Perturbations of the excited quantum oscillator: From number states to statistical distributions

D. K. Sunko
Department of Physics, Faculty of Science, University of Zagreb, 10000 Zagreb, Croatia

B. Gumhalter[a)]
Institute of Physics of the University, P.O. Box 304, 10001 Zagreb, Croatia

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We discuss the transitions that an external time-dependent perturbation can induce upon a quantum harmonic oscillator in an excited initial state. In particular, we show how to describe transitions of the oscillator from initial states characterized by statistical distributions. These results should be useful for interpretations of the properties of weakly dispersive bosonic excitations in quantum systems whose dynamics is investigated by time or energy resolved spectroscopies. © *2004 American Association of Physics Teachers.*

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I. INTRODUCTION: THE FORCED OSCILLATOR MODEL

In many areas of quantum physics the forced oscillator model provides a paradigm for demonstrating exact nonperturbative solutions to systems subjected to strong time-dependent perturbations. Many authors have used this model as a testing ground for nearly all the many-body techniques developed during that time.[1–5] In addition, a number of realistic problems can be fruitfully treated within this simple model[5–16] or its generalizations.[7,17–21]

The present work revisits the same problem once more with a different agenda. Our primary aim is to describe how to use the forced oscillator model to calculate the excitation probabilities for an oscillator whose initial state is characterized by a statistical distribution. In Sec. II we first introduce the forced oscillator model in second quantization notation. We begin with the usual exact solution for the evolution operator of the oscillator subject to an external time-dependent force coupled linearly to the oscillator displacement; the corresponding scattering operator is derived by a limiting procedure. We present in Sec. III some useful novel derivations of the expressions for state-to-state transition probabilities of the oscillator. By observing the symmetries that these probabilities should satisfy, we are able to demonstrate some interconnections between the present formulas and the ones obtained in various earlier treatments of the same problem, giving more insight into the temporal evolution of the studied system. In particular, in the last part of Sec. III and Appendix A we show the equivalence of the results of two alternative operator disentanglements that appear naturally in the calculation of transition matrix elements.

The main body of the paper is in Sec. IV, which gives the first systematic study of transitions of the forced quantum oscillator whose initial state is characterized by a statistical distribution. We derive two interesting results. The first is the generating function for the transition probabilities, obtained using the results of Sec. III. As a by-product, we also find a surprisingly simple shortcut to the spectral density of a perturbed oscillator initially in thermal equilibrium, derived in textbooks by a rather involved time-dependent Green's function formalism.[5] Our second result is a general expression for the transition probabilities as a functional of the initial distribution of oscillator states, which should enable the tackling of a wide class of physical problems that can be described by the forced oscillator model. Although this expression cannot be generally evaluated in a closed form, it efficiently provides a formula for the probability of a transition into any particular final state. All that is required is that the generating function of the initial distribution be known. As an example, we introduce the two-pulse problem: one pulse excites the oscillator from its ground state into a known distribution of excited states. The second pulse then acts on this distribution, and the above-mentioned formula provides the excitation probability of any final state. The calculation is worked out in some detail, showing for a concrete example that the result is symmetric with respect to the order of application of the two pulses.

II. PROBLEM AND NOTATION

The Hamiltonian corresponding to the forced oscillator model can be written in the simple form:

$$H = H_0 + V(t), \tag{1}$$

where

$$H_0 = \hbar\omega_0(a^\dagger a + \tfrac{1}{2}) \tag{2}$$

is the Hamiltonian of an unperturbed quantized harmonic oscillator of mass m and characteristic frequency ω_0, with $a^\dagger$ and a denoting the usual creation and annihilation operators of noninteracting bosons, respectively. The eigenstates $|n\rangle$ of H_0 are the eigenstates of the number operator $\hat{N} = a^\dagger a$, that is, they satisfy

$$\hat{N}|n\rangle = n|n\rangle, \tag{3}$$

and

$$H_0|n\rangle = \hbar\omega_0(n + \tfrac{1}{2})|n\rangle, \tag{4}$$

with the eigenvalues $n = 0,1,2,\ldots$, which are the number of bosons in the state $|n\rangle$ with the total energy $n\hbar\omega_0$.

