Generalized Calogero model in arbitrary dimensions

S.Meljanac[a1], M.Mileković [b2], A. Samsarov[a3]

[a] Rudjer Bošković Institute, Bijenička c.54, HR-10002 Zagreb, Croatia

[b] Theoretical Physics Department, Faculty of Science, P.O.B. 331, Bijenička c.32, HR-10002 Zagreb, Croatia

Abstract

We define a new multispecies model of Calogero type in D dimensions with harmonic, two-body and three-body interactions. Using the underlying conformal $SU(1,1)$ algebra, we indicate how to find the complete set of the states in Bargmann-Fock space. There are towers of states, with equidistant energy spectra in each tower. We explicitely construct all polynomial eigenstates, namely the center-of-mass states and global dilatation modes, and find their corresponding eigenenergies. We also construct ladder operators for these global collective states. Analysing corresponding Fock space, we detect the universal critical point at which the model exhibits singular behavior. The above results are universal for all systems with underlying conformal $SU(1,1)$ symmetry.

PACS number(s): 03.65.Fd, 03.65.Sq, 05.30.Pr

Keywords: multispecies Calogero model, SU(1,1) algebra.

[1]e-mail: meljanac@irb.hr
[2]e-mail: marijan@phy.hr
[3]e-mail: a.samsarov@irb.hr

1 Introduction

The (rational) Calogero model describes N identical particles on the line which interact through an inverse-square two-body interaction and are subjected to a common confining harmonic force. Starting from the inception [1], the model and its various descendants (also known as Calogero-Sutherland-Moser systems [2]) continue to be of interest for both physics and mathematics community, primarly because they are connected with a number of mathematical and physical problems, ranging from random matrices and symmetric polynomials [3] to condensed matter systems and black hole physics [4]. The model is also connected to Haldane's exclusion statistics [5]. The role of Haldane statistical parameter is played by (universal) coupling constant in the two-body interaction. In Haldane's formulation there is the possibility of having particles of different species with a mutual statistical coupling parameter depending on the species coupled. This suggest the possible generalization of the ordinary one-dimensional Calogero model with identical particles to the one-dimensional Calogero model with non-identical particles. Distinguishabillity can be introduced by allowing particles to have different masses and different couplings to each other. In this way a one-dimensional multispecies Calogero model is obtained [6,7].

Further generalization can be achieved by formulating the model in dimensions higher than one. In the case of single-species model(s), it was shown that some exact eigenstates (including the ground state) can be obtained for a D dimensions provided that a long-range three-body interaction is added [8]. The inevitable appearance of three-body interaction in $D > 1$ makes any analysis of such a model(s) highly nontrivial and very little is known about their exact solvability. Some progress has been achieved only recently for a class of two-dimensional models with identical particles [9].

In a present Letter we propose a new type of partially solvable multispecies model of Calogero type in D dimensions. In addition to the harmonic potential, it contains two-body and three-body interactions with coupling constants depending on the particle's species. We also allow particles to have different masses. In this way we incorporate both generalizations mentioned above into a single model. We indicate how to obtain (in principle) all eigenstates of the model Hamiltonian in Bargmann representation. The spectrum of states shows a remarkable simplicity. There are towers of states with equidistant energies. We are able to find all polynomial eigenstates and corresponding eigenenergies of the Hamiltonian, describing global collective states. Closer inspection of the Fock space, corresponding to the relative motion of particles, reveals the existence of the universal critical point at which system exhibits singular behaviour. This result generalizes that mentioned in [7,10]. Our results are universal and applicable to all systems with underlying $SU(1,1)$ algebra.

2 A model Hamiltonian

We start the analysis with observation that the exact wave functions of the Calogero model are highly correlated. These correlations are encoded in the wave functions in the form of a Jastrow prefactor $(x_i - x_j)^\nu$ for any pair of particles i, j. The exponent of the correlator is related to the strength of the two-body interaction. It is then plausible to make an ansatz for the most general ground state wave function for the N *distinguishable* Calogero-like particles in D dimensions in the form ($\hbar = 1$)

$$\Psi_0(\vec{r}_1, ..., \vec{r}_N) = \Delta e^{-\frac{\omega}{2}\sum_{i=1}^N m_i \vec{r}_i^2} \tag{1}$$

where the Jastrow prefactor is generalized to

$$\Delta = \prod_{i<j} |\vec{r}_i - \vec{r}_j|^{\nu_{ij}}, \quad \nu_{ij} = \nu_{ji}, \quad i, j = 1, 2, \cdots N. \tag{2}$$

