrt{M\omega}X \mp \frac{1}{\sqrt{M\omega}}\frac{\partial}{\partial X}),$$

$$A_2^{\pm} = \frac{1}{2}(\omega T_+ + \frac{1}{\omega}T_-) \mp T_0, \tag{9}$$

which satisfy the following commutation relations:

$$[A_1^-, A_1^+] = 1, \qquad [A_2^-, A_2^+] = \frac{1}{\omega}\tilde{H},$$

$$[A_1^-, A_2^-] = [A_1^+, A_2^+] = 0, \qquad [A_1^-, A_2^+] = A_1^+, \qquad [A_1^+, A_2^-] = -A_1^-,$$

$$[\tilde{H}, A_1^{\pm}] = \pm\omega A_1^{\pm}, \tag{10}$$

$$[\tilde{H}, A_2^{\pm}] = \pm 2\omega A_2^{\pm}.$$

They act on the Fock vacuum $|\tilde{0}\rangle \propto \tilde{\Psi}_0(x_1, x_2, \cdots x_N)$ as

$$A_1^-|\tilde{0}\rangle = A_2^-|\tilde{0}\rangle = 0, \qquad \langle\tilde{0}|\tilde{0}\rangle = 1.$$

The excited states are built as

$$|n_1, n_2\rangle \propto A_1^{+n_1} A_2^{+n_2}|\tilde{0}\rangle, \qquad \forall n_1, n_2 = 0, 1, \cdots \tag{11}$$

The repeated action of the operators A_1^+ (A_2^+) on the vacuum $|\tilde{0}\rangle$ reproduces in the coordinate representation Hermite polynomials (hypergeometric function), respectively.

The states $|n_1, n_2\rangle$ are eigenstates of $\tilde{H}$, Eq.(4), with the eigenvalues (cf. Eq.(10))

$$E_{n_1,n_2} = \omega(n_1 + 2n_2) + E_0. \tag{12}$$

The energy spectrum is *linear* in quantum numbers n_1, n_2. This result is universal, i.e. it holds for all parameters m_i and ν_{ij} in the Hamiltonian $\tilde{H}$ (or H). Notice that the energy of the ground state and the energy of the first excited state are non-degenerate whereas the higher energy levels are degenerate. The structure of degeneracy is as follows:

n_1	n_2	$n = n_1 + 2n_2$	Degenerate states
0	0	0	$\|\tilde{0}\rangle$
1	0	1	$A_1^+\|\tilde{0}\rangle$
2	0	2	$A_1^{+2}\|\tilde{0}\rangle$
0	1	2	$A_2^+\|\tilde{0}\rangle$
1	1	3	$A_1^+A_2^+\|\tilde{0}\rangle$
3	0	3	$A_1^{+3}\|\tilde{0}\rangle$
0	2	4	$A_2^{+2}\|\tilde{0}\rangle$
2	1	4	$A_1^{+2}A_2^+\|\tilde{0}\rangle$
4	0	4	$A_1^{+4}\|\tilde{0}\rangle$
5	0	5	$A_1^{+5}\|\tilde{0}\rangle$
3	1	5	$A_1^{+3}A_2^+\|\tilde{0}\rangle$
1	2	5	$A_1^+A_2^{+2}\|\tilde{0}\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

It is evident that for $n = even$, the degeneracy is $(\frac{n}{2}+1)$ and for $n = odd$, the degeneracy is $(\frac{n+1}{2})$.

In order that the two models described by Hamiltonian (2), with statistical parameters ν_{ij} and ν'_{ij}, have the same tower of states (11) and the same spectrum (12), the necessary conditions are $\sum_{i<j}\nu_{ij} = \sum_{i<j}\nu'_{ij}$, $\omega = \omega'$ and the number of particles should be the same, i.e. $N = N'$.

The dynamical symmetry algebra of the model is defined as maximal algebra commuting with the Hamiltonian $\tilde{H}$. The dynamical symmetry of the ordinary Calogero model is complicated polynomial algebra denoted by $C_N(\nu)$ in [15]. In our case, owing to the fact that $\tilde{H}$ (10) can be rewritten in terms of two independent, uncoupled oscillators (see bellow Eqs.(13) and (14)), this polynomial algebra can be linearized to the ordinary $SU(2)$ algebra. This is the minimal symmetry that remains in the generic case, i.e. for general ν_{ij} and m_i. In fact, this is the same dynamical symmetry underlying the two-body Calogero model [15,16].