The perturbation $V(t)$ is taken to describe the linear coupling of the oscillator displacement to an external time-dependent force $F(t)$:

$$V(t) = -u_0(a + a^\dagger)F(t), \tag{5}$$

where $u_0=\sqrt{\hbar/2m\omega_0}$ is the root-mean-square displacement of the oscillator. $F(t)$ may represent the force acting on the oscillator due to a potential source, such as an external electromagnetic field or a massive moving particle.

The final state of an oscillator subjected to an external perturbation $V(t)$ is obtained as a result of the action of the scattering operator S on the initial oscillator state $|\psi_i\rangle$. S is obtained by a limiting procedure:

$$S=\lim_{t\to\infty,\ t'\to-\infty} U(t,t')=\lim_{t\to\infty,\ t'\to-\infty} e^{-iH(t-t')/\hbar}, \tag{6}$$

where $U(t,t')$ is the evolution operator of the system. The transition amplitudes are most conveniently calculated from the evolution operator in the interaction or Dirac picture, $U_I(t,t')$, which is defined by

$$U(t,t')=e^{-iH_0(t-t')}U_I(t,t'). \tag{7}$$

As was first published by Dyson[22] and is now in many textbooks,[23–26] U_I can be represented in the form:

$$U_I(t,t')=\mathcal{T}\exp\left(-\frac{i}{\hbar}\int_{t'}^{t}d\tau V_I(\tau)\right), \tag{8}$$

where $\mathcal{T}$ stands for Dyson's chronological or time ordering operator and $V_I(\tau)=e^{iH_0\tau/\hbar}V(\tau)e^{-iH_0\tau/\hbar}$. The scattering operator in the interaction picture is then obtained from the limiting procedure

$$S_I=\lim_{t\to\infty,t'\to-\infty} U_I(t,t'). \tag{9}$$

Of particular interest are nonperturbative solutions to the forced oscillator model for which the external perturbations vanish in the remote past and in the remote future, that is, solutions that satisfy the scattering boundary conditions:

$$F(t\to\pm\infty)\to 0. \tag{10}$$

It has been shown by many authors using very different approaches, such as the Green's function method,[1,2] Wick's theorem,[3,4] the canonical transformation method,[6] and nested commutator expansion,[7] that for the particular interaction (5) and the scattering boundary conditions (10), the operator S_I takes the form:

$$S_I=e^{-i(G_1+G_2)}=e^{-i(G_++G_-)-iG_2}, \tag{11}$$

where $G_+=ga^\dagger$ and $G_-=g^*a=(G_+)^\dagger$ are linear in the boson operators $a^\dagger$ and a, respectively. The coupling functions are given by

$$g=\frac{u_0}{\hbar}\int_{-\infty}^{\infty}e^{i\omega_0\tau}F(\tau)d\tau, \tag{12}$$

and

$$G_2=-\frac{u_0^2}{\hbar^2}\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\tau}d\tau'\,F(\tau)F(\tau')\sin\omega_0(\tau-\tau') \tag{13}$$

is a phase shift which is independent of the operators a and $a^\dagger$. If the initial state of the oscillator $|\psi_i\rangle$ at the time $t\to-\infty$ is known, the final state $|\psi_f\rangle$ at the time $t\to\infty$ is obtained from

$$|\psi_f\rangle=S_I|\psi_i\rangle. \tag{14}$$

In practice, $|\psi_i\rangle$ may represent either an eigenstate $|n\rangle$ of the oscillator or a mixed state characterized by a given initial distribution of the eigenstates. In the following we shall consider both cases and find the oscillator final states for several initial state distributions.

III. PROBABILITY AMPLITUDES FOR STATE-TO-STATE TRANSITIONS OF A QUANTUM OSCILLATOR PERTURBED BY AN EXTERNAL FORCE

The goal in calculations of various transition probabilities involving a perturbed quantum oscillator is the probability $P_{m,n}$ of the transition from an unperturbed oscillator eigenstate $|n\rangle$ to another eigenstate $|m\rangle$. $P_{m,n}$ is given in terms of the matrix elements of S_I as

$$P_{m,n}=|S_{m,n}|^2=|\langle m|S_I|n\rangle|^2. \tag{15}$$

The problem of calculating $S_{m,n}$ has been extensively treated in the literature (see, for example, Refs. 1–7, and references therein). In this section, we shall rederive the basic expression for $S_{m,n}$, using a somewhat different approach that is better suited to our purpose.

