IEEE MELECON 2004, May 12-15, 2004, Dubrovnik, Croatia

Low-Sensitivity, Low-Power 4^{th}-Order Low-Pass Active-*RC* Allpole Filter Using Impedance Tapering

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***Abstract*—The analytical design procedure of low-sensitivity, low-power, low-pass (LP) 2^{nd}-and 3^{rd}-order class-4 active-*RC* allpole filters, using impedance tapering, has already been published [1][2]. In this paper the desensitisation using impedance tapering is applied to the design of LP 4^{th}-order filters. The numerical design procedure was performed by Newton's iterative method. Analytically designed unity-gain LP 4^{th}-order filters [3] can provide initial values for Newton's method. The sensitivities of a filter transfer function to passive component tolerances, as well as active gain variations are examined by the Schoeffler sensitivity and Monte Carlo PSpice simulation. Butterworth and Chebyshev 0.5dB filter examples illustrate the design method.**

I. Introduction

A procedure for the *analytical* design of low-sensitivity class-4 2^{nd}-and 3^{rd}-order Sallen-and-Key [4] active resistance-capacitance (*RC*) low-pass (LP) allpole filters was presented in [1], with the realizability constraints in [2]. It was shown in [1] that by the use of "impedance tapering", in which L-sections of the *RC* network are successively impedance scaled upwards, from the driving source to the positive amplifier input, the sensitivity of the filter characteristics to *passive* component tolerances can be significantly decreased.

In this paper, the design method based on "impedance tapering" is extended to the design of 4^{th}-order LP active-*RC* filters each with a single operational amplifier (opamp). It is also demonstrated that obtaining an *analytical* solution is possible only for lower-than-4^{th}-order filters and for a special case of 4^{th}-order unity-gain filter (β=1) [3]. The examples of Butterworth and Chebyshev filters with 0.5dB pass-band ripple illustrate optimal filter design with minimum passive and active sensitivities.

II. Fourth-Order Allpole Filter

Consider the 4^{th}-order single-amplifier LP filter shown in Fig. 1. It is a low-power circuit, insofar as it uses only one opamp. Its voltage transfer function $T(s)$, is given by:

$$T(s)=\frac{V_2}{V_1}=\frac{\beta a_0}{s^4+a_3s^3+a_2s^2+a_1s+a_0}, \qquad (1)$$

where coefficients a_i (i=0,…,3) as a function of components of the circuit are given by (2).

Transfer function $T(s)$ in (1), can be written in terms of pole Q-factors, q_{pi}, and pole frequencies ω_{pi}; (i=1, 2) as

$$T(s)=\frac{\beta\omega_{p1}^2\omega_{p2}^2}{\left[s^2+(\omega_{p1}/q_{p1})s+\omega_{p1}^2\right]\left[s^2+(\omega_{p2}/q_{p2})s+\omega_{p2}^2\right]}. \qquad (3)$$

Note that the gain for the class-4 circuit is given by:

$$\beta=1+R_F/R_G\geq 1. \qquad (4)$$

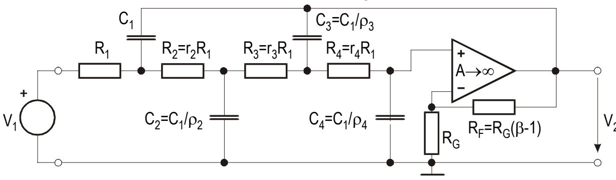


Fig. 1. 4^{th}-order LP filter with single opamp and impedance scaling factors r_i and ρ_i; (i=2,3,4).

Introducing the design frequency ω_0 and impedance scaling factors r_i and ρ_i as in Fig. 1, defined by

$$\omega_0=(R_1C_1)^{-1},\ R_i=r_iR_1,\ C_i=C_1/\rho_i;\ i=2,3,4; \qquad (5)$$

into (2), and using

$$\alpha_i=a_i/\omega_0^{n-i};\ i=0,1,2,\dots,n-1;\ n=4, \qquad (6)$$

we obtain a system of four equations with eight unknowns. To design the 4^{th}-order LP filter we have to solve this system, and therefore we must choose four variables, and then calculate the remaining four. For example, we can calculate the resistive tapering factors r_i (i=2, 3, 4) and gain β from given coefficients a_i (i=0, …, 3), chosen capacitive factors ρ_i (i=2, 3, 4) and the design frequency ω_0. Capacitive scaling factors ρ_i should geometrically progress, providing "capacitive tapering" in the filter design. Note that, alternatively, we could have started by choosing resistive scaling factors r_i, thus providing a "resistive tapering" design procedure. We can express values of r_4 and β explicitely, but we obtain a nonlinear relation between r_2 and r_3. The new system of four nonlinear equations is given by:

