**Interacting families of Calogero-type particles and SU(1,1) algebra**

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We study a one-dimensional model with F interacting families of Calogero-type particles. The model includes harmonic, two-body and three-body interactions. We emphasize the universal SU(1,1) structure of the model. We show how SU(1,1) generators for the whole system are composed of SU(1,1) generators of arbitrary subsystems. We find the exact eigenenergies corresponding to a class of the exact eigenstates of the F-family model. By imposing the conditions for the absence of the three-body interaction, we find certain relations between the coupling constants. Finally, we establish some relations of equivalence between two systems containing F families of Calogero-type particles.

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# 1 Introduction

Since its inception, the ordinary Calogero model [1] continues to be of interest for both physics and mathematics community [2]. The model describes $N$ identical (single-species) particles on the line which interact through an inverse-square two-body interaction and are subjected to a common confining harmonic force. The inverse-square potential can be regarded as a pure statistical interaction [3] and the model maps to an ideal gas of particles obeying fractional Haldane statistics [4]. The role of Haldane statistical parameters is played by a universal coupling constant in the two-body interaction. However, in Haldane's formulation of statistics there is a possibility of having particles of different species with a mutual statistical coupling parameter depending on the species coupled. This suggests the generalization of the single-species Calogero model to the multispecies Calogero model. Distinguishability of the species can be introduced by allowing particles to have different masses and different couplings to each other. While the single-species Calogero model is completely solvable [1,5], very little is known about spectra and wave functions of the multispecies Calogero model [6].

Recently, we used an operator method to analyse a one-dimensional multispecies Calogero model with two- and three-body interactions [7]. We succeeded in finding a class of, but not all, exact eigenstates and eigenenergies of the model Hamiltonian. The analysis relied heavily on the SU(1,1) algebraic structure of the Hamiltonian and once more stressed the importance of the conformal symmetry of the quantum singular oscillator [8]. We were also able to generalize the model of Ref.[7] to arbitrary dimensions [9].

In the present Letter, which is in a sense a continuation of our investigation of the ordinary Calogero model [10] and the multispecies Calogero model [7,9], we turn to the important problem of interacting families of Calogero-type particles in one dimension, a theme which is already announced in [7]. We consider a model with a potential that generally includes harmonic, two-body and three-body interactions acting between particles belonging to different families, as well as the interaction between particles belonging to the same family with the coupling constant that may be different for different families. In Section 2 we prepare all necessary tools for handling the problem of interacting families. We collect, without rederiving, the main results of the analysis of the one-dimensional multispecies Calogero model [7]. In Section 3 we apply these results to the case of two interacting families of Calogero particles. We display the model Hamiltonian and find the ground state energy. We construct generators of SU(1,1) algebra for interacting families and underline the importance of the dilatation part of the algebra, i.e. generator $T_0$. Furthermore, by imposing the conditions for the absence of the three-body interaction in the initial Hamiltonian, we find certain relations between the coupling constants. In Section 4 we extend these results to three and more interacting families of Calogero particles. We show that the underlying SU(1,1) structure is universal, i.e. holds for an arbitrary number of families, arbitrary masses of Calogero particles and arbitrary coupling constants. We particularly show how to obtain SU(1,1) generators of the whole system from SU(1,1) generators of arbitrary subsystem, i.e. we establish composition rules for SU(1,1) generators. We also find the *exact* eigenenergies corresponding to a class of the *exact* eigenstates of the model with F interacting families. We discuss the relations between the coupling constants in the case when a three-body interaction vanishes. Finally, we establish some relations of equivalence between two systems containing F families. Section 5 is a short conclusion.