The calculation of $S_{m,n}$ starts with the application of the Weyl or Baker–Hausdorff formula to the right-hand side of Eq. (11). This formula states that if the commutator $[A,B]$ of the operators A and B commutes with A and B, the identity,[27]

$$e^{A+B}=e^Ae^Be^{-(1/2)[A,B]}, \tag{16}$$

holds. To apply Eq. (16) to Eq. (11), we have to assume a definite order of the operators G_+ and G_-. If we associate A with iG_+ and B with $-iG_-$, Eq. (16) yields:

$$S_I=e^{-(1/2)|g|^2}e^{-iG_+}e^{-iG_-}, \tag{17}$$

where for later convenience we have omitted the phase shift $-iG_2$ because it does not contribute to the absolute square in Eq. (15).

The derivation of $S_{m,n}$ for the opposite order of the operators G_+ and G_- is also instructive and is discussed in Appendix A. If we make use of the disentangled form (17), we have:

$$S_{m,n}=e^{-(1/2)|g|^2}\langle m|e^{-iga^\dagger}e^{-ig^*a}|n\rangle, \tag{18}$$

which is a standard expression that has been used as a point of departure in calculations of the transition probabilities.[2–4] Although these calculations are instructive in their own right, the simplest and most direct way to proceed is to expand the operators e^{-iG_+} and e^{-iG_-} in Eq. (18) in a power series. The property $a|0\rangle=0$ cuts off the exponential series in $-iG_-$ or $-iG_+$, that is,

$$e^{-iG_-}|n\rangle=\sum_{s=0}^{n}\frac{(-ig^*a)^s}{s!}|n\rangle. \tag{19}$$

We assume for definiteness $m\geqslant n$ and obtain

$$S_{m,n}=e^{-(1/2)|g|^2}\sum_{p=0}^{\infty}\sum_{s=0}^{n}\left(\frac{m!n!}{(m-p)!(n-s)!}\right)^{1/2}\times\frac{(-ig)^p}{p!}\frac{(-ig^*)^s}{s!}\langle m-p|n-s\rangle$$
$$=e^{-(1/2)|g|^2}\left(\frac{n!}{m!}\right)^{1/2}(-ig)^{m-n}\times\sum_{s=0}^{n}\frac{(-|g|^2)^s m!}{s!(m-n+s)!(n-s)!},\tag{20}$$

where we have used the orthogonality of the number states, that is, $\langle m-p|n-s\rangle=\delta_{m-n+s,p}$, to eliminate the sum over p. The sum on the right-hand side of Eq. (20) is the associated Laguerre polynomial $L_n^{m-n}(|g|^2)$, and hence:

$$S_{m,n}=e^{-(1/2)|g|^2}\left(\frac{n!}{m!}\right)^{1/2}(-ig)^{m-n}L_n^{m-n}(|g|^2)\quad(m\geqslant n).\tag{21}$$

We could obtain the case $m\leqslant n$ in an analogous fashion, but we can avoid the additional work by exploiting the property,

$$(-1)^m\frac{m!}{x^m}L_m^{n-m}(x)=(-1)^n\frac{n!}{x^n}L_n^{m-n}(x),\tag{22}$$

which means that the scattering matrix is Hermitian. This property extends Eq. (21) to the case $m\leqslant n$, allowing it to be written in an alternative form with m and n interchanged, and g replaced by g^*. Consequently, $P_{m,n}=P_{n,m}$.

These results are identical to the ones quoted in Refs. 2–4.[28] It should be observed that the appearance of the polynomials in Eq. (21) is due to the specific order of the operators e^{-iG_+} and e^{-iG_-} in S_I, because this order introduces the cutoff in Eq. (19). If we start from the reverse order of the operators in Eq. (16), we must first carry out a resummation of an infinite series in order to retrieve the result (21) (see Appendix A).