$$\begin{aligned}&r_4=\rho_2\rho_3\rho_4/(r_2r_3\alpha_0);\\&a\cdot r_2^2r_3^2+b\cdot r_2r_3^2+c\cdot r_2r_3+d\cdot r_3^2+e\cdot r_3+f\cdot r_2=0; \qquad (7)\\&g\cdot r_2^3r_3^2+h\cdot r_2^2r_3^2+i\cdot r_2r_3^2+j\cdot r_2^2r_3+k\cdot r_3^2+l\cdot r_2r_3+m\cdot r_2^2=0;\\&\beta=1+\rho_3/\rho_4-(r_4/\rho_4)[\alpha_3-1-(\rho_2+\rho_3)/r_3-(1+\rho_2)/r_2],\end{aligned}$$

where the constants a to m in the 2^{nd} and 3^{rd} equations can readily be calculated from eqs. (2) to (6). The next step,

$$a_0=(R_1R_2R_3R_4C_1C_2C_3C_4)^{-1},\ a_1=a_0\{R_1(C_1+C_2+C_3+C_4)+R_2(C_2+C_3+C_4)+R_3(C_3+C_4)+R_4C_4-\beta[R_1C_1+C_3(R_1+R_2+R_3)]\},$$

$$\begin{aligned}a_2=a_0\{&R_1R_2C_1(C_2+C_3+C_4)+R_1R_3C_3(C_1+C_2)+R_1R_3C_4(C_1+C_2)+R_1R_4C_4(C_1+C_2+C_3)+\\&+R_2R_3C_2(C_3+C_4)+R_2R_4(C_2+C_3)C_4+R_3R_4C_3C_4-\beta[(R_1+R_2)R_3C_2C_3+R_1(R_2+R_3)C_1C_3]\},\end{aligned} \qquad (2)$$

$$a_3=a_0\{R_1R_2R_3C_1C_2(C_3+C_4)+R_1R_2R_4C_1C_4(C_2+C_3)+R_1R_3R_4C_3C_4(C_1+C_2)+R_2R_3R_4C_2C_3C_4-\beta R_1R_2R_3C_1C_2C_3\}.$$

which is the only one that seems to be possible in trying to find an analytical solution, is to merge and factorise the 2nd and 3rd equations of the system (7) in the form given by:

$$(r_2^3 + nr_2^2 + or_2 + p)\cdot(r_3^2 + qr_3 + t) = 0\,. \qquad (8)$$

The form (8) is obviously impossible for representation and we can't solve the system (7) analytically. Instead, the second and third equations in (7) can be solved for the values of r_2 and r_3 only numerically, using, for example Newton's iterative method. Once we have the values of r_2 and r_3, the remaining values of r_4 and β readily follow from the first and last equations of the system, respectively.

In the following examples, a 4th-order LP active-*RC* filter will be solved using the program "Mathematica".

III. Example

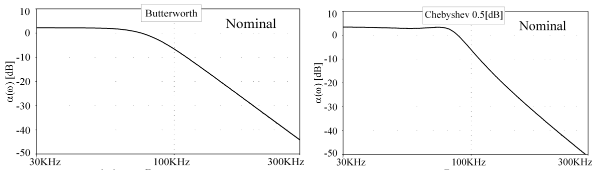
Consider the Butterworth and Chebyshev filter with a pass-band ripple R_p=0.5dB having the coefficients shown in Table I. The corresponding amplitude responses $\alpha(\omega)$[dB] are shown in Fig. 2.