Here, m_i are masses of the particles, ω is the frequency of the harmonic potential and ν_{ij} are the statistical parameters between particles i and j. In principle, one could start with any wave function with no nodes, except at the coincidence points, and which is continuosly connected with Gausss function when parameters $\nu_{ij} \to 0$. Note that for $\nu_{ij} = \nu$, $m_i = m$ and $D \neq 1$, Eq.(1) smoothly goes to exact ground state of the Calogero-Marchioro model [11], so the wave function (1) is a natural choice. Adopting the reasoning from Ref.[12], we can ask for what kind of Hamiltonian is the wave function (1) the exact ground state. It turns out that $\Psi_0(\vec{r}_1, ..., \vec{r}_N)$ will be, for sufficiently small deformations ν_{ij}, the exact ground state of the Hamiltonian

$$\begin{aligned} H = & -\frac{1}{2}\sum_{i=1}^N \frac{1}{m_i}\vec{\nabla}_i^2 + \frac{\omega^2}{2}\sum_{i=1}^N m_i \vec{r}_i^2 \\ & +\frac{1}{2}\sum_{i<j} \frac{\nu_{ij}(\nu_{ij} + D - 2)}{|\vec{r}_i - \vec{r}_j|^2}(\frac{1}{m_i} + \frac{1}{m_j}) \\ & +\frac{1}{2}\sum_{i\neq j, i\neq k} \frac{\nu_{ij}\nu_{ik}(\vec{r}_i - \vec{r}_j)(\vec{r}_i - \vec{r}_k)}{m_i|\vec{r}_i - \vec{r}_j|^2|\vec{r}_i - \vec{r}_k|^2}, \end{aligned} \tag{3}$$

such that

$$H\Psi_0 = E_0\Psi_0, \tag{4}$$

$$E_0 = \omega(\frac{ND}{2} + \sum_{i<j}\nu_{ij}) \equiv \omega\epsilon_0. \tag{5}$$

The ground state (1) and the Hamiltonian (3) are invariant under the group of permutation of N elements, S_N, generated by exchange operators K_{ij} [13]. Operators

K_{ij} interchange indices $i \leftrightarrow j$ in all quantities, i.e. $m_i \leftrightarrow m_j$, $\nu_{ik} \leftrightarrow \nu_{jk}$, $\vec{r}_i \leftrightarrow \vec{r}_j$, $\vec{p}_i \leftrightarrow \vec{p}_j$.

For $D = 1$ the three-body term in (3) identically vanish if $\nu_{ij} = \nu$, $m_i = m$ or if $\nu_{ij} = \alpha m_i m_j$, α beeing some universal constant [7]. Unlike in one dimension, however, it does not vanish in higher dimensions and plays a crucial role in the analysis that is to follow.

Let us perform the non-unitary transformation on Ψ_0, namely $\tilde{\Psi}_0 = \Delta^{-1}\Psi_0$. It generates a similarity transformation which leads to another S_N invariant Hamiltonian $\tilde{H} = \Delta^{-1} H \Delta$. We find $\tilde{H}$ as

$$\begin{aligned}\tilde{H} &= -\frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\vec{\nabla}_i^2 + \frac{\omega^2}{2}\sum_{i=1}^{N} m_i \vec{r}_i^{\,2} - \sum_{i<j}\nu_{ij}\frac{(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2}\left(\frac{1}{m_i}\vec{\nabla}_i - \frac{1}{m_j}\vec{\nabla}_j\right) \\ &= \omega^2 T_+ - T_- . \end{aligned} \tag{6}$$

where we have introduced

$$T_- = \frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\vec{\nabla}_i^2 + \sum_{i<j}\nu_{ij}\frac{(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2}\left(\frac{1}{m_i}\vec{\nabla}_i - \frac{1}{m_j}\vec{\nabla}_j\right), \tag{7}$$

$$T_+ = \frac{1}{2}\sum_{i=1}^{N} m_i \vec{r}_i^{\,2}, \qquad T_0 = \frac{1}{2}\left(\sum_{i=1}^{N}\vec{r}_i\vec{\nabla}_i + \varepsilon_0\right).$$