## 2 A multispecies Calogero model: main results

The model of Ref.[7] is specified by masses of particles, $m_i$, and the coupling constants $\omega$ and $\nu_{ij},\ i,j=1,2,...,N.$ The Hamiltonian is

$$H(\omega)=-\frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\frac{\partial^2}{\partial x_i^2}+\frac{\omega^2}{2}\sum_{i=1}^{N}m_ix_i^2+\frac{1}{4}\sum_{i\neq j}\frac{\nu_{ij}(\nu_{ij}-1)}{(x_i-x_j)^2}(\frac{1}{m_i}+\frac{1}{m_j})+$$
$$+\frac{1}{2}\sum_{i\neq j,i\neq k,j\neq k}\frac{\nu_{ij}\nu_{jk}}{m_j(x_j-x_i)(x_j-x_k)}. \tag{1}$$

The ground state wave function is of the Calogero type:

$$\Psi_0(x_1,...,x_n)=\prod_{i<j}|x_i-x_j|^{\nu_{ij}}e^{-\frac{\omega}{2}\sum_{i=1}^{N}m_ix_i^2}\equiv\Delta e^{-\frac{\omega}{2}\sum_{i=1}^{N}m_ix_i^2} \tag{2}$$

and the corresponding ground state energy is

$$E_0=\omega\epsilon_0=\omega(\frac{N}{2}+\sum_{i<j}\nu_{ij}). \tag{3}$$

When all couplings $\nu_{ij}$ are equal, Eq.(3) reduces to the well-known Calogero result $\epsilon_0=\frac{N}{2}+\nu\frac{N(N-1)}{2}$.

After performing a similarity transformation

$$\tilde{H}(\omega)=\Delta^{-1}H(\omega)\Delta,$$
$$\tilde{\Psi}=\Delta^{-1}\Psi,$$

one obtains a non-Hermitean Hamiltonian $\tilde{H}(\omega)$ with a hidden three-body interaction:

$$\tilde{H}(\omega)=-\frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\frac{\partial^2}{\partial x_i^2}+\frac{\omega^2}{2}\sum_{i=1}^{N}m_ix_i^2-\frac{1}{2}\sum_{i\neq j}\frac{\nu_{ij}}{(x_i-x_j)}(\frac{1}{m_i}\frac{\partial}{\partial x_i}-\frac{1}{m_j}\frac{\partial}{\partial x_j})=$$
$$=\omega^2T_+-T_-, \tag{4}$$

where

$$T_-=-\tilde{H}(\omega=0),\qquad T_+=\frac{1}{2}\sum_{i=1}^{N}m_ix_i^2,$$

$$T_0 = \frac{1}{2}(\sum_{i=1}^{N} x_i \frac{\partial}{\partial x_i} + \epsilon_0). \tag{5}$$

The set of operators $\{T_\pm, T_0\}$ satisfy the SU(1,1) algebra:

$$[T_-, T_+] = 2T_0, \quad [T_0, T_\pm] = \pm T_\pm. \tag{6}$$

Note that $T_0$ serves as a dilatation operator. One can deduce that

$$T_0 \Delta = (\frac{1}{2}\sum_{i<j} \nu_{ij} + \frac{\epsilon_0}{2})\Delta$$

$$T_- \Delta = 0. \tag{7}$$

It is convenient to introduce the centre-of-mass coordinate $X = \frac{1}{M}\sum_{i=1}^{N} m_i x_i$ ( where $M = \sum_{i=1}^{N} m_i$ ) and relative coordinates $\xi_i = x_i - X$. In terms of these coordinates, the Hamiltonian $\tilde{H}(\omega)$, Eq.(4), separates into parts which describe its centre-of-mass motion (CM) and its relative motion (R), namely $\tilde{H}(\omega) = \tilde{H}(\omega)_{CM} + \tilde{H}(\omega)_R$. In the same way one can split the generators $T_\pm$ and $T_0$ into the centre-of-mass and relative parts, i.e. $T_{\pm,0} = T_{\pm,0_{(CM)}} + T_{\pm,0_{(R)}}$.

In the next section we apply these results to the case of two interacting families.