IV. PROBABILITY AMPLITUDES FOR TRANSITIONS FROM A STATISTICAL DISTRIBUTION OF INITIAL STATES

A. Direct method

Let us now assume that before the action of $F(t)$ in the remote past $t'\to-\infty$, the oscillator was in thermal equilibrium with a heat bath. In this case the initial state of the oscillator is determined by a statistical distribution, and the probability that after the action of the external perturbation the oscillator will end up in a state $|m\rangle$ is given by

$$P_m=\sum_{n=0}^{\infty}P_{m,n}w_n=\sum_{n=0}^{\infty}|S_{m,n}|^2w_n.\tag{23}$$

Here w_n is the initial thermal distribution:

$$w_n=e^{-n\beta\hbar\omega_0}\Big/\sum_{p=0}^{\infty}e^{-p\beta\hbar\omega_0}=(1-e^{-\beta\hbar\omega_0})e^{-n\beta\hbar\omega_0},\tag{24}$$

where $\beta=1/kT$, T is the temperature, and k is Boltzmann's constant. To evaluate Eq. (23), we proceed directly by exploiting the symmetry of the S-matrix elements arising from Eq. (22), and write

$$P_m=\sum_{n=0}^{\infty}|S_{m,n}|^2w_n=(1-u)e^{-|g|^2}\frac{(|g|^2)^m}{m!}\sum_{n=0}^{\infty}n!\left(\frac{u}{|g|^2}\right)^n[L_n^{m-n}(|g|^2)]^2,\tag{25}$$

where $u=e^{-\beta\hbar\omega_0}$. The summation over n can be carried out using standard tables [see, for example, Ref. 29, Eq. (8.976.1), and Ref. 30, Eq. (10.12.20)] to yield

$$P_m=(1-u)e^{-|g|^2(1-u)}u^mL_m(-s),\tag{26}$$

where $s=|g|^2(u+1/u-2)$, and $L_m(x)$ is the ordinary Laguerre polynomial,

$$L_m(x)\equiv L_m^0(x)=\sum_{k=0}^{m}\frac{1}{k!}\binom{m}{k}(-x)^k.\tag{27}$$

This result goes beyond the immediate physical problem it solves. For an oscillator in equilibrium with a heat bath, the w_n are pure powers, so Eq. (26) also may be interpreted as a generating function for the probabilities themselves ($\kappa=|g|^2$):

$$F_m(\kappa,\lambda)\equiv\sum_n P_{m,n}\lambda^n=\sum_n P_{n,m}\lambda^n=e^{-\kappa(1-\lambda)}\lambda^mL_m\left(-\kappa\left(\lambda+\frac{1}{\lambda}-2\right)\right).\tag{28}$$

This interpretation allows a simple calculation of the average energy shift:

$$\Delta E=\hbar\omega_0\sum_n(n-m)P_{mn}=\hbar\omega_0\left(\left.\frac{\partial F_m}{\partial\lambda}\right|_{\lambda=1}-mF_m|_{\lambda=1}\right)=\hbar\omega_0(\kappa+m-m)=|g|^2\hbar\omega_0,\tag{29}$$

another well known result.[1–4] The fluctuation of ΔE is obtained just as easily, which is left to the reader as an exercise:

$$\sqrt{\Delta E^2}\equiv\hbar\omega_0\sqrt{\sum_n(n-m)^2P_{mn}-\left(\sum_n(n-m)P_{mn}\right)^2}=\hbar\omega_0|g|\sqrt{2m+1}.\tag{30}$$

The relative fluctuation decreases as the coupling constant increases.