The operators $T_\pm$, T_0 satisfy the $SU(1,1)$ algebra

$$[T_-, T_+] = 2T_0, \qquad [T_0, T_\pm] = \pm T_\pm. \tag{8}$$

The following identity (i.e. similarity transformation) holds for $\omega \neq 0$:

$$\tilde{H} = \omega^2 T_+ - T_- = 2\omega S\, T_0\, S^{-1}, \tag{9}$$

$$S = e^{-\omega T_+}\, e^{-\frac{1}{2\omega}T_-}.$$

Owing to this identity, we can employ Bargmann representation and construct iteratively Bargmann-Fock space of eigenstates. We begin with state Φ_0, which is the lowest weight vector of the operator T_- and also an eigenstate of T_0:

$$T_-\Phi_0 = 0, \qquad T_0\Phi_0 = \frac{\epsilon_0}{2}\Phi_0. \tag{10}$$

In our case, $\Phi_0 = 1$ and ϵ_0 is given in Eq.(5).
The tower of excited states (level 0-tower) is obtained by succesive application of T_- operator:

$$T_-\Phi_{2p} = \Phi_{2p-2}, \qquad 2\omega T_0\Phi_{2p} = \omega(2p+\epsilon_0)\Phi_{2p}, \qquad p = 0, 1, 2, 3.... \tag{11}$$

The states Φ_{2p} are either polynomials or irrational functions of homogenity $2p$, and are eigenstates of T_0. Two succesive states differ in energy by an amount 2ω.
Similarly, one can construct towers of states at level 1, $\Phi^{I_1}_{2p+1}$, $p = 0, 1, 2...$,
$I_1 = (i_1 = 1, 2, ...N; \alpha_1 = 1, 2, ...D)$, using

$$T_-\Phi^{I_1}_1 = 0, \qquad T_-\Phi^{I_1}_{2p+1} = \Phi^{I_1}_{2p-1}, \qquad 2\omega T_0\Phi^{I_1}_{2p+1} = \omega(2p+\epsilon^{I_1}_1)\Phi^{I_1}_{2p+1}. \tag{12}$$

Here, $\epsilon^{I_1}_1$ is energy of the first excited state which tends to $(1+\frac{ND}{2})$ in the limit $\nu_{ij} \to 0$. Two succesive states also differ in energy by an amount 2ω.
Following the procedure, one gets the towers of states at level k, $0 \le k \le ND$, using

$$T_-\Phi^{I_1,...,I_k}_k = 0, \qquad T_-\Phi^{I_1,...,I_k}_{2p+k} = \Phi^{I_1,...,I_k}_{2p+k-2}, \qquad 2\omega T_0\Phi^{I_1,...,I_k}_{2p+k} = \omega(2p+\epsilon^{I_1,...,I_k}_k)\Phi^{I_1,...,I_k}_{2p+k}.$$

Here, the energies $\epsilon^{I_1,...,I_k}_k$ tends to $(k+\frac{ND}{2})$ in the limit $\nu_{ij} \to 0$. The states $\Phi^{I_1,...,I_k}_{2p+k}$ are eigenstates of the Hamiltonian $2\omega T_0 = S^{-1}\,\tilde{H}\,S$, Eq.(9). Particularly, the state $\Phi_0 = 1$ is a ground state (i.e. the lowest energy eigenstate) for all towers if $\epsilon_0 < \epsilon^{I_1,...I_k}_k$, $\forall I_1, ..., I_k$ and for all indices k.
Notice that the operator T_+ of Eq.(7), acting on the particular state in the given

tower, gives an another state in the same tower with energy greater by an amount 2ω (see also Sec. 3, Eq.(17)).