We point out that one can construct the creation and annihilation operators $\{B_2^+, B_2^-\}$:

$$B_2^{\pm} = A_2{}^{\pm} - \frac{1}{2} A_1^{\pm 2}. \tag{13}$$

In terms of $A_1^{\pm}$ and $B_2^{\pm}$, the above-mentioned $SU(2)$ algebra ($[J_+, J_-] = 2J_0$, $[J_0, J_{\pm}] = \pm J_{\pm}$) is generated by

$$J_+ = A_1^{+2} B_2^- \frac{1}{\sqrt{2(\hat{N}_2 - 1 + \frac{E_{0R}}{\omega})(\hat{N}_1 + 1)}},$$

$$J_- = B_2^+ A_1^{-2} \frac{1}{\sqrt{2(\hat{N}_2 + \frac{E_{0R}}{\omega})(\hat{N}_1 - 1)}},$$

$$J_0 = \frac{1}{4}(\hat{N}_1 - 2\hat{N}_2).$$

Here, $\hat{N}_1$ and $\hat{N}_2$ are number operators counting A_1- and B_2- modes, respectively.

The benefit of the construction (13) is that the operators $A_1^{\pm}$, corresponding to the CM motion, decouple completely (cf. Eq.(10)), i.e.

$$[A_1^{\pm}, B_2^{\mp}] = 0.$$

Hence, we get

$$\tilde{H}_{CM} = \frac{1}{2}\omega \{A_1^-, A_1^+\}_+ \equiv \omega(\hat{N}_1 + \frac{E_{0CM}}{\omega}),$$

$$\tilde{H}_R = \omega [B_2^-, B_2^+] \equiv \omega(2\hat{N}_2 + \frac{E_{0R}}{\omega}),$$

$$[\tilde{H}_R, B_2^{\pm}] = \pm 2\omega B_2^{\pm}. \tag{14}$$

The Fock space now splits into the CM-Fock space, spanned by $A_1^{+n_1}|\tilde{0}\rangle_{CM}$, and the R-Fock space, spanned by $B_2^{+n_2}|\tilde{0}\rangle_R$, where $|\tilde{0}\rangle_{CM} \propto \tilde{\Psi}_0(X)$ and $|\tilde{0}\rangle_R \propto \tilde{\Psi}_0(\xi_1, \xi_2 \cdots \xi_N)$.

At this point it is useful to make a contact with the Fock space of the ordinary Calogero model. In the ordinary Calogero model the S_N-symmetric subspace of the whole Fock space is spanned by the states $\mathcal{A}_1^{+n_1}|\tilde{0}\rangle_{CM}$ and $(\mathcal{B}_2^{+n_2}\cdots\mathcal{B}_N^{+n_N})|\tilde{0}\rangle_R$ [7,16], where

$$\mathcal{A}_1=\sum_{i=1}^N a_i, \qquad \mathcal{B}_k=\sum_{i=1}^N (a_i-\frac{1}{N}\mathcal{A}_1)^k, \qquad 2\le k\le N.$$

The so-called Dunkl-Polychronakos operators a_i satisfy the algebra [7]

$$[a_i,a_j^\dagger]=\delta_{ij}(1+\nu\sum_{k=1}^N K_{ik})-\nu K_{ij}, \tag{15}$$

$$[a_i^\dagger,a_j^\dagger]=[a_i,a_j]=0.$$

Our operators A_1 and B_2 correspond exactly to the operators $\mathcal{A}_1$ and $\mathcal{B}_2$ in the ordinary Calogero model. Within our algebraic treatment, we are unable to construct the eigenstates of (2) which correspond to the Calogero-states $(\mathcal{B}_3^{+n_3}\cdots\mathcal{B}_N^{+n_N})|\tilde{0}\rangle_R$.

Reducing the problem (2) to the (4) and (14), i.e. $H\to\tilde{H}\to\tilde{H}_R$, has an interesting consequence, namely the existence of the universal critical point, defined by the null-vector

$$\frac{{}_R\langle\tilde{0}|B_2^-B_2^+|\tilde{0}\rangle_R}{{}_R\langle\tilde{0}|\tilde{0}\rangle_R}=\frac{N-1}{2}+\frac{1}{2}\sum_{i\neq j}\nu_{ij}=0. \tag{16}$$

The above relation (16) follows directly from Eq.(14) by demanding ${}_R\langle\tilde{0}|\tilde{H}_R|\tilde{0}\rangle_R=E_{0R}=0$. More generally, from Eq.(14) (see also Ref.[17]) immediately follows that

$$B_2^+B_2^-=\hat{N}_2(\hat{N}_2-1+\frac{E_{0R}}{\omega})\equiv\phi(\hat{N}_2),$$

$$B_2^-B_2^+=(\hat{N}_2+1)(\hat{N}_2+\frac{E_{0R}}{\omega})\equiv\phi(\hat{N}_2+1),$$

$$\frac{{}_R\langle\tilde{0}|B_2^{-m}B_2^{+m}|\tilde{0}\rangle_R}{{}_R\langle\tilde{0}|\tilde{0}\rangle_R}=\prod_{k=1}^m\phi(k). \tag{17}$$

Since Eq.(16) implies that $\phi(1) = \frac{E_{0R}}{\omega} = 0$, it is obvious that the critical point (Eq.(16)) is unique, i.e. there are no similar critical points when norms of states involving higher powers of the operators B_2 are involved.