TABLE I

BUTTERWORTH* AND CHEBYSHEV SPECIFICATIONS COEFFICIENTS

No.	Specif.	R_p [dB]	a_0 $\times10^{22}$	a_1 $\times10^{17}$	a_2 $\times10^{11}$	a_3 $\times10^{5}$	ω_{p1} $\times10^{5}$	ω_{p2} $\times10^{5}$	q_{p1}	q_{p2}
1.*	1.0/40dB; 50/250kHz	0.0	6.09	3.20	8.42	13.0	4.97	4.97	0.54	1.31
2.	0.5/38dB; 80/300kHz	0.5	2.42	1.13	4.34	6.02	3.00	5.18	0.71	2.94

We start with the Butterworth transfer function coefficients in line 1) of Table I. In the design process, various ways of impedance tapering have been applied, i.e. capacitive and resistive, and the resulting component values are presented in Table II with resistors in [kΩ] and capacitors in [pF]. A sensitivity analysis was performed. The standard deviations $\sigma_\alpha(\omega_p)$ in [dB] (related to the Shoeffler sensitivities) of the variation of the log gain $\Delta\alpha=8.68588\Delta|T(j\omega)|/|T(j\omega)|$[dB], with respect to zero mean and 1% standard deviation of *passive* components, was calculated at the dominant pole frequency $\omega_p=\omega_{p2}$. This is the frequency vicinity in which the magnitude spread is the highest; it is shown in the last column of Table II.



(a) (b)

Fig. 2. Magnitude of transfer functions for filters in Table I. **(a)** Butterworth. **(b)** Chebyshev with R_p=0.5[dB].

TABLE II

COMPONENT VALUES OF IMPEDANCE-TAPERED 4TH-ORDER LP FILTER

No.	Tap.	R_1	C_1	r_2	r_3	r_4	ρ_2	ρ_3	ρ_4	β	$\sigma_\alpha(\omega_p)$
1)	Cap.	15.6	100	3.20	5.03	0.17	1	1	1	2.0	2.227
2)		1.59	800	12.6	5.47	5.84	2	4	8	2.0	0.765
3)		0.46	2700	18.7	10.8	25.1	3	9	27	2.0	0.638
1)	Res.	10	193	1	1	1	0.28	2.12	1.43	4.0	1.605
2)		10	149	2	4	8	0.41	7.02	6.68	4.0	1.198
3)		10	134	3	9	27	0.54	14.7	18.1	4.0	1.079

Beside Schoeffler's sensitivity, Monte Carlo runs (MC) are also performed as a double-check and presented in Figs. 4 and 6. The corresponding PSpice circuit model of a 4th-order LP filter with 1% Gaussian distributed, zero-mean resistors and capacitors is shown in Fig. 3. Note that the opamp is simulated by a voltage-controlled-voltage-source with high and constant gain value of $A=10^{10}$.

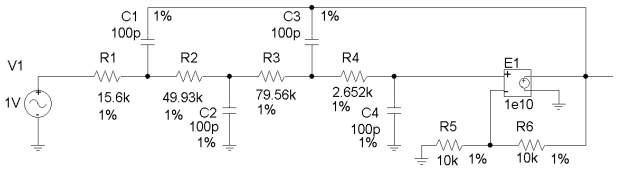
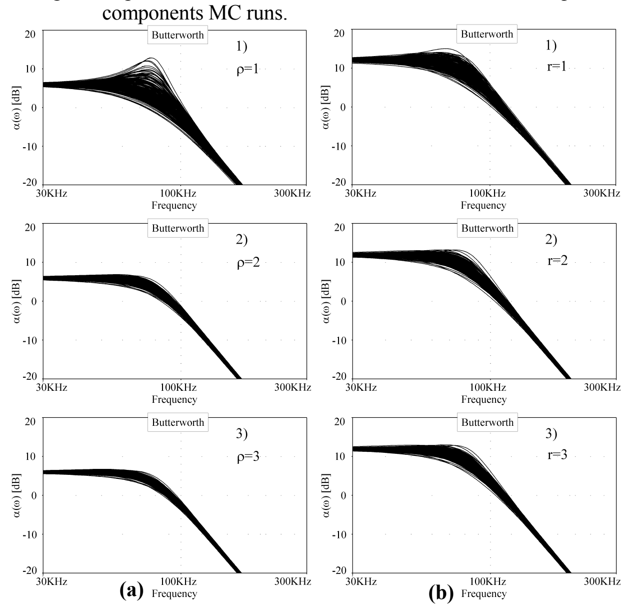


Fig. 3. PSpice circuit model of 4th-order LP filter for *passive* components MC runs.