# 3 Two interacting families

Let us consider two families, $\mathcal{F}_1$ and $\mathcal{F}_2$, of Calogero particles. The first one, denoted by $\mathcal{F}_1 = \{m_1, \nu_1, N_1\}$, is described by $N_1$ particles of mass $m_1$, the coupling constant $\nu_1$ and the coordinates of the particles are $\{x_i\} = \{x_1, x_2, ..., x_{N_1}\}$. Similarly, the second one, denoted by $\mathcal{F}_2 = \{m_2, \nu_2, N_2\}$, is described by $N_2$ particles of mass $m_2$, the coupling constant $\nu_2$ and the coordinates of the particles are $\{z_\alpha\} = \{z_1, z_2, ..., z_{N_2}\}$. The interaction strength between the first and the second family is $\nu_{12} = \kappa$.

The full Hamiltonian now reads

$$H(\omega) = H_1(\omega) + H_2(\omega) + H_{int}, \tag{8}$$

where $H_{int}$ is given by

$$H_{int} = \frac{1}{4}\sum_{i}^{N_1}\sum_{\alpha}^{N_2}\frac{\kappa(\kappa-1)}{(x_i - z_\alpha)^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+$$

$$+\frac{1}{4}\sum_{i}^{N_1}\sum_{\alpha\neq\beta}^{N_2}\left(\frac{\kappa^2}{m_1(x_i-z_\alpha)(x_i-z_\beta)}\right)+\frac{1}{2}\sum_{i}^{N_1}\sum_{\alpha\neq\beta}^{N_2}\left(\frac{\nu_2\kappa}{m_2(z_\alpha-x_i)(z_\alpha-z_\beta)}\right)+$$

$$+\frac{1}{4}\sum_{i\neq j}^{N_1}\sum_{\alpha}^{N_2}\left(\frac{\kappa^2}{m_2(z_\alpha-x_i)(z_\alpha-x_j)}\right)+\frac{1}{2}\sum_{i\neq j}^{N_1}\sum_{\alpha}^{N_2}\left(\frac{\nu_1\kappa}{m_1(x_i-z_\alpha)(x_i-x_j)}\right), \tag{9}$$

and $H_1(\omega)$ ( $H_2(\omega)$ ) are Calogero Hamiltonians, Eq.(1), for the first and the second family, respectively.

The corresponding ground state wave function of the Hamiltonian (8) is

$$\Psi_0(x_1,...,x_{N_1},z_1,...,z_{N_2}) = \prod_{i,\alpha}(x_i - z_\alpha)^\kappa \Psi_{0,1}(x_1,...,x_{N_1})\Psi_{0,2}(z_1,...,z_{N_2})$$

$$\equiv \Delta_{12}\Psi_{0,1}(x_1,...,x_{N_1})\Psi_{0,2}(z_1,...,z_{N_2}), \tag{10}$$

where $\Psi_{0,1}$ and $\Psi_{0,2}$ are the Calogero ground states (when $\kappa = 0$), Eq.(2), for the families $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

We can perform a similarity transformation with a $\Delta_1\Delta_2$ part of the full Jastrow prefactor $\Delta = \Delta_1\Delta_2\Delta_{12}$ in (8,10), to obtain

$$\Delta_1{}^{-1}\Delta_2{}^{-1}H(\omega)\Delta_1\Delta_2 = \tilde{H}_1(\omega) + \tilde{H}_2(\omega) + H_{int},$$

$$\Delta_1{}^{-1}\Delta_2{}^{-1}\Psi_0 = \prod_{i,\alpha}(x_i - z_\alpha)^\kappa \tilde{\Psi}_{0,1}\tilde{\Psi}_{0,2}. \tag{11}$$

The ground state energy of the Hamiltonian (8) can be split into three terms:

$$\epsilon_0 = \epsilon_{0,1} + \epsilon_{0,2} + \kappa N_1 N_2, \tag{12}$$

describing the ground state energies of each family and the interaction between them, respectively.