The procedure outlined above in Eqs.(10-12) is exaustive, i.e. it gives all eigenstates of the S-transformed Hamiltonian $S^{-1}\,\tilde{H}\,S$ (cf. Eq.(9)) , provided one is able to solve differential equations (11) and (12). This is a non-trivial task, even in $D=1$. However, one can readily show that this procedure, when applied to the system of N D-dimensional free harmonic oscillators ($\nu_{ij}=0$ in Eqs.(6) and (3)) yields the following set of eigenstates for $\tilde{H}=H$:

$$S\,\cdot\,(\prod_{i=1}^{N}\prod_{\alpha=1}^{D} r_{i,\alpha}^{n_{i,\alpha}}), \qquad n_{i,\alpha}=0,1,2...,, \tag{13}$$

where (cf. Eqs.(7) and (9))

$$S=e^{-\omega T_+}\,e^{-\frac{1}{2\omega}T_-}=e^{-\omega\frac{1}{2}\sum_{i=1}^{N}m_i\vec{r}_i^{\,2}}\,e^{-\frac{1}{2\omega}\frac{1}{2}\sum_{i=1}^{N}\frac{1}{m_i}\vec{\nabla}_i^2}.$$

The corresponding eigenenergies are

$$\omega(\frac{ND}{2}+\sum_{i=1}^{N}\sum_{\alpha=1}^{D}n_{i,\alpha}). \tag{14}$$

For the convenience of the reader, we describe in the next Table the structure of the few lowest towers of states at level k (Eqs.(10)-(12)) in this simple case.

k	level-k tower	indices
0	1	-
1	r_I	$I=(i,\alpha)$
2	$r_{I_1}r_{I_2}$	$I_1\neq I_2$
3	$r_{I_1}r_{I_2}r_{I_3}$	$I_1\neq I_2\neq I_3\neq I_1$
$\vdots$	$\vdots$	$\vdots$

For the general case, Eqs.(10)-(12), the towers of states at level k, $\Phi_k^{I_1,\ldots,I_k}$, need not have such a simple monomial structure since they can be , in principle, homogenious irrational functions. From Eq.(13) one can count the number of states at each level of given homogenity. For example, there are ND states of homogenity one, $ND + \frac{ND(ND-1)}{2}$ states of homogenity two, etc. There are 2^{ND} towers in total. Now, one can put an interesting question, namely is there, for sufficiently small deformation parameters ν_{ij}, one-to-one correspondence between our multispecies model $\tilde{H}(\nu_{ij})$, Eq.(6), and N D-dimensional free harmonic oscillators $H(\nu_{ij}=0) = \tilde{H}(\nu_{ij}=0)$. According to our analysis, there is no unique similarity transformation between these two systems. However, there is similarity transformation between given tower in the interacting system ($\nu_{ij} \neq 0$) and analogous tower in the free system ($\nu_{ij}=0$), up to the constant $\epsilon_k^{I_1,\ldots I_k}$. Particularly, this was shown for $D=1$ and identical particles ($\nu_{ij}=\nu$) in Ref.[9]. In that case, the eigenstates are restricted to S_N-symmetric representations.

We are unable to find towers of states by solving differential equations (11-12) in general. However, as we will show in the next section, we are able to construct global collective states for the Hamiltonian (6). These states represent all states of the polynomial type in Bargmann representation in generic case. Moreover, these states are universal for all systems with underlying conformal $SU(1,1)$ symmetry.

3 Ladder operators and Fock space representation for global collective states

It is convenient to introduce the center-of-mass coordinate $\vec{R}$ and the relative coordinates $\vec{\rho}_i$ [14]:

$$
\begin{aligned}
\vec{R} &= \frac{1}{M}\sum_{i=1}^{N} m_i \vec{r}_i, \qquad \vec{\nabla}_R = \sum_{i=1}^{N} \vec{\nabla}_i \\
\vec{\rho}_i &= \vec{r}_i - \vec{R}, \qquad \vec{\nabla}_{\rho_i} = \vec{\nabla}_i - \frac{m_i}{M}\vec{\nabla}_R
\end{aligned} \tag{15}
$$

They satisfy identities $\sum_{i=1}^N m_i\vec{\rho}_i = \sum_{i=1}^N \vec{\nabla}_{\rho_i} = 0$. In terms of the variables just introduced, the Hamiltonian $\tilde{H}$ and wave function $\tilde{\Psi}_0$ separate into parts which describe center-of-mass motion (CM) and relative motion (R), namely $\tilde{H} = \tilde{H}_{CM} + \tilde{H}_R$ and $\tilde{\Psi}_0(\vec{r}_1, ..., \vec{r}_N) = \tilde{\Psi}_0(\vec{R})\tilde{\Psi}_0(\vec{\rho}_1, ..., \vec{\rho}_N)$.

