GLASNIK MATEMATIČKI
Vol. 40(60)(2005), 1 - 11

BIPLANES (56, 11, 2) WITH A FIXED-POINT-FREE INVOLUTORY AUTOMORPHISM

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ABSTRACT. The aim of this article is to prove that exactly four biplanes with parameters (56, 11, 2) admit a fixed-point-free action of an involutory automorphism. These are: Hall's biplane B_{20}, Salwach and Mezzaroba's biplane B_{22}, Denniston's biplane B_{24} and Denniston's biplane B_{26}.

1. INTRODUCTION AND PRELIMINARIES

A biplane with parameters (56, 11, 2) is a symmetric design having 56 points and lines, each line consisting of 11 points, each point lying on 11 lines, each two lines intersecting in 2 points and each pair of points lying on 2 lines. Five different biplanes with parameters (56, 11, 2) were discovered until 1985. These biplanes are known in literature and denoted by B_{20}, B_{22}, B_{24}, B_{26} and the Janko-Tran van Trung biplane, which we shall denote by $J\text{-}T$ (see [8] and [5]). The existence of them all was proved using the computer and assuming additionally an action of an automorphism group $G \leq Aut\mathcal{D}$. Until now, it has been unknown whether there exists a biplane for (56, 11, 2) allowing an action of a nontrivial automorphism group and not isomorphic to one of the already listed ones. The only remaining possibility is the case of an involution acting fixed-point-freely on such a biplane. The reason why this case has not been solved before lies in the combinatorial expansion that arises during the construction, which is by a multiple larger than in any other case.

If $G \leq Aut\mathcal{D}$ is an automorphism group of the biplane $\mathcal{D}$ for (56, 11, 2) and if p is a prime divisor of $|G|$, then it holds:

$$p \in \{2, 3, 5, 7, 11\}.$$

2000 *Mathematics Subject Classification.* 05B05.
Key words and phrases. Biplane, design, automorphism group, orbit, orbit structure.

All the cases for $p > 2$ can easily be solved using the program by V. Ćepulić described in [2]. The following results came out, which V. Ćepulić hasn't published yet:

$p = 11$,	no biplane;
$p = 7$,	the biplane B_{20};
$p = 5$,	the biplane B_{20};
$p = 3$,	the biplanes B_{20}, B_{22}, B_{26}, J-T.

For $p = 2$, all possibilities for which an involution acts with $F > 0$ fixed points have already been examined. Using the known bounds for the number of fixed points and lines

$$1 + \frac{k-1}{\lambda} \leq F \leq k + \sqrt{k - \lambda}\,, \quad F \neq 0\,,$$

by putting $k = 11$, $\lambda = 2$ in it, one gets

$$F \in \{6, 8, 10, 12, 14\}\,.$$

The case $F = 6$ produces the biplanes B_{20}, B_{22}, B_{24} and J-T, what can be found in [7]. Other cases have been solved again by the program by V. Ćepulić with the following outcome. The case $F = 8$ gives all the 5 known biplanes, the cases $F = 10$ and $F = 12$ fail, while the case $F = 14$ gives biplanes B_{20}, B_{22} and B_{24}. The last case has been published in [3], and the other cases haven't been published yet.

From the short summary presented above, it is clear that the solution of the case $F = 0$, together with a publication of the cases solved by V. Ćepulić, completes the classification of all biplanes $\mathcal{D}(56, 11, 2)$ with nontrivial automorphism groups. For all the details considering the constructions and classifications mentioned above the reader is referred to the forthcoming paper [4].

Hence, it is our intention now to construct all designs $\mathcal{D}$ with the parameter triple $(56, 11, 2)$, which admit an involutory automorphism ρ acting without any fixed points and lines. We shall use the fundamental construction idea introduced by Janko and Tran van Trung in [6]. At first we shall build all possible orbit structures $\mathcal{S}$ of $\mathcal{D}$ admitting such a ρ. After that we shall build the biplanes themselves by "indexing" the "big points" of $\mathcal{S}$.