For each family, one can define $SU(1,1)$ generators $T_\pm^{(I)}, T_0^{(I)}$, $I = 1,2$. These two sets of generators, i.e. the corresponding $SU(1,1)$ algebras, mutually commute. From the following two relations:

$$T_0 = T_0^{(1)} + T_0^{(2)} + \frac{1}{2}\kappa N_1 N_2,$$

$$T_0\Delta_{12} = \frac{1}{2}(\kappa N_1 N_2 + \epsilon_0)\Delta_{12},$$

we find

$$(T_0^{(1)} + T_0^{(2)})\Delta_{12} = \frac{\epsilon_0}{2}\Delta_{12}. \tag{13}$$

Furthermore, from Eq. (7) and after multiplication by $\Delta_1{}^{-1}\Delta_2{}^{-1}$ from the left, it follows that

$$T_-\Delta_{12} \equiv (T_-^{(1)} + T_-^{(2)} - H_{int})\Delta_{12} = 0. \tag{14}$$

Note that $T_+ = T_+^{(1)} + T_+^{(2)}$.

As we have already shown in [7], for the general $\nu_{ij}$ and $m_j$ the three-body interactions in the initial Hamiltonian (1) vanish identically if the following conditions are satisfied for all triples of indices $i,j,k$:

$$\frac{\nu_{ij}\nu_{jk}}{m_j} = \frac{\nu_{ji}\nu_{ik}}{m_i} = \frac{\nu_{ik}\nu_{kj}}{m_k}. \tag{15}$$

In this case, the Hamiltonian contains the two-body interactions (i.e. inverse-square interactions) only. The unique solution of these conditions is $\nu_{ij} = \lambda\, m_i\, m_j$, $\lambda$ being some universal constant.

In our two-family system this corresponds to the condition

$$\nu_{ij} = \lambda m_i m_j = \kappa, \qquad \forall\, i, j \tag{16}$$

or explicitly

$$\nu_1 = \lambda m_1^2, \qquad \nu_2 = \lambda m_2^2, \qquad \nu_{12} = \kappa = \lambda m_1 m_2, \tag{17}$$

from which it follows

$$\nu_1 \nu_2 = \kappa^2,$$
$$\nu_2 = (\frac{m_2}{m_1})^2 \nu_1. \tag{18}$$

Note that Eqs.(16-18) imply that the couplings $\nu_1, \nu_2$ and $\kappa$ have to be simultaneously positive, negative or zero.

The connection between the coupling constants $\{\nu_1, \nu_2, \kappa\}$, Eqs.(18), is ascribed to the weak-strong coupling duality in [11], but it is *de facto* a simple consequence of the absence of the three-body interaction in the starting Hamiltonian (1). We also point out that all the above relations for $T_0^{(1)} + T_0^{(2)}$ and $T_-^{(1)} + T_-^{(2)}$ (Eqs.(13,14)) hold generally for arbitrary masses $m_1, m_2$ and arbitrary coupling constants $\nu_1, \nu_2, \kappa$, i.e. irrespectively of the presence/absence of the three-body interaction.

# 4 Three and more interacting families, exact eigenstates and equivalences between models

We extend the above analysis to the case of three families, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, of Calogero particles. The families are characterized by mutually distinct numbers of particles, masses of particles and different coupling constants. One can immediatelly generalize the results (13,14):

$$\begin{gathered}
\Delta = \Delta_1\Delta_2\Delta_3\Delta_{12}\Delta_{13}\Delta_{23}, \\
T_0 = T_0^{(1)} + T_0^{(2)} + T_0^{(3)} + \frac{1}{2}(\epsilon_0 - \epsilon_{0,1} - \epsilon_{0,2} - \epsilon_{0,3}), \\
T_+ = T_+^{(1)} + T_+^{(2)} + T_+^{(3)}, \\
T_- = T_-^{(1)} + T_-^{(2)} + T_-^{(3)} - H_{int}, \\
(T_0^{(1)} + T_0^{(2)} + T_0^{(3)})\Delta_{12}\Delta_{13}\Delta_{23} = \frac{\epsilon_0}{2}\Delta_{12}\Delta_{13}\Delta_{23}, \\
(T_-^{(1)} + T_-^{(2)} + T_-^{(3)} - H_{int})\Delta_{12}\Delta_{13}\Delta_{23} = 0.
\end{gathered} \tag{19}$$