Using Eqs.(7) and (15) we define creation (+) and annihilation (-) operators

$$\vec{A_1}^{\pm} = \frac{1}{\sqrt{2}}(\sqrt{M\omega}\vec{R} \mp \frac{1}{\sqrt{M\omega}}\vec{\nabla}_R),$$

$$A_2{}^{\pm} = \frac{1}{2}(\frac{T_-}{\omega} + \omega T_+) \mp T_0, \tag{16}$$

which satisfy the following commutation relations $(\alpha, \beta = 1, 2, \cdots D)$:

$$[A_{1,\alpha}^-, A_{1,\beta}^+] = \delta_{\alpha\beta}, \qquad [A_{1,\alpha}^-, A_{1,\beta}^-] = [A_{1,\alpha}^+, A_{1,\beta}^+] = 0,$$

$$[\vec{A_1}^-, A_2{}^+] = \vec{A_1}^+, \qquad [A_2^-, \vec{A_1}^+] = \vec{A_1}^-,$$

$$[A_2^-, A_2{}^+] = \frac{\tilde{H}}{\omega}, \qquad [\tilde{H}, \vec{A_1}^{\pm}] = \pm\omega\vec{A_1}^{\pm},$$

$$[\tilde{H}, A_2{}^{\pm}] = \pm 2\omega A_2{}^{\pm}. \tag{17}$$

Notice that $A_2{}^{\pm} = S\,T_{\pm}\,S^{-1}$, $\vec{A_1}^+ = S\,\vec{R}\,S^{-1}$ and $\vec{A_1}^- = S\,\vec{\nabla}_R\,S^{-1}$, with S defind in Eq.(9). They act on the Fock vacuum $|\tilde{0}\rangle \propto \tilde{\Psi}_0(\vec{r}_1, ..., \vec{r}_N)$ as

$$\vec{A_1}^-|\tilde{0}\rangle = A_2^-|\tilde{0}\rangle = 0, \qquad \langle\tilde{0}|\tilde{0}\rangle = 1. \tag{18}$$

The excited states in the Fock space, corresponding to global collective states, are of the form

$$A_{1,1}^+{}^{n_{1,1}} \cdots A_{1,D}^+{}^{n_{1,D}} A_2^{+n_2}|\tilde{0}\rangle \equiv \prod_{\alpha=1}^{D}(A_{1,\alpha}^+)^{n_{1,\alpha}} A_2^{+n_2}|\tilde{0}\rangle, \tag{19}$$

where $n_{1,\alpha} = 0, 1, 2...(\forall\alpha)$ and $n_2 = 0, 1, 2...$

The repeated action of the operators $A_{1,\alpha}^+$ on the vacuum $|\tilde{0}\rangle$ reproduces, in the coordinate representation, Hermite polynomials $H_{n_{1,\alpha}}(R_\alpha\sqrt{M\omega})$. Similarly, the repeated action of the operator A_2^+ on the vacuum $|\tilde{0}\rangle$ reproduces hypergeometric function, which reduces to associated Laguerre polynomials $L_{n_2+\varepsilon_0-1}^{\varepsilon_0-1}(2\omega T_+)$ for certain values of parameters. The states (19) are eigenstates of the $\tilde{H}$ with the energy eigenvalues (cf. last two equations in Eqs.(17))

$$E_{n_{1,\alpha};n_2} = \omega(\sum_{\alpha=1}^{D} n_{1,\alpha} + 2n_2 + \varepsilon_0). \tag{20}$$

This is the part of the complete spectrum which corresponds to center-of-mass states and global dilatation states, respectively.

Now we show that the states (19) are perfectly normalizable,i.e. quadratically integrable and physically acceptable for both Hamiltonians $\tilde{H}$ and H, provided that $\epsilon_0 > \frac{D}{2}$. First, we completely decouple CM- and R-motion by introducing another set of the creation and annihilation operators $\{B_2^+, B_2^-\}$:

$$B_2^{\pm} = A_2{}^{\pm} - \frac{1}{2}(\vec{A_1}{}^{\pm})^2, \tag{21}$$

such that

$$[A_{1,\alpha}{}^{\pm}, B_2{}^{\mp}] = 0. \tag{22}$$

Hence, we get

$$\begin{aligned} \tilde{H}_R = \omega[B_2^-, B_2^+], \qquad [\tilde{H}_R, B_2^{\pm}] = \pm 2\omega B_2^{\pm}, \\ \tilde{H}_{CM} = \frac{1}{2}\omega \sum_{\alpha=1}^{D} \{A_{1,\alpha}^-, A_{1,\alpha}{}^+\}_+. \end{aligned} \tag{23}$$