2. CONSTRUCTION OF ORBIT STRUCTURES FOR A FIXED-POINT-FREE INVOLUTION

First we introduce some notation. The automorphism group $\langle \rho \rangle$ has 28 orbits on the set of points of $\mathcal{D}$. We denote these orbits and their points by:

$$\mathcal{P}_1 \equiv \{1_0, 1_1\}, \ldots, \mathcal{P}_j \equiv \{j_0, j_1\}, \ldots, \mathcal{P}_{28} \equiv \{28_0, 28_1\},$$

and ρ maps j_s onto $j_{(s+1)(mod\,2)}$ for every $1 \leq j \leq 28$. The $\langle\rho\rangle$- orbits of lines of $\mathcal{D}$ we denote by:

$$\mathcal{B}_1 \equiv \{x_1, x_1\rho\}, \ldots, \mathcal{B}_i \equiv \{x_i, x_i\rho\}, \ldots, \mathcal{B}_{28} \equiv \{x_{28}, x_{28}\rho\}.$$

Let us consider the form of any line of $\mathcal{D}$ in terms of the number of occurrences of symbols for $\langle\rho\rangle$ - orbits. We refer to the well-known formulae for the multiplicities of orbit symbols when a group of prime order p acts fixed-point-freely. If μ_{ij} is the number of occurrences of an orbit symbol j on any line from $\mathcal{B}_i$, we have

$$\text{(1)} \qquad \sum_{j=1}^{28} \mu_{ij}(\mu_{ij} - 1) = \lambda(p-1) = 2 \cdot (2-1) = 2\,, \qquad 1 \leq i \leq 28\,.$$

Since $k = 11$, this means that on each line there is one orbit symbol which occurs twice, and nine orbit symbols which occur once.

If μ_{ij} and μ_{rj} are the numbers of occurrences of an orbit symbol j on two representatives of distinct $\langle\rho\rangle$ - orbits $\mathcal{B}_i$ and $\mathcal{B}_r$ then it holds

$$\text{(2)} \qquad \sum_{j=1}^{28} \mu_{ij}\mu_{rj} = \lambda p = 2 \cdot 2 = 4\,, \qquad 1 \leq i, r \leq 28,\ i \neq r\,.$$

We say briefly that the "game product" of two lines from distinct $\langle\rho\rangle$-orbits is equal to 4.

DEFINITION 1. *We call the* 28×28 *matrix* $\mathcal{S} = [\mu_{ij}]$, *satisfying conditions (1) and (2), the multiplicity matrix (or orbit structure) of* $\mathcal{D}$ *for* $\langle\rho\rangle$.

Dualizing our arguments we obtain:

LEMMA 2. *A multiplicity matrix* $\mathcal{S} = [\mu_{ij}]$ *of* $\mathcal{D}(56, 11, 2)$ *for a fixed-point-free involution* ρ *consists of* 28 *rows*

$$\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{28}$$

and 28 *columns* $1, 2, \ldots, 28$. *Every of them consists of one entry equal to* 2, *nine* $1's$ *and eighteen* $0's$, *satisfying:*

$$\text{(3)} \qquad \sum_{r=1}^{28} \mu_{ir}\mu_{jr} = \sum_{t=1}^{28} \mu_{ti}\mu_{tj} = 4\,, \qquad 1 \leq i, j \leq 28,\ i \neq j\,.$$

For two orbit structures $\mathcal{S}_1 = [\mu'_{ij}]$ and $\mathcal{S}_2 = [\mu''_{ij}]$, an isomorphism σ from $\mathcal{S}_1$ onto $\mathcal{S}_2$ is a bijection which maps rows of $\mathcal{S}_1$ onto rows of $\mathcal{S}_2$, and columns of $\mathcal{S}_1$ onto columns of $\mathcal{S}_2$, preserving the entries: $\mu''_{\sigma(i)\sigma(j)} = \mu'_{ij}$. Next we define a precedence relation for rows of orbit structures, and then for orbit structures themselves.

DEFINITION 3. *Suppose that there is given an order among the columns of* $\mathcal{S}$. *For two rows* $\hat{x} = [\mu_{ij}]_i$ *and* $\hat{y} = [\mu'_{ij}]_i$ *we define that* $\hat{x}$ *precedes* $\hat{y}$, $\hat{x} \preceq \hat{y}$, *if there is some* r, $1 \leq r < 28$, *such that* $\mu_{ij} = \mu'_{ij}$ *for* $j < r$ *and* $\mu_{ir} > \mu'_{ir}$. *As usual,* $\hat{x} \prec \hat{y}$ *will stand for* $\hat{x} \preceq \hat{y}$ *and* $\hat{x} \neq \hat{y}$.