For the initial $N$ -body multispecies Calogero model we can write composition laws for the SU(1,1) generators:

$$T_0 = \sum_{i=1}^{N} T_0^{(i)} + \frac{1}{2}\sum_{i<j} \nu_{ij},$$

$$T_+ = \sum_{i=1}^{N} T_+^{(i)}, \qquad T_- = \sum_{i=1}^{N} T_-^{(i)} - H_{int},$$

from which it follows

$$\begin{gathered}
T_0\Delta = \frac{1}{2}(\sum_{i<j}\nu_{ij} + \epsilon_0)\Delta, \\
(\sum_{i=1}^{N} T_0^{(i)})\Delta = \frac{\epsilon_0}{2}\Delta, \\
(\sum_{i=1}^{N} T_-^{(i)} - H_{int})\Delta = 0.
\end{gathered} \tag{20}$$

These relations are general, valid for an arbitrary number of families F (i.e. for an arbitrary partition of a multispecies Calogero model), and for an arbitrary choice of masses $m_i$ and coupling constants $\nu_{ij}$.

As we elaborated in [7], the infinite set of exact eigenstates of the Hamiltonian (1) can be constructed by applying ladder operators

$$A_1^{\pm} = \frac{1}{\sqrt{2}}(\sqrt{M\omega}X \mp \frac{1}{\sqrt{M\omega}}\frac{\partial}{\partial X}) \tag{21}$$

and

$$B_2^{\pm} = \frac{1}{2}(\omega T_+ + \frac{T_-}{\omega}) \mp T_0 - \frac{1}{2}A_1^{\pm 2} \tag{22}$$

to the vacuum

$$\tilde{\Psi}_0(x_1, x_2, \cdots x_N) = \tilde{\Psi}_0(X)\tilde{\Psi}_0(\xi_1, \xi_2 \cdots \xi_N) = e^{-\frac{M\omega}{2}X^2} e^{-\frac{\omega}{2}\sum_{i=1}^N m_i \xi_i^2}.$$

The exact eigenstates are

$$\tilde{\Psi}_{n_1 n_2} = A_1^{+n_1} B_2^{+n_2}, \tilde{\Psi}_0. \qquad n_1, n_2 = 0, 1, 2, , , \tag{23}$$

The exact eigenenergies corresponding to these states are $(I, J = 1, 2, ...F)$

$$\begin{aligned} E_{n_1 n_2} &= \omega(n_1 + 2n_2 + \epsilon_0), \\ \epsilon_0 &= \sum_{I=1}^{F} \epsilon_{0,I} + \sum_{I<J;I,J=1}^{F} \epsilon_{0,IJ}, \\ \epsilon_{0,I} &= \frac{N_I}{2} + \nu_I \frac{N_I(N_I - 1)}{2}, \\ \epsilon_{0,IJ} &= \nu_{IJ}\, N_I\, N_J. \end{aligned} \tag{24}$$

In the special case, when there is no three-body interaction (i.e. relations (15) are satisfied), we can identify

$$\nu_I = \lambda m_I^2, \qquad \forall I = 1, 2...F,$$

$$\nu_{IJ} = \lambda m_I m_J, \qquad \forall I, J = 1, 2...F. \tag{25}$$

Since the masses are positive, the couplings $\nu_I$ and $\nu_{IJ}$ have the same sign, depending on the sign of the free parameter $\lambda$.

Now we establish some relations of equivalence between the two systems containing F families of Calogero particles.

**Case 1.** *Complete equivalence of the two systems.*
Let $S = \{\omega, m_I, \nu_I, \nu_{IJ}, N_I\}$ and $S' = \{\omega, m'_I, \nu'_I, \nu'_{IJ}, N'_I\}$ be two Calogero systems with F families. We call them completely equivalent if

$$\epsilon_{0,I} = \epsilon'_{0,I}, \qquad \epsilon_{0,IJ} = \epsilon'_{0,IJ}, \qquad N_I = N'_I.$$