At the critical point the system described by $\tilde{H}_R$ collapses completely. This means that the relative coordinates, the relative momenta and the relative energy are all zero at this critical point. There survives only one oscillator, describing the motion of the centre-of-mass. This singular behaviour was first noticed in [14] for the case $\nu_{ij} = \nu$ and $m_i = m$. Of course, for the initial Hamiltonian H, which is not unitary (i.e. physically) equivalent to $\tilde{H}$, this corresponds to some $\nu_{ij} < 0$ and the norm of the wave function (1) blows up at the critical point. For ν_{ij} negative but greater than the critical values (16), the wave function is singular at coincidence points but still quadratically integrable.

4 Discussion and outlook

In this Letter we have studied the most general multispecies Calogero model on the line, Eq.(2) , with a three-body interaction and an extended S_N invariance. By applying the similarity transformation Δ, we have obtained the Hamiltonian (4), on which we have performed the Fock space analysis (9,10) and found some of its (but not all) excited states, Eq.(11), and their energies E_{n_1,n_2}, (12). It turns out that the energy (12) is *linear* in quantum numbers n_1 and n_2 and there is a dynamical $SU(2)$ symmetry responsible for the degeneracy of higher-energy levels with $n = n_1 + 2n_2 \geq 2$. By splitting the Fock space into the CM-Fock space and the R-Fock space (14), we have detected the universal critical point (16) at which the system exhibits singular behaviour.

To conclude this analysis, let us consider the last term in (2) more closely, namely

$$\frac{1}{2}\sum_{i,j,k\neq}\left(\frac{\nu_{ij}\nu_{jk}}{m_j}\right)\frac{1}{(x_j-x_i)(x_j-x_k)}. \tag{18}$$

If we put $m_j = m = const.$ in (18), $\forall j$, symmetrize under the cyclic exchange of the indices $(i \to j \to k \to i)$ and reduce the sum to a common denominator using the identity

$$\sum_{cycl.}\frac{1}{(x_i-x_j)(x_i-x_k)} = 0,$$

we obtain that the necessary condition for vanishing of the three-body interaction is $\nu_{ij} = \nu = const.$, $\forall i, j$. In this way, the problem (2) is reduced to the ordinary N-body Calogero model with two-body interactions only [1].

For the general ν_{ij} and m_j, the above procedure yields the following necessary conditions for the absence of the three-body interaction (18):

$$\frac{\nu_{ij}\nu_{jk}}{m_j} = \frac{\nu_{jk}\nu_{ki}}{m_k} = \frac{\nu_{ki}\nu_{ij}}{m_i}. \qquad \forall (i,j,k) \tag{19}$$

The unique solution of these conditions is $\nu_{ij} = \alpha\, m_i\, m_j$, α being some universal constant. This particular connection between masses and interaction parameters was also displayed in [10,12]. In [12], the condition (19) arose from the demand that the asymptotic Bethe ansatz should be applicable to the ground state of a multispecies many-body quantum system obeying mutual statistics, while in [10] its origin was not obvious. In our approach, it has the simplest possible interpretation. The results presented here are easily applied to the model with F distinct families of particles. For example, for $\alpha = 1$, the ground state energy becomes

$$E_0 = \omega\left(\frac{N}{2} + \sum_{a=1}^{F} m_a^2\frac{N_a(N_a-1)}{2} + \frac{1}{2}\sum_{a\neq b}^{F} m_a m_b N_a N_b\right),$$

$$N = \sum_{a=1}^{F} N_a. \tag{20}$$

Two systems characterized by $\{\omega, m_a, N_a\}$ and $\{\omega', m'_a, N'_a\}$ are identical if $\sum N_a = \sum N'_a$, $\omega = \omega'$ and $E_0 = E'_0$.

The open problem that still remains is the construction of (generalized) Dunkl-Polychronakos operators a_i and $a_i^\dagger$ (15), which may help in finding the complete set of eigenstates of the Hamiltonian (2) (or (4)), in one-to-one correspondence with the ordinary Calogero model.

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