DEFINITION 4. *Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be two orbit structures of* $\mathcal{D}$ *for* $\langle \rho \rangle$. *We define that* $\mathcal{S}_1$ *precedes* $\mathcal{S}_2$, $\mathcal{S}_1 \preceq \mathcal{S}_2$, *if* $\mathcal{S}_1$ *precedes* $\mathcal{S}_2$ *in terms of rows precedence.* $\mathcal{S}_1 \prec \mathcal{S}_2$ *will stand for* $\mathcal{S}_1 \preceq \mathcal{S}_2$ *and* $\mathcal{S}_1 \neq \mathcal{S}_2$.

Now we sketch our algorithm for constructing all orbit structures $\mathcal{S}$ of $\mathcal{D}$ for $\langle \rho \rangle$ (see also [1]). We produce the structures by building up the rectangular schemes level by level. The i-th layer of $\mathcal{S}$, denoted $\hat{x}^{(i)}$, consists of all possible rows with one entry equal to 2, nine entries equal to 1, and the remaining eighteen entries being 0. We can easily compute the number of possibilities for $\hat{x}^{(i)}$:

$$N_i = |\hat{x}^{(i)}| = \frac{28!}{18!9!} = 131\,231\,000\,, \qquad i = 1, 2, \ldots, 28\,.$$

A *partial orbit structure of l-th level,* denoted by $\mathcal{S}(l)$, is any matrix with l rows from $\hat{x}^{(i)}$, $i = 1, 2, \ldots, l$, satisfying the consistence condition (3) for rows, and not violating the consistence condition (3) for columns. Let $\mathcal{S}^{(l)}$ be the set of all possible partial structures $\mathcal{S}(l)$, $\mathcal{S}(1) = [2111111111000000000000000000]$ being obviously the only member of $\mathcal{S}^{(1)}$. We construct $\mathcal{S}^{(l)}$ from $\mathcal{S}^{(l-1)}$, $2 \leq l \leq 28$, in the following way. For each partial orbit structure $\mathcal{S}(l-1) \in \mathcal{S}^{(l-1)}$ we exhaust all 131 231 000 possibilities for the l-th level, by generating the corresponding rows $[\mu_{lr}]_l$ in the lexicographical order defined above. For a particular $[\mu_{lr}]_l$, after testing the condition (3), we include

$$\mathcal{S}(l) = \mathcal{S}(l-1) \cup [\mu_{lr}]_l$$

into $\mathcal{S}^{(l)}$, if it cannot be eliminated by finding some automorphism σ such that a scheme $\mathcal{S}(l)\sigma$ precedes $\mathcal{S}(l)$. If $\mathcal{S}(l)\sigma \prec \mathcal{S}(l)$ in terms of the precedence of partial schemes considered as parts of the whole orbit structures $\mathcal{S}$, $\mathcal{S}(l)$ is omitted. In this way, we ensure the elimination of a lot of isomorphic orbit structures, retaining only those among them which are the first in terms of the defined precedence. At the end of this procedure $\mathcal{S}^{(28)}$ will be the set of all possible orbit structures for our particular problem.

Applying the algorithm we have obtained as the only solutions (up to isomorphism) ten orbit structures: $\mathcal{S}_1 - \mathcal{S}_{10}$. This result has been achieved after nearly 4 000 hours of continuous computing on a computer "*DynatechDCS* – 1/320". The greatest number of schemes we have gotten on level 12, where we have counted approximately 80 000 000 (not necessarily non-isomorphic) schemes. Below we enclose all the 10 solutions.

STRUCTURE $\mathcal{S}_1$

```
2111111111000000000000000000
1200000000111111110000000000
1020000000111000001111100000
1002000000110100001000011110
1000200000100011000110011001
1000020000100110000001100111
1000002000001100100101010101
1000000200001100010010101011
1000000020000010111100101100
1000000002000001111011010010
0111100000000000200010100111
0111010000000000020101011001
0110001100000000200110000111
0101001100000002000101111000
0100110010000200001111000010
0100110001002000001000111100
0100001011200000000001101011
0100000111020000000110010101
0011000011001111001000000002
0010101010011010010000010020
0010100101100110010001000200
0010011001010110100010002000
0010010110100101100000120000
0001101001010101010100200000
0001100110011001100002001000
0001011010101001010020000100
0001010101101010100200000010
0000111100110000112000000001
```