The Fock space now splits into the CM-Fock space, spanned by $\prod_{\alpha=1}^{D}(A_{1,\alpha}^+)^{n_{1,\alpha}}|\tilde{0}\rangle_{CM}$ and the R-Fock space, spanned by $B_2^{+n_2}|\tilde{0}\rangle_R$, where $|\tilde{0}\rangle_{CM} \propto e^{-\frac{\omega}{2}M\vec{R}^2}$ and $|\tilde{0}\rangle_R \propto e^{-\frac{\omega}{2}\sum_i m_i \vec{\rho}_i^2}$. We point out that the R-modes are universal for all systems with underlying conformal $SU(1,1)$ symmetry, i.e. for the Hamiltonians of the form $H = -T_- + \omega^2 T_+ + \gamma T_0$, where $T_{\pm}, T_0$ satisfy $SU(1,1)$ algebra (8).

Closer inspection of the R-Fock space of the Hamiltonian $\tilde{H}_R$, Eq.(23), reveals the existence of the universal critical point defined by the zero-energy condition

$$E_{0R} = \frac{(N-1)D}{2} + \frac{1}{2}\sum_{i\neq j} \nu_{ij} = 0. \tag{24}$$

At the critical point the system described by $\tilde{H}_R$ collapses completely. This means that the relative coordinates, the relative momenta and the relative energy are all zero at this critical point. There survives only one oscillator, describing the motion of the centre-of-mass. Such behaviour resembles some features of the Bose-Einstein condensate. It was first noticed in Ref.[9] for the case $D = 1$, $\nu_{ij} = \nu$ and $m_i = m$. In that case the critical point (24) is simply at $\nu = -\frac{1}{N}$. (Notice that there is also critical point at $\nu = 1 + \frac{1}{N}$ for this case). Of course, for the initial Hamiltonian H, Eq.(3), which is not unitary (i.e. physically) equivalent to $\tilde{H}$, this corresponds to some $\nu_{ij} < 0$, satsfying Eq.(24), and the norm of the wave function (1) blows up at the critical point. For ν_{ij} negative but greater than the critical values (24), the wave function is singular at coincidence points but still quadratically integrable. Out of the critical point we have one-to-one correspondence between our multispecies system (6) and the system of N D-dimensional free oscillators, at least for the dilatation states $B_2^{+n_2}|\tilde{0}\rangle_R$.

4 Conclusion

In summary, we have defined a nontrivial many-body Hamiltonian H (Eq.(3)) of Calogero type in D dimensions with two- and three-body interactions among non-identical particles. Strength of the interactions, ν_{ij}, depends on the particle's species and this feature makes any analysis of such a model nontrivial, even in $D = 1$. Using underlying $SU(1,1)$ structure of the transformed Hamiltonian $\tilde{H}$ (Eq.(6)) and

Bargmann representation we outlined a procedure which gave in principle all eigenstates of the Hamiltonian. While we were unable to solve corresponding differential equations (11,12), we were able to find some general features of the solutions. There are towers of states with equidistant energy spectra. In each tower two neighbouring states differ in energy by 2ω. Moreover, we managed to solve $\tilde{H}$ partially, i.e. we explicitely found its global collective states, corresponding to the center-of-mass motion and the relative motion of particles. Those are all polynomial solution in the Bargmann representation in generic case. We also found their eigenenergies. The spectrum of collective modes, Eq.(20), is linear, equidistant and degenerate. It is also found that, for $\sum_{i\neq j}\nu_{ij} = -(N-1)D$, the Fock space, corresponding to the relative motion of particles, contained states of zero norm and the whole system exhibited singular behaviour. At this critical point the ground state wave function of the Hamiltonian H, Eq.(3), posseses infinite norm.

If we consider identical Bose (Fermi) particles, with $m_i = m$, and $\nu_{ij} = \nu$, the eigenstates are restricted to S_N-symmetric (antisymmetric) functions and the critical point is at $\nu = -\frac{D}{N}$. Our analysis of multispecies Calogero model gives deeper insight on the single-species Calogero models in higher dimensions.