A closely related problem, also solvable by the direct approach, is the calculation of the spectral function of a perturbed oscillator that was initially in thermal equilibrium with a heat bath. Spectral functions of this kind are encountered in various spectroscopic studies of the dynamical properties of systems exhibiting nondispersive bosonic excitations.[5,7,20,31] We are interested in the experimentally observed spectrum $N_T(\Delta E)$, so we must collect transitions with a given energy transfer $\Delta E=l\hbar\omega_0$, in contrast to transitions to a fixed final state, that is,

$$N_T(\Delta E)=\sum_l A_l(T)\,\delta(\Delta E-l\hbar\omega_0), \tag{31}$$

where

$$A_l(T)=\sum_n P_{n+l,n}w_n=(1-u)\sum_n P_{n+l,n}u^n, \tag{32}$$

in the same notation as before. The energy transfer may be positive or negative, corresponding to absorption and emission spectra, respectively. If we use Eqs. (21) and (22), we have

$$P_{n+l,n}=|g|^{2l}e^{-|g|^2}\frac{n!}{(n+l)!}[L_n^l(|g|^2)]^2. \tag{33}$$

If we substitute this result in Eq. (32), the summation can be carried out by using tabulated expressions [see, for example, Ref. 32, Eq. (5.11.3.7)] to yield:

$$\sum_{n=0}^{\infty}\frac{n!}{(n+l)!}[L_n^l(|g|^2)]^2u^n =\frac{(|g|^2)^{-l}u^{-l/2}}{1-u}e^{-2|g|^2u/(1-u)}I_l\left(2|g|^2\sqrt{\frac{u}{1-u}}\right), \tag{34}$$

where $I_l(x)$ is the modified Bessel function of the first kind. After a little rearrangement, Eq. (34) becomes

$$A_l(T)=\exp\left(-|g|^2\frac{1+u}{1-u}\right)I_l\left(2|g|^2\frac{\sqrt{u}}{1-u}\right)u^{-l/2}. \tag{35}$$

If we replace u by the Bose–Einstein distribution of the oscillator states, $N=u/(1-u)=1/[\exp(\beta\hbar\omega_0)-1]$, we find:

$$A_l(T)=e^{-(2N+1)|g|^2}\left(\frac{N+1}{N}\right)^{l/2}I_l(\sqrt{4N(N+1)|g|^4}). \tag{36}$$

Equation (36) is identical to Eq. (4.3.15) of Ref. 5 and to Eq. (3.42) of Ref. 31 derived by the method of thermal Green's functions.

B. The operator approach

The obvious disadvantage of the direct method is that it depends on fortuitous circumstances in that an appropriate summation formula was found in tables. Here we construct a more general method that can in principle treat arbitrary initial distributions once they are characterized by their own generating functions.

First, we observe that the matrix elements (18) may be written in the form:

$$S_{m,n}=\frac{e^{-|g|^2/2}}{\sqrt{m!n!}}(-i)^{m+n}\left(g-\frac{d}{dg^*}\right)^m(g^*)^n, \tag{37}$$

which is derived in Appendix B. This form enables us to represent the absolute squares of the S-matrix elements as

$$|S_{m,n}|^2=\frac{e^{-|g|^2}}{m!n!}\left(g^*-\frac{d}{dx}\right)^m\left(g-\frac{d}{dy}\right)^m\Bigg|_{x=g,y=g^*}(xy)^n, \tag{38}$$

where x and y are introduced to "protect the symbols," because one operator factor should not act on the other.

As our first exercise in using Eq. (38), we rederive the generating function obtained previously by the direct approach. If we substitute Eq. (38) into Eq. (23) and carry out the summation, we obtain an expression for P_m that is easily evaluated to yield

$$P_m=(1-u)\frac{e^{-|g|^2}}{m!}\left(g^*-\frac{d}{dx}\right)^m\left(g-\frac{d}{dy}\right)^m\Bigg|_{x=g,y=g^*}e^{uxy} =(1-u)e^{-|g|^2(1-u)}u^mL_m(-s), \tag{39}$$

in accord with Eq. (26). Hence, we have derived the above-mentioned summation formula for the Laguerre polynomials. In effect, the quantity $(xy)^n$ in Eq. (38) has been summed in Eq. (23) to give e^{uxy} in Eq. (39) before the action of the differential operators. All the information of the initial oscillator distribution has been coded in e^{uxy}.