STRUCTURE $\mathcal{S}_2$

```
2111111111000000000000000000
1200000000111111110000000000
1020000000111000001111100000
1002000000100110001100011100
1000200000100001100011011010
1000020000010101001010000111
1000002000010010010001110110
1000000200001010101000101011
1000000020001001010101001101
1000000002000100110110110001
0111100000100000010000100112
0110011000010000100100012001
0110000110001100000010020110
0101001100000011000121000001
0101000011010000101101000020
0100110010000011001100210000
0100101001001000012010001100
0100010101100100000002101100
0011010100000000112100101000
0011001001001102000000101010
0010101010000210101001000001
0010100101010011100100000200
0010010011100020010010001010
0001110001012010000001010001
0001100110020100010010101000
0001011010101000200010100100
0000111100101100010200000010
0000001111210001001000010001
```

STRUCTURE $\mathcal{S}_3$

```
2111111111000000000000000000
1200000000111111110000000000
1020000000111000001111100000
1002000000100110001100011100
1000200000100001100011011010
1000020000010101001010000111
1000002000010010010001110110
1000000200001010101000101011
1000000020001001010101001101
1000000002000100110110110001
0111100000100000010000100112
0110011000010000100100012001
0110000110001100000010020110
0101010001001100000002101010
0101001100000011000121000001
0100110010000011001100210000
0100100101010000101101000200
0100001011100000012010001010
0011010100000000112100101000
0011000011010011100100000020
0010110001001020010010001100
0010101010000210101001000001
0010001101100102000000101100
0001101001012001001000010001
0001100110020100010010101000
0001011010101000200010100100
0000111100101100010200000010
0000010111210010000001010001
```

STRUCTURE $\mathcal{S}_4$

```
2111111111000000000000000000
1200000000111111110000000000
1020000000111000001111100000
1002000000110100001000011110
1000200000001011000110011100
1000020000001000110001111010
1000001100100011000101000021
1000001100010000110010100201
1000000011000210000011110001
1000000011000001112100001001
0111001000000010100011002001
0111000100000001010100120001
0110110000001100001000000112
0101000020001000010111000110
0100100110010010001000201010
0100100101100000101002010100
0100011001100100000200101100
0100011001010001001020010010
0011000002001011100000100110
0010101010010100200100010010
0010100101100100020010001010
0010011010100020011000010100
0010010110010102000001001100
0001110010200001100010100001
0001110001020010010101000001
0001102000001101011001100000
0001010200001110101110000000
0000001111112000000000011001
```

STRUCTURE S_5

```
2111111111000000000000000000
1200000000111111110000000000
1020000000111000000111100000
1002000000100110001100011100
1000200000100001100011011010
1000020000010100010010110110
1000001100010001011000102001
1000001100002000100100010111
1000000011000100111002000101
1000000011000021000110100011
0111100000100000010000100112
0110011000000011001001020001
0110000200000110000011001110
0101010010001000101020001001
0101000002011000000001111010
0100101001010001001110000200
0100100110000100101100210000
0100011010100000010201001010
0011001010010101101000000020
0011000101000001120110010000
0010110001001201000100001001
0010100020011010010000011100
0010011001100010200000101100
0001110100020010100101000001
0001102000001110010011100000
0001010110101002000001100100
0000110101101010012000000010
0000001111210100000010010001
```

STRUCTURE S_6

```
2111111111000000000000000000
1200000000111111110000000000
1020000000111000000111100000
1002000000100110001100011100
1000110000110100000010010021
1000110000000001112001001001
1000001100010001100101000210
1000001100000020010011110001
1000000011002001000010011101
1000000011000100110100201010
0111100000000000200010110101
0110101000000001100011000201 0
0110000020000110001001000111
0101100001001000010102010010
0101000200011000001000101011
0100011010010001001100120000
0100011001200000000001101101
0100010101000100011120000100
0011020000001011010000100110
0011001001010101010100000002
0010100110110000020000011100
0010011100001200100001011000
0010000102100011101000010010
0001100110100102000011100000
0001010011020010100011001000
0001002010101000111010000010
0000201001011110001000100100
0000110110101010100200000001
```

