A Multispecies Calogero Model: Some Novel Results

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It is well known that the Calogero model for N identical prticles on the line [1] is connected to Haldane's exclusion statistics [2]. The role of statistical parameter is played by the strenght of the two-body inverse-square interaction ν. In Haldane's formulation, however, there is the possibility of having particles of different species with a mutual statistical coupling parameter depending on the species coupled. This suggest the possible generalization of the Calogero system to the system of N distinguishable particles, distinguishability beeing introduced through the replacements $\nu \to \nu_{ij} = \nu_{ji}$ and $m \to m_i, m$ beeing a mass of the Calogero particles.

A one-dimensional multispecies Calogero model is defined by the following Hamiltonian $(\hbar = \omega = 1; i, j = 1,2 \ldots N)$[3]:

$$H = -\frac{1}{2}\sum_i \frac{1}{m_i}\frac{\partial_i^2}{\partial x_i^2} + \frac{1}{2}\sum_i m_i x_i^2 + \frac{1}{4}\sum_{i \neq j} \frac{\nu_{ij}(\nu_{ij}-1)}{(x_i - x_j)^2}\left(\frac{1}{m_i} + \frac{1}{m_j}\right)$$

$$+\frac{1}{2}\sum_{i,j,k\neq} \frac{\nu_{ij}\nu_{jk}}{m_j (x_j - x_i)(x_j - x_k)}. \tag{1}$$

The ground state of H is, at least for the small «deformations» ν_{ij} and m_i,

$$\Psi_0(x_1, x_2, \cdots x_n) = \Delta e^{-\frac{1}{2}\sum_i m_i x_i^2} = \prod_{i<j}(x_i - x_j)^{\nu_{ij}} e^{-\frac{1}{2}\sum_i m_i x_i^2}$$

$$E_0 = \left(\frac{N}{2} + \frac{1}{2}\sum_{i\neq j}\nu_{ij}\right). \tag{2}$$

Notice the appearance of the thre-body term in (1). It can be shown that in the limit $\nu_{ij} \to \nu$ and $m_i \to m$, the last term in (1) identically vanish and the Hamiltonian (1) smoothly goes to the Calogero Hamiltonian [1]. (Another nontrivial condition for dissapearance of the three-body interaction is $v_{ij} = \alpha m_i m_j$ [3]). Performing the non-unitary transformation on $\Psi_0, \Psi_0 = \Delta^{-1}\Psi_0$, one generates Hamiltonian $\widetilde{H} = \Delta^{-1} H \Delta$ for which the three-body term apparently dissapears:

$$\tilde{H} = -\frac{1}{2}\sum_i \frac{1}{m_i}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_i m_i x_i^2 - \frac{1}{2}\sum_{i\neq j}\frac{v_{ij}}{\left(x_i - x_j\right)}\left(\frac{1}{m_i}\frac{\partial}{\partial x_i} - \frac{1}{m_j}\frac{\partial}{\partial x_j}\right). \tag{3}$$

For later convenience we define operators {T+,T-,T0} as

$$T_+ = \frac{1}{2}\sum_i m_i x_i^2, \qquad T_0 = \frac{1}{2}\left(\sum_i x_i \frac{\partial}{\partial x_i} + E_0\right)$$

$$T_- = \frac{1}{2}\sum_i \frac{1}{m_i}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}\sum_{i\neq j}\frac{v_{ij}}{\left(x_i - x_j\right)}\left(\frac{1}{m_i}\frac{\partial}{\partial x_i} - \frac{1}{m_j}\frac{\partial}{\partial x_j}\right). \tag{4}$$

Operators (4) generate SU (1,1) algebra. In terms of these operators, the Hamiltonian (3) reads $\widetilde{H} = T_+ - T_-$. Next, we separate center-of-mass motion (CM) and relative motion (R) of particles by defining variables X and ξ_i:

$$X = \frac{\sum_i m_i x_i}{\sum_i m_i} \equiv \frac{1}{M}\sum_i m_i x_i, \qquad \xi_i = x_i - X.$$

Hamiltonian $\widetilde{H}$ then separates as $\widetilde{H} = \widetilde{H}_{CM} + \widetilde{H}_R$ and wave function $\widetilde{\Psi}_0$ separates as $\widetilde{\Psi}_0(x_1, x_2 \cdots x_N) = \widetilde{\Psi}_0(X)\widetilde{\Psi}_0(\xi_1, \xi_2 \cdots \xi_N)$.

Now we introduce creation and annihilation operators $\{A_1^+, A_1^-\}, \{A_2^+, A_2^-\}$ and $\{B_2^+ B_2^-\}$:

$$A_1^{\pm} = \frac{1}{\sqrt{2}}\left(\sqrt{M}X \mp \frac{1}{\sqrt{M}}\frac{\partial}{\partial X}\right), \qquad A_2^{\pm} = \frac{1}{2}\left(T_+ + T_-\right) \mp T_0,$$

$$B_2^{\pm} = A_2^{\pm} - \frac{1}{2}A_1^{\pm 2}. \tag{5}$$

such that

$$\widetilde{H} = \left[A_2^-, A_2^+\right], \qquad \left[\widetilde{H}, A_1^{\pm}\right] = \pm A_1^{\pm}, \qquad \left[\widetilde{H}, A_2^{\pm}\right] = \pm 2A_2^{\pm},$$

$$\left[A_1^-, A_1^+\right] = 1, \qquad \left[A_1^-, A_2^+\right] = A_1^+, \qquad \left[A_1^-, A_2^-\right] = \left[A_1^+, A_2^+\right] = 0.$$