The generalization to an arbitrary initial distribution w_n is obvious: if its generating function, $W(\lambda)\equiv\Sigma_n w_n\lambda^n/n!$, is known, the final distribution is

$$P_m[W]=\frac{e^{-|g|^2}}{m!}\left(g^*-\frac{d}{dx}\right)^m\left(g-\frac{d}{dy}\right)^m\Bigg|_{x=g,y=g^*}W(xy). \tag{40}$$

An interesting and illustrative application of the above results is the two-pulse problem mentioned in Sec. I. For the oscillator initially in the ground state, $w_n=\delta_{n0}$ and $W(\lambda)=1$, and the derivatives in Eq. (40) vanish, leaving the oscillator in a Poisson distribution after the first pulse:

$$w_n=e^{-|\gamma|^2}\frac{|\gamma|^{2n}}{n!}, \tag{41}$$

where γ is the coupling function for the first pulse. By using Eq. (41) as the initial distribution for the second pulse, the corresponding $W(\lambda)$ is a modified Bessel function of the first kind:

$$W(\lambda)=\sum_n\frac{\lambda^n}{n!}w_n=e^{-|\gamma|^2}I_0(2|\gamma|\sqrt{\lambda}). \tag{42}$$

After the second pulse, the probability that the oscillator ends up in the final state $|m\rangle$ is given by

$$P_m^{\text{Poiss}}=\frac{e^{-|g|^2-|\gamma|^2}}{m!}\left(g^*-\frac{d}{dx}\right)^m \times\left(g-\frac{d}{dy}\right)^m\Bigg|_{x=g,y=g^*}I_0(2|\gamma|\sqrt{xy}), \tag{43}$$

where g is the coupling function for the second pulse. We have so far been unable to push this expression further, but note that it easily gives expressions for any given m with the aid of a symbolic algebra program. Thus for $m=3$,

$$P_3^{\text{Poiss}}=\frac{e^{-|g|^2-|\gamma|^2}}{6}[(|\gamma|^6+15|\gamma|^4|g|^2+15|\gamma|^2|g|^4+|g|^6 +4|\gamma|^2|g|^2)I_0(2|\gamma g|)-(6|\gamma|^5|g|+20|\gamma|^3|g|^3 +6|\gamma||g|^5+4|\gamma||g|+6|\gamma||g|^3 +6|\gamma|^3|g|)I_1(2|\gamma g|)]. \tag{44}$$

The two-pulse problem is always symmetric in the pulses, $\gamma\leftrightarrow g$. This symmetry is obvious in our formulation, because it amounts to the successive application of commuting operators, when iterating Eq. (40).

V. CONCLUSION

In the present study of perturbations of an excited quantum oscillator, we have obtained several new results, two of which should be of special interest. The first is the generating function for the transition probabilities, which arguably closes the problem of transitions between fixed initial and final states. As an example, we derived the fluctuation in the average energy transfer, a simple exercise once the generating function is known.

The second result treats the transition problem at the level of statistical distributions of states. We have obtained an expression that reduces the studied scattering problem to the action of a differential operator on the generating function of the initial distribution. The process can be iterated, thus providing a nontrivial new type of problem, the two-pulse problem. Here the first probe excites the oscillator into a nonthermal statistical distribution of states, while the second one acts on this distribution. A closed formula for a particular final state was given as an example.

In the course of this work, we have also demonstrated some compact new derivations of known results for the transition probabilities and spectral functions of the oscillator [see the derivation of Eqs. (21) and (36), respectively]. These have proven useful in the interpretations of photoemission and photoabsorption spectra of solids that support nondispersive bosonic excitations,[5] such as long wavelength optical phonons or collective electronic excitations (plasmons). Particular cases include inelastic neutron scattering from optical phonons in solids,[33] inelastic scattering of electrons from bulk and surface optical phonons in crystals,[31] and inelastic scattering of atoms from nondispersive surface phonons.[7]

In general, the forced oscillator model is applicable to the study of weakly dispersive bosonic excitations that are detected in interactions with external time-dependent probes, such as moving semiclassical or classical particles or electromagnetic fields. We hope the reader interested in such problems will find the present work to be a useful reference that provides new and additional insights into a textbook example of a simple quantum system perturbed by external probes.