These conditions imply

$$\nu_I = \nu'_I, \qquad \nu_{IJ} = \nu'_{IJ}. \tag{26}$$

**Case 2.** *Partial equivalence of the two systems.*
We call the two systems $S$ and $S'$ partially equivalent if

$$\epsilon_0 = \epsilon'_0,$$

while the number of particles, $N$ and $N'$, may be the same or different. For example, in the case of one-family systems ($F = 1$) and $N \neq N'$, the above condition implies that

$$N + \nu N(N-1) = N' + \nu' N'(N'-1). \tag{27}$$

**Case 3.** *Special case: the single system.*
Consider a single system with F families of Calogero particles. We can demand that

$$\epsilon_{0,1} = \epsilon_{0,I}, \qquad \epsilon_{0,12} = \epsilon_{0,IJ}, \qquad \forall I, J = 1, 2...F.$$

In the case of the two-family system ($F = 2$), we have

$$N_1 + \nu_1 N_1(N_1 - 1) = N_2 + \nu_2 N_2(N_2 - 1). \tag{28}$$

We obtain very interesting relations between the couplings $\nu_1$ and $\nu_2$ if we impose the strong-weak duality condition on the couplings, namely $\nu_1\nu_2 = 1$. (We fix $\kappa^2 = 1$ in Eq.(18)).

The quadratic equation (28) then has two solutions:

$$(i) \qquad \nu_1 = \frac{N_2 - 1}{N_1 - 1} > 0,$$

$$(ii) \qquad \nu_1 = -(\frac{N_2}{N_1}) < 0.$$

Their physical implications are summarized in Table 1.

Table 1.

| $\nu_1$ | $\kappa$ | $\lambda$ | $\epsilon_0$ | Comments |
|---|---|---|---|---|
| $\frac{N_2-1}{N_1-1}$ | $+1$ | $\lambda > 0$ | $2N_1N_2 > 0$ | Physical solution, no three-body interaction. |
| $-(\frac{N_2}{N_1})$ | $+1$ | $-$ | $N_1 + N_2 > 0$ | Physical solution, with a three-body interaction. |
| $\frac{N_2-1}{N_1-1}$ | - 1 | $-$ | 0 | Unphysical solution, with a three-body interaction. |
| $-(\frac{N_2}{N_1})$ | -1 | $\lambda < 0$ | $N_1 + N_2 - 2N_1N_2 < 0$ | Unphysical solution, no three-body interaction. |

A few remarks are in order.

*Remark 1.* Notice that solution $(i)$ requires $N_{1,2} \geq 2$.

*Remark 2.* If the generalized strong-weak duality condition is imposed, i.e. $\nu_I\nu_J = 1$, $(I \neq J = 1, 2, ...F)$, then it follows that it can be satisfied for $F = 2$ only.

*Remark 3.* In Refs.[7,9,10] we showed that there existed a critical point $\epsilon_{0R} = 0$ at which the system described by $\tilde{H}(\omega)_R$ collapsed completely, i.e. the relative momenta, the relative energy and the relative coordinates were all zero at this critical point. The ground state was a square-integrable function only for $\epsilon_{0R} > 0$. This is the reason why we ascribe the term 'unphysical' to the last solutions in Table 1.

# 5 Conclusion

In this Letter we have studied the most general Calogero model on the line with a three-body interaction possessing an arbitrary number of mutually interacting families of Calogero particles. We have found the exact eigenenergies corresponding to a class of the exact eigenstates of the model. We have established relations of equivalence between two systems with F families which imply a certain connection between the coupling constants. Particularly interesting appear the relations between the coupling constants in the single system with F families of Calogero particles when a strong-weak duality condition is imposed. We have paid special attention to the SU(1,1) structure of the model. We have found certain relations between SU(1,1) generators that are universal for all choices of masses and coupling constants. We particularly show how to obtain SU(1,1) generators of the whole system from SU(1,1) generators of arbitrary subsystem. Moreover, the same relations are valid for an arbitrary number of dimensions and for all potentials that behave as a kinetic energy term under the dilatation represented by the generator $T_0$. There is only one difference between one and higher dimensions. In the case of one dimension, one can exclude the three-body interaction between particles from the beginning, while there is no known way how to do this in dimensions higher than one. Our results can also be extended to other systems with the underlying conformal or superconformal symmetry [12].

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