They act on the Fock vacuum $\left|\widetilde{O}\right\rangle \propto \widetilde{\Psi}_0(x_1, x_2, \cdots x_N)$ as:

$$A_1^-\left|\tilde{0}\right\rangle = A_2^-\left|\tilde{0}\right\rangle = B_2^-\left|\tilde{0}\right\rangle = 0\,,\ \left\langle\tilde{0}\,|\,\tilde{0}\right\rangle = 1.$$

The excited states are built as

$$\left|n_1, n_2\right\rangle \propto A_1^{+n_1} A_2^{+n_2}\left|\tilde{0}\right\rangle, \quad \forall n_1, n_2 = 0.1, \cdots \tag{6}$$

The states $\left|n_1, n_2\right\rangle$ are eigenstates of $\tilde{H}$, Eq. (3), with the eigenvalues

$$E_{n_1, n_2} = (n_1 + 2n_2) + E_0 = n + E_0 \tag{7}$$

Thus the energy spectrum is linear in quantum numbers n_1, n_2 and degenerate. This result is universal, i.e. it holds for all parameters m_i and ν_{ij}. It is evident that for *n=even*, the degeneracy is $\left(\frac{n}{2}+1\right)$ and for *n=odd*, the degeneracy is $\left(\frac{n+1}{2}\right)$. The dynamical symmetry of degenerate energy levels is SU(2) algebra [4], generated by $A_1^{+2}B_2^-, B_2^+A_1^{-2}$ and $A_1^+A_1^-$. This is the minimal symmetry that remains in the generic case.

Introduction of the operators $\{B_2^+, B_2^-\}$ help us to decouple inessential CM-motion, described by the operators $A_1^\pm$, i.e.

$$\begin{gathered}
\tilde{H}_{CM} = \frac{1}{2}\left\{A_1^-, A_1^+\right\}_+, \qquad \tilde{H}_R = \left[B_2^-, B_2^+\right], \\
\left[\tilde{H}_{CM}, A_1^\pm\right] = \pm A_1^\pm, \quad \left[\tilde{H}_R, B_2^\pm\right] = \pm 2B_2^\pm, \\
\tilde{H}_{CM}\left|\tilde{0}\right\rangle_{CM} = \frac{1}{2}\left|\tilde{0}\right\rangle_{CM}, \qquad \tilde{H}_R\left|\tilde{0}\right\rangle_R = \left(\frac{N-1}{2} + \frac{1}{2}\sum_{i\neq j}\nu_{ij}\right)\left|\tilde{0}\right\rangle_R.
\end{gathered} \tag{8}$$

The Fock space now splits into the CM-Fock space, spanned by $A_1^{+n_1}\left|\tilde{0}\right\rangle_{CM}$, and the R-Fock space, spanned by $B_1^{+n_2}\left|\tilde{0}\right\rangle_R$, with $\left|\tilde{0}\right\rangle_{CM} \propto \tilde{\Psi}_0(X)$ and $\left|\tilde{0}\right\rangle_R \propto \tilde{\Psi}_0(\xi_1, \xi_2 \cdots \xi_N)$.

Closer inspection of the R-Fock space reveals the existence of the universal critical point at which the system exhibits singular behaviour [3,5]. The critical point is defined by the null-vector

$$ {}_R\langle \tilde{0} | B_2^- B_2^+ | \tilde{0} \rangle_R = \frac{N-1}{2} + \frac{1}{2} \sum_{i \neq j} \nu_{ij} = 0. \qquad (9) $$

At the critical point the system described by $\widetilde{H}_R$ collapses completely, i.e. the relative coordinates, the relative momenta and the relative energy are all zero. There survives only one oscillator, describing the motion of the centre-of-mass.

The analysis performed here can be generalized to the arbitrary dimensions. However, the inevitable appearance of the three-body interaction in D>1 makes any analysis of such a model(s) highly nontrivial. We have shown in [6] that the essential features of D=1 model (1) still persist for D>1. For example, the excited states and their energies for the D-dimensional analogue of the Hamiltonian (3) are given as:

$$ \left(\prod_{\alpha=1}^{D} \left(A_{1,\alpha}^{+} \right)^{n_{1,\alpha}} \right) \left(A_2^{+n_2} \right) \left| \widetilde{0} \right\rangle, \qquad E_{n_{1,\alpha};n_2} = \left(\sum_{\alpha=1}^{D} n_{1,\alpha} + 2n_2 + E_0 \right) $$

where $\alpha = 1,2,...D$, $n_{1,\alpha} = 0,1,2...(\forall \alpha)$ and $n_2 = 0,1,2...$ The underlying dynamical symmetry responsible for the degeneracy of spectra is SU (D+1) and the critical point happens to be at

$$ \sum_{i \neq j} \nu_{ij} = -(N-1)D. $$

The open problem that still remains is the question of integrability and full solvability of the model(s). It would be also interesting to incorporate supersymmetry into the model (s). Finally, the most important question is whather there is a realistic system in physics or in other cross-disciplinary areas where such a model(s) play a role.

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