(a) **(b)**

Fig. 4. Passive components Monte Carlo runs of impedance-tapered 4th-order LP filters as given in Table II. **(a)** Capacitively. **(b)** Resistively.

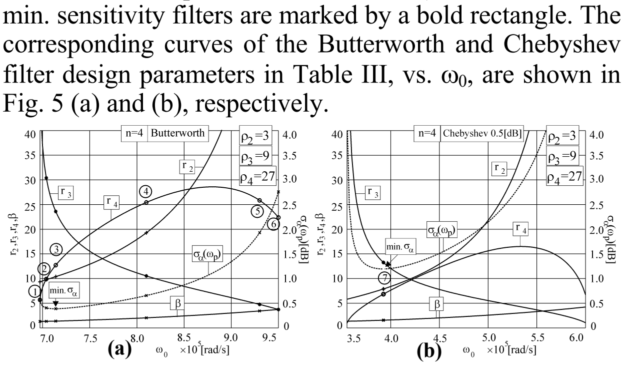
Observing Fig. 4 we conclude that impedance tapering significantly reduces sensitivity to passive components. For the next example, we choose 6 different values of ω_0 with tapered capacitors. We obtain 6 different filters with the design values in Table III and MC runs shown in Fig. 6.

TABLE III

DEPENDENCE OF DESIGN PARAMETERS ON SELECTION OF ω_0

No.	Spec.	R_1	C_1	r_2	r_3	r_4	ρ_2	ρ_3	ρ_4	β	$\sigma_\alpha(\omega_p)$
1)	Bu.	0.531	2700	9.31	53.6	5.67	3	9	27	1.29	0.569
2)		0.526	2700	9.77	30.4	9.91	3	9	27	1.32	0.404
3)		0.519	2700	10.4	23.6	12.7	3	9	27	1.37	**0.388**
4)		0.457	2700	19.3	10.5	25.4	3	9	27	2.04	0.655
5)		0.398	2700	73.0	4.75	25.8	3	9	27	3.43	1.928
6)		0.390	2700	117	3.72	22.4	3	9	27	3.73	2.756
7)	Ch.	0.926	2700	7.89	13.3	6.84	3	9	27	1.56	**1.195**

Using the Chebyshev transfer function coefficients in line 2) of Table I, we designed the minimum sensitivity circuit, with component values in line 7) of Table III. The min. sensitivity filters are marked by a bold rectangle. The corresponding curves of the Butterworth and Chebyshev filter design parameters in Table III, vs. ω_0, are shown in Fig. 5 (a) and (b), respectively.



(a) (b)

Fig. 5. Design parameters r_2, r_3, r_4 and β in Table III as a function of ω_0.

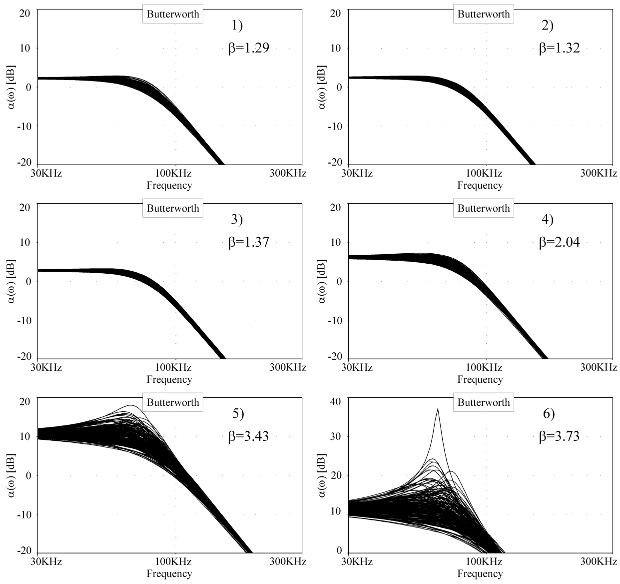


Fig. 6. Passive components Monte Carlo runs of amplitude response of capacitively-tapered 4th-order LP filter as a function of ω_0 (Table III).

In Fig. 7 we introduced a model of the open-loop gain A device (inside dashed rectangle) to use with PSpice for active component gain variation in the 4th-order LP filter. R_7 and R_9 are supposed to be tracking resistors, as well as R_8 and R_{10}. A-variations are performed by variations of both R_7 and R_8 and theirs tracking counterparts. The gain A values are log distributed in the range from $1.3{\cdot}10^2 \div 4.5{\cdot}10^5$. In practice, the gain A changes with temperature and at higher frequencies the gain A becomes smaller. Note that all passive components take their nominal values. The corresponding MC runs of an active component are presented in Figs. 8 and 9.