All results presented here are common and universal for all systems with underlying conformal $SU(1,1)$ symmetry. The potentially most interesting applications of our results might be in two dimensions and quantum Hall efect.

Acknowledgment

This work was supported by the Ministry of Science and Technology of the Republic of Croatia under contracts No. 0098003 and No. 0119261.

References

[1] F. Calogero, J. Math. Phys.**10** (1969) 2191;
F. Calogero, J. Math. Phys.**10** (1969) 2197;
F. Calogero, J. Math. Phys.**12** (1971) 419.

[2] See e.g. in:Calogero-Moser-Sutherland Models, in: J.F.Van Dijen and L. Vinet (Eds.), CRM Series in Mathematical Physics, Springer, Berlin 2000.

[3] B.D.Simmons, P.A.Lee, and B.L.Altshuler, Phys. Rev. Lett. **72** (1994) 64;
L.Lapointe and L. Vinet, Comm.Math.Phys. **178** (1996) 425.

[4] N. Kawakami and S.K.Yang, Phys. Rev. Lett. **67** (1991) 2493;
N.F.Johnson and L.Quiroga, Phys. Rev. Lett. **74** (1995) 4277;
N.F.Johnson and L.Quiroga, J.Phys. Condens. Matter **9** (1997) 5889;
G.W. Gibbons and P.K. Townsend, Phys. Lett. B **454** (1999) 187;
D.Birmingham, K.S. Gupta, and S. Sen, Phys. Lett. B **505** (2001) 191.

[5] F.D.M. Haldane, Phys. Rev. Lett. **67** (1991) 937;
Y.S.Wu, Phys. Rev. Lett. **73** (1994) 922;
M.V.N.Murthy and R.Shankar, Phys. Rev. Lett. **73** (1994) 3331;
S.Mashkevich, Phys.Lett. A **233** (1997) 30.

[6] P. Forrester, J. Phys. A: Math. Gen. **25** (1992) L607;
C. Furtlehner, S. Ouvry, Mod.Phys.Lett. B**9** (1995) 503;
D. Sen, Nucl. Phys. B **479** (1996) 554;
A. Dasniers de Veigy, Nucl. Phys. B **483** (1997) 580.

[7] S. Meljanac, M. Mileković A.Samsarov, Phys. Lett. B**573** (2003) 202.

[8] G.Date, P.K.Ghosh, M.V.N.Murthy, Phys. Rev. Lett. **81** (1998) 3051;
M.V.N.Murthy, R.K.Bhaduri, D.Sen, Phys. Rev. Lett. **76** (1996) 4103;
A.Khare, K.Ray, Phys.Lett.A **230** (1997) 139;
R.K.Ghosh, S.Rao, Phys.Lett.A **238** (1998) 213;
P.K.Ghosh, Phys.Lett.A **229** (1997) 203;
P.K.Ghosh, J. Phys. A: Math. Gen. **34** (2001) 5583.

[9] N. Gurappa, P.K. Panigrahi, Phys.Rev. B **59** (1999) R2490;
N. Gurappa, P.K. Panigrahi, Phys.Rev. B **67** (2003) 155323.

[10] V. Bardek, L. Jonke, S. Meljanac, M. Mileković, Phys.Lett. B **531** (2002) 311.

[11] F.Calogero, C.Marchioro, J. Math. Phys. **14** (1973) 182.

[12] C.L.Kane, S.Kivelson, D.H.Lee, S.C.Zhang, Phys.Rev. B **43** (1991) 3255.

[13] M. Vasiliev, Int.J. Mod. Phys. A **6** (1991) 1115;
A.P. Polychronakos, Phys. Rev. Lett. **69** (1992) 703;
L. Brink, T.H. Hansson, M. Vasiliev, Phys.Lett. B **286** (1992) 109;
L. Brink, T.H. Hansson, S.E. Konstein, M. Vasiliev, Nucl. Phys. B **384** (1993) 591;
N. Gurappa, P.K. Panigrahi, V. Srinivasan, Mod.Phys.Lett. A**13** (1998) 339;
S. Meljanac, M. Mileković , M. Stojić, Eur. Phys. J. C **60** (2002) 331.

[14] A.M. Perelomov, Generalized Coherent States and Their Applications, Springer, Berlin, 1986.