STRUCTURE S_7

```
2111111111000000000000000000
1200000000111111110000000000
1020000000111000000111100000
1002000000000111001110011000
1000110000110000101000020110
1000110000000110010102000101
1000001100101001000001002110
1000001100000100200110100011
1000000011010000021010001011
1000000011001011000000210101
0110101000100010000010011002
0110100010000200001000101110
0110001001000001010200010110
0101010100020000000100101101
0101010001100010000011100020
0101001001001000102001000101
0100100200000001011011110000
0100010020001000100111011000
0011100010010002100001000011
0011010100002100010000010011
0011000110100010110010000200
0010021000000011111000101000
0010000102010110100001011000
0001200001101000110100101000
0001002010110100010001110000
0000111001011101000020000100
0000101110011020001100000010
0000010111200101001100000001
```

STRUCTURE S_8

```
2111111111000000000000000000
1200000000111111110000000000
1020000000111000000111100000
1001100000200100001000011110
1001010000000120000111010001
1000101000000002101110001001
1000010100010000021001001101
1000001100001010100100100210
1000000011010100100000211001
1000000011001001010011010020
0111001000000010010010102010
0110110000000001010100120100
0110001100100000100001010012
0101001010000101001002100100
0101000101010000101020010100
0100110001001010002000100011
0100100101010100000201001010
0100010020101000000110001101
0011100010020011000000000111
0011010010000100211100000010
0010101001001200010010000101
0010010002100011100001001100
0010000210001111001000011000
0002000101101001010100100001
0001111000012000100001011000
0000200110100010110011100000
0000021100110101000010100010
0000002011110010011100010000
```

STRUCTURE $\mathcal{S}_9$	**STRUCTURE** $\mathcal{S}_{10}$
2111111111000000000000000000	2111111111000000000000000000
1200000000111111110000000000	1200000000111111110000000000
1020000000111000001111100000	1011000000210000001111100000
1001100000200100001000011110	1011000000001110001000021100
1001010000000120000111010001	1000110000110001000000011021
1000101000000002101110001001	1000110000001000201110000101
1000010100010001010101000210	1000001100010001010101001200
1000001100001100100000210101	1000001100000210000110100011
1000000011010010100010101020	1000000011001010100002101010
1000000011001000021001011001	1000000011000001021010110001
0111001000000010010100102100	0120100000000010010011000111
0110110000000001010010120010	0110101000000001100100211000
0110010100100000100001001012	0101010010000010011200001010
0101001010000101001002100010	0101000200000001101011010010
0101000101001000101200010010	0101000110101000000000101102
0100110001010010002000100101	0100101001010100002001001001
0100100101001100000021001100	0100020010010100000011110100
0100001020110000000110010101	0100001002101000000110010110
0011100010002011000000000111	0011010001001102000101000001
0011010010000100211010000100	0010011100012000011000100010
0010101001010200010100000011	0010010101100100110010002000
0010001002100011100001010100	0010001020100101101000000110
0010000210010111001000011000	0010000111020010100100010001
0002000101110001010010100001	0002100001010100110000100110
0001111000021000100001011000	0001101010011011000020001000
0000200110100010110101100000	0001012000100010110001010001
0000020011101101000100101000	0000200110101100010101010000
0000012100101010011010000010	0000110101100021001000100100

3. FINAL RESULTS

Applying the first step of our algorithm we have found all possible solutions for $\mathcal{S}$. In the following, we shall write every orbit structure $\mathcal{S} = [\mu_{jr}]$ as a set of 28 orbit lines $\hat{x}_j = [\mu_{jr}]_j$, $j = 1, \ldots, 28$, represented as sequences of their $k = 11$ "big points". If $\mu_{jr_0} = 2$ and $\mu_{jr_i} = 1$ for $i = 1, 2, \ldots, 9$, then we write

$$\hat{x}_j \equiv r_0 r_0 r_1 r_2 r_3 r_4 r_5 r_6 r_7 r_8 r_9 \,,$$

the numbers $r_0, r_1, \ldots, r_9 \in \{1, 2, \ldots, 28\}$ being the "big points" of $\mathcal{S}$. Now we try to construct the biplanes by "indexing" $\mathcal{S}$, e.g. supporting its big points with appropriate indices from $\{0, 1\}$. We give a brief description of our algorithm.