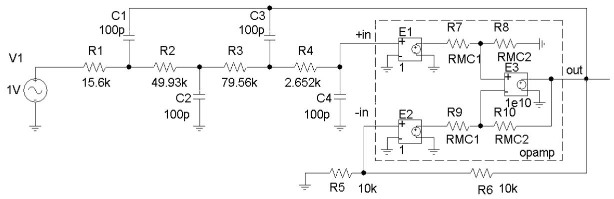


Fig. 7. PSpice circuit model for an *active* component MC runs.

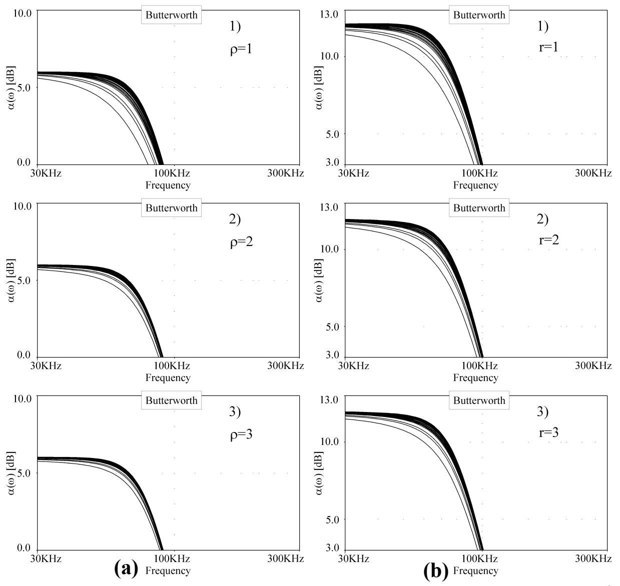


Fig. 8. Active component (opamp) MC runs of impedance-tapered 4th-order LP filters as given in Table II. **(a)** Capacitively. **(b)** Resistively.

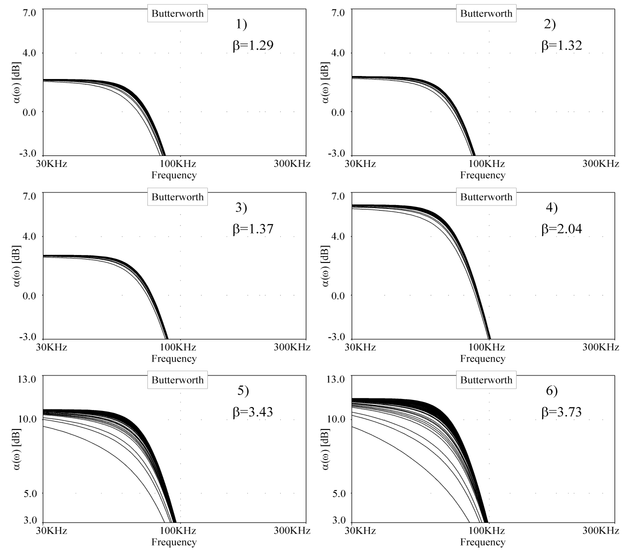


Fig. 9. Active component (opamp) MC runs of amplitude response of capacitively-tapered 4th-order LP filter as a function of ω_0 (Table III).

Observing all above MC runs one can conclude that the impedance tapering significantly decreases sensitivity with respect to *passive* and *active* component variations, compared with the non-tapered standard circuit. The min. passive sensitivity of the filter is achieved by capacitive tapering and selecting the appropriate value of ω_0 (filter no. 3). This also corresponds to the min. active sensitivity. Finally, MC runs of Chebyshev filter no. 7) is shown in Fig. 10.

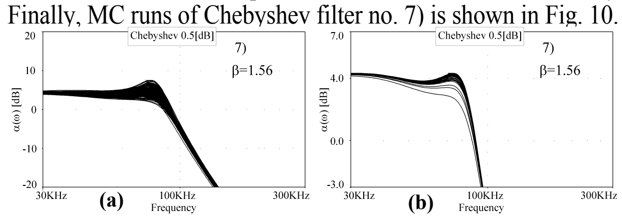


Fig. 10. **(a)** Passive, and **(b)** active sensitivity represented by MC runs of amplitude response of Chebyshev filter no. 7) in Table III.