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APPENDIX A: ALTERNATIVE DERIVATION OF THE SCATTERING MATRIX ELEMENTS

In this appendix we present the treatment of operator disentanglement, encountered in the calculation of matrix elements $S_{m,n}$ when starting from the opposite order of boson operators in the Weyl or Baker–Hausdorff formula, Eq. (16). If we substitute $A=-iG_-$ and $B=-iG_+$ in the latter, we have:

$$S_I=e^{(1/2)|g|^2}e^{-iG_-}e^{-iG_+}. \tag{A1}$$

In this case,

$$S_{m,n}=e^{(1/2)|g|^2}\frac{(-ig)^{m-n}}{\sqrt{m!n!}}\sum_{p=0}^{\infty}\frac{(m+p)!}{p!(m-n+p)!}(-|g|^2)^p \qquad (m\geqslant n). \tag{A2}$$

Note the positive sign in the exponent of $\exp(\frac{1}{2}|g|^2)$ in front of the now infinite sum on the right-hand side of Eq. (A2). An analogous expression can be obtained for $m<n$ by interchanging m with n and replacing g with its complex conjugate. The infinite number of terms in the sum on the right-hand side of Eq. (A2) means that the creation operator, which now acts first in Eq. (A1), gives rise to an arbitrary number of virtual excitations (bosons) in the intermediate state. This situation is encountered in the treatment of the forced oscillator model by the propagator technique of time-dependent perturbation theory.[7] The infinite series on the right-hand side of Eq. (A2) sums to an expression involving the Kummer or degenerate hypergeometric function,[34]

$$\sum_{p=0}^{\infty}\frac{(m+p)!}{p!(m-n+p)!}(-|g|^2)^p = \frac{m!}{(m-n)!}{}_1F_1(m+1,m-n+1;-|g|^2). \tag{A3}$$

If we exploit the functional properties [see, for example, Ref. 29, Eqs. (9.212.1) and (8.972.1)]:

$${}_1F_1(m+1,m-n+1;-z)=e^{-z}{}_1F_1(-n,m-n+1;z), \tag{A4}$$

and

$$\binom{m}{n}{}_1F_1(-n,m-n+1;z)=L_n^{m-n}(z), \tag{A5}$$

and substitute them in Eq. (A2), we again retrieve Eq. (21). In effect, the operator disentanglement relating the two ways of calculating $S_{m,n}$ by using either Eq. (17) or Eq. (A1) has been reduced to a transformation of hypergeometric functions.

APPENDIX B: PROOF OF EQ. (37)

We proceed by the direct evaluation of the expression on the right-hand side:

$$\left(g-\frac{d}{dg^*}\right)^m(g^*)^n=\sum_k\binom{m}{k}g^k(-1)^{m-k}\times\frac{n!}{(n-m+k)!}(g^*)^{n-m+k} \propto (g^*)^{n-m}L_n^{m-n}(|g|^2), \tag{B1}$$

and using the symmetry in Eq. (22), we easily recover Eq. (21). This derivation can be reduced to the statement that

$$x^m\left(1-\frac{d}{dx}\right)^m x^n=(-1)^m m!x^nL_m^{n-m}(x), \tag{B2}$$

which, due to Eq. (22), is symmetric in m and n.

To elucidate this derivation a bit further, we observe that the boson operators are characterized by their commutator, $[a,a^\dagger]=1$. Essentially the same commutator is found between x and d/dx, operating on some functional space. This equivalence of the commutators means that all operations in Fock space can be mapped to corresponding operations in functional space. The coherent state formalism is an explicit realization of such a mapping.[35] To make a connection with the forced-oscillator problem, all that is needed is to interpret the matrix elements $S_{m,n}$ as operations on coherent states. In particular,[2]

$$S_{m,n} \propto (-ig|(a+ig)^m|n\rangle, \tag{B3}$$

where $|\alpha)$ is a coherent state characterized by the complex number α. The operator $(a+ig)^m$ maps onto $(g-d/dg^*)^m$ up to constant factors, because a acts as a creation operator to the left, and creation operators map onto derivatives by the coherent state mapping.[35]

[a]Electronic mail: branko@ifs.hr

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