Let $\mathcal{S} = [\mu_{jr}]$ be the orbit structure under consideration. For the j-th row of $\mathcal{S}$ we construct lines x_j from the line orbit $\mathcal{B}_j$, by supplying the orbit numbers of $\hat{x}_j$ with indices from $\{0, 1\}$. For x'_j, x''_j corresponding to the same $\hat{x}_j$ we define: x'_j *precedes* x''_j, $x'_j \prec x''_j$, if the sequence of indices of big points corresponding to x'_j precedes that of x''_j lexicographically. Among two lines of the orbit $\mathcal{B}_j$ we take out as its representative the first in terms of the defined precedence, thus obtaining $\tilde{x}_j$ - the canonical form of x_j. In the following we identify $\tilde{x}_j$ with x_j and call it the *canonical line.* The set of all j-th level canonical lines we denote $x^{(j)}$.

After finding $x^{(j)}$, we build the partial designs. A *partial design of j-th level*, denoted by Δ_j, consists of j canonical lines which satisfy the design conditions. By $\mathcal{D}^{(j)}$ we denote the set of j-th level partial designs Δ_j which we construct in our procedure, $\Delta_1 \equiv 1_0 1_1 2_0 3_0 4_0 5_0 6_0 7_0 8_0 9_0 10_0$ being obviously the only member of $\mathcal{D}^{(1)}$. For two partial designs Δ_j' and Δ_j'' we say that Δ_j' *precedes* Δ_j'', $\Delta_j' \prec \Delta_j''$, if there exists some q, $q \leq j$, such that: *(i)* corresponding i-th level canonical lines of Δ_j' and Δ_j'' coincide for $1 \leq i < q$, and *(ii)* q-th level canonical line of Δ_j' precedes that of Δ_j''.

We construct $\mathcal{D}^{(j)}$ from $\mathcal{D}^{(j-1)}$, $2 \leq j \leq 28$, in the following way. To each partial design $\Delta_{j-1} \in \mathcal{D}^{(j-1)}$ we join all possible j-th level canonical lines x_j which intersect each line of Δ_{j-1} in exactly two points. In such a way we obtain one by one potential partial designs $\Delta_j = \Delta_{j-1} \cup x_j$ of the j-th level. Now, we include Δ_j into $\mathcal{D}^{(j)}$ if it cannot be eliminated by finding some automorphism σ such that $\Delta_j \sigma \prec \Delta_j$ in terms of the precedence of partial designs. At the end of this procedure, $\mathcal{D}^{(28)}$ will be the set of all possible biplanes with the orbit structure $\mathcal{S}$, admitting the action of the given involution ρ.

The described procedure has been carried out by computer as well, the computing time being only 20 minutes. It has turned out that among 10 orbit structures $\mathcal{S}_1$-$\mathcal{S}_{10}$ only five of them can be supplied by indices. Namely, we have obtained the following biplanes:

($\mathcal{S}_1$) none
($\mathcal{S}_2$) B_{22} (2 copies), B_{24} (2 copies)
($\mathcal{S}_3$) B_{20} (2 copies), B_{22} (2 copies)
($\mathcal{S}_4$) none
($\mathcal{S}_5$) B_{24} (2 copies)
($\mathcal{S}_6$) none
($\mathcal{S}_7$) B_{22} (1 copy), B_{26} (3 copies)
($\mathcal{S}_8$) none
($\mathcal{S}_9$) none
($\mathcal{S}_{10}$) B_{24} (16 copies)

The first appearances of the obtained biplanes are enclosed below. We shall write down only the 28 line orbit representatives. The remaining 28 lines one can get by changing all the indices of these representatives modulo 2. We were able to identify the resulting biplanes as B_{20}, B_{22}, B_{24} or B_{26} using their chain representations (see [8] for details).