From the Chebyshev example in Fig. 10 we conclude that for higher pole Qs, q_p, both the active and passive sensitivities are increased.

A. Design of 4th-order Filter Starting from β=1 Filter

We use β=1 filter elements in [3] for the starting values in the Newton's iterative method, when numerically solving the set of non-linear eqs. (7). We obtain the filters in Table IV and the corresponding MC runs shown in Figs. 11 and 12.

TABLE IV
DEPENDENCE OF DESIGN PARAMETERS ON SELECTION OF ω_0

No.	Tap.	R_1	C_1	r_2	r_3	r_4	ρ_2	ρ_3	ρ_4	β	$\sigma_\alpha(\omega_p)$
1)	β=1	0.611	5	1.08	5.24	2.76	1.34	1.0	62.1	1.0	0.168
2)	Res.	10	289	1.08	5.24	2.76	1.17	2.47	23.1	1.1	0.258
3)		10	267	1.08	5.24	2.76	0.96	3.60	14.0	1.3	0.406
4)		10	228	1.08	5.24	2.76	0.63	5.47	7.44	2.0	0.780
5)	Cap.	1.288	5	0.40	0.70	2.83	1.34	1.0	62.1	1.1	0.342
6)		5.780	5	0.06	0.07	0.48	1.34	1.0	62.1	1.3	1.433

We chose resistors as starting values and then increased the values of β by changing the design frequency ω_0, thus obtaining the circuits no. 1)-4). Our goal is to reach, for example, the min. active sensitivity (filter no. 3 in Table IV, see Fig. 12). Note that for the 4th-order filter, we do not have any analytical expression by which we can calculate the min. GSP, as is possible for the 2nd-order case [5].

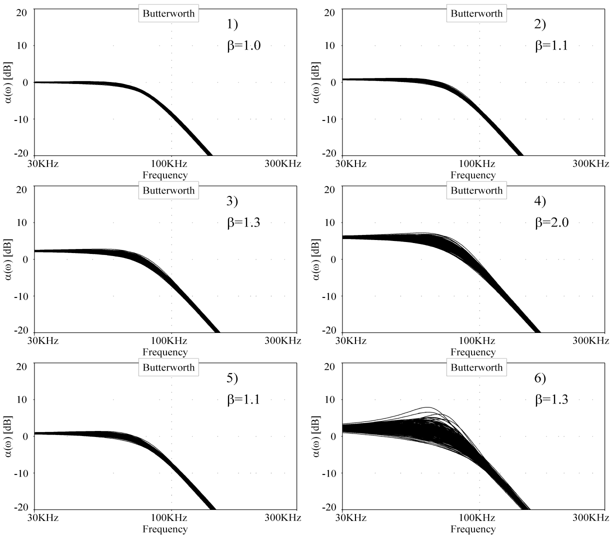


Fig. 11. Passive components Monte Carlo runs of amplitude response of capacitively-tapered 4^{th}-order LP filter as a function of β (Table IV).

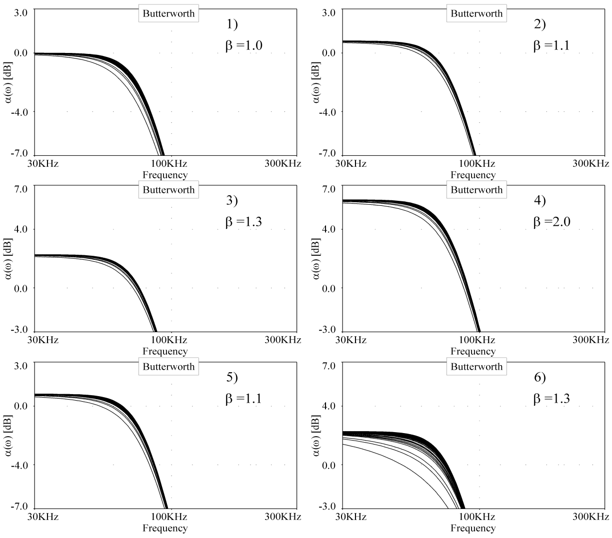


Fig. 12. Active component (opamp) MC runs of amplitude response of capacitively-tapered 4^{th}-order LP filter as a function of β (Table IV).

It follows from Table IV that we must be careful not to choose capacitors as starting values, because we will then obtain bad circuits: no. 5)-6). The corresponding curves for design parameters in Table IV, vs. ω_0, which can be represented by β, are shown in Fig. 13. As follows from [3] it is not possible to design the Chebyshev filter starting from the β=1 filter.

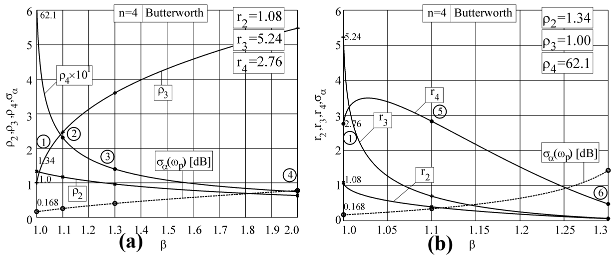


Fig. 13. Design parameters in Table IV as a function of ω_0. Note that the abscise-axis is represented by β. **(a)** R-Tapering. **(b)** C-Tapering.

Note that the unity gain circuit no. 1) has min. passive sensitivity, but on the other hand, it is very sensitive with respect to the gain *A*. This is evident from the curves in Fig. 13, in which the gradients of the component ratios are very large in the vicinity of the gain β=1. Unity gain filter is even more sensitive for higher pole Q-factors.

B. *Comparison of 4^{th}-order Filter with 2-Biquad CAS*

In this section we compare the performance of the single opamp, (1-OA) 4^{th}-order LP filter with a cascade (CAS) of two 2^{nd}-order LP filter "biquads" as in [1]. We apply capacitive impedance tapering with resistor values selected for GSP-minimization, to minimize both *passive* and *active* sensitivity, to the two biquads in cascade [5]. The component values are given in Table V.

TABLE V

COMPONENT VALUES OF 2^{ND}-ORDER LP SECTIONS IN CAS

No.	Design	Blocks)	R_1	C_1	r_2	ρ_2	β	$\sigma_\alpha(\omega_p)$
1)	min.	I)	3.506	5	1.58	1.20	1.03	0.208
	GSP.	II)	5.125	5	2.47	4	1.27	

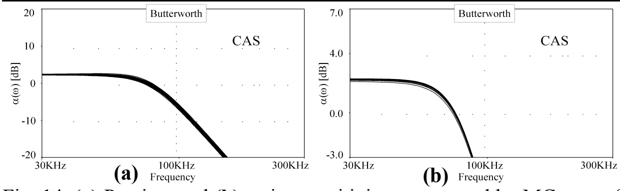


Fig. 14. **(a)** Passive, and **(b)** active sensitivity represented by MC runs of amplitude response of CAS 4^{th}-order LP filter (Table V).

To examine sensitivities we perform MC runs as in the examples above, and present them in Fig. 14. Clearly, the CAS realization is easier to design, and has much lower sensitivities than the 1-OA filter. On the other hand the 1-OA filter has one opamp and two resistors less. Thus, the reduction in power and component count achieved with the 1-OA filter is obtained at a price: cascading of two impedance-tapered biquads is better both with regard to *passive* and *active* sensitivities.

IV. CONCLUSION

In this paper we presented the optimal design of single opamp 4^{th}-order LP filters. Unfortunately, the design equations for higher-than–third order filters defy any analytical solution, thus we presented a numerical design procedure. In one example we used unity gain filter elements as the starting values. It is shown in numerous examples that impedance tapering reduces both *passive* and *active* sensitivities. The unity gain tapered filter is shown to have min. passive sensitivity, but the active sensitivity is still too high. Therefore, an optimal design procedure is to apply capacitive tapering, and then to change the design frequency ω_0 to minimize active sensitivity, and by that, to keep the passive sensitivity reduced, as well. Finally, we compared the tapered single opamp 4^{th}-order LP filter to a cascade of two 2^{nd}-order tapered LP "biquads". The latter has substantially reduced sensitivities, and the design equations for the min. GSP are available in closed form. Thus, the decision on which approach o take is typically one of tradeoffs: low power and element count vs. low sensitivity and design simplicity.

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