BIPLANE B_{20}

1_0	1_1	2_0	3_0	4_0	5_0	6_0	7_0	8_0	9_0	10_0
1_0	2_0	2_1	11_0	12_0	13_0	14_0	15_0	16_0	17_0	18_0
1_0	3_0	3_1	11_1	12_0	13_1	19_0	20_0	21_0	22_0	23_0
1_0	4_0	4_1	11_1	14_1	15_0	19_1	20_1	24_0	25_0	26_0
1_0	5_0	5_1	11_0	16_1	17_1	21_1	22_0	24_0	25_1	27_0
1_0	6_0	6_1	12_1	14_0	16_1	19_1	21_0	26_1	27_1	28_0
1_0	7_0	7_1	12_1	15_1	18_0	22_1	23_0	24_1	26_0	27_0
1_0	8_0	8_1	13_0	15_1	17_1	19_0	23_1	25_0	27_1	28_1
1_0	9_0	9_1	13_1	16_0	18_1	20_1	22_1	25_1	26_1	28_1
1_0	10_0	10_1	14_1	17_0	18_1	20_0	21_1	23_1	24_1	28_0
2_0	3_1	4_1	5_0	11_1	18_0	23_1	26_1	27_0	28_0	28_1
2_0	3_0	6_1	7_1	12_1	17_0	20_0	24_0	25_0	25_1	28_1
2_0	3_1	8_0	9_1	13_1	14_0	21_1	24_0	24_1	26_0	27_1
2_0	4_1	6_0	10_1	13_0	14_1	22_0	22_1	23_0	25_1	27_1
2_0	4_0	7_1	8_1	15_1	16_0	20_1	21_0	21_1	22_0	28_0
2_0	5_1	6_1	9_0	15_0	16_1	19_0	20_1	23_0	23_1	24_1
2_0	5_1	8_1	10_0	12_0	17_1	19_1	20_0	22_1	26_0	26_1
2_0	7_0	9_1	10_1	11_0	18_1	19_0	19_1	21_0	25_0	27_0
3_0	4_1	6_1	8_0	16_0	17_1	18_0	18_1	19_1	22_0	24_1
3_0	4_1	9_1	10_0	12_0	15_1	16_1	17_0	20_1	27_0	27_1
3_0	5_1	6_0	10_1	13_1	15_0	15_1	18_0	21_1	25_0	26_1
3_0	5_0	7_1	9_1	14_0	14_1	15_0	17_1	19_0	22_1	28_0
3_0	7_0	8_1	10_1	11_1	14_0	16_0	16_1	23_1	25_1	26_0
4_0	5_0	7_1	10_1	12_0	13_0	13_1	16_1	19_1	24_1	28_1
4_0	5_1	8_0	9_1	12_0	12_1	14_1	18_0	21_0	23_1	25_1
4_0	6_1	7_0	9_1	11_1	13_0	17_0	17_1	21_1	23_0	26_1
5_0	6_1	7_0	8_1	11_0	13_1	14_1	18_0	20_0	20_1	27_1
6_0	8_1	9_1	10_0	11_0	11_1	12_1	15_0	22_0	24_1	28_1

BIPLANE B_{22}

1_0	1_1	2_0	3_0	4_0	5_0	6_0	7_0	8_0	9_0	10_0
1_0	2_0	2_1	11_0	12_0	13_0	14_0	15_0	16_0	17_0	18_0
1_0	3_0	3_1	11_1	12_0	13_1	19_0	20_0	21_0	22_0	23_0
1_0	4_0	4_1	11_1	14_1	15_0	19_1	20_1	24_0	25_0	26_0
1_0	5_0	5_1	11_0	16_1	17_1	21_1	22_0	24_0	25_1	27_0
1_0	6_0	6_1	12_1	14_0	16_1	19_1	21_0	26_1	27_1	28_0
1_0	7_0	7_1	12_1	15_1	18_0	22_1	23_0	24_1	26_0	27_0
1_0	8_0	8_1	13_0	15_1	17_1	19_0	23_1	25_0	27_1	28_1
1_0	9_0	9_1	13_1	16_0	18_1	20_1	22_1	25_1	26_1	28_1
1_0	10_0	10_1	14_1	17_0	18_1	20_0	21_1	23_1	24_1	28_0
2_0	3_1	4_0	5_1	11_1	18_0	23_1	26_1	27_0	28_0	28_1
2_0	3_0	6_1	7_1	12_1	17_0	20_0	24_0	25_0	25_1	28_1
2_0	3_1	8_1	9_0	13_1	14_0	21_1	24_0	24_1	26_0	27_1
2_0	4_1	7_1	8_0	15_1	16_0	20_1	21_0	21_1	22_0	28_0
2_0	4_1	9_1	10_0	12_0	17_1	19_1	20_0	22_1	27_0	27_1
2_0	5_0	6_1	9_1	15_0	16_1	19_0	20_1	23_0	23_1	24_1
2_0	5_1	7_0	10_1	13_0	18_1	19_0	19_1	21_0	25_1	26_0
2_0	6_0	8_1	10_1	11_0	14_1	22_0	22_1	23_0	25_0	26_1
3_0	4_0	6_1	8_1	16_0	17_1	18_0	18_1	19_1	22_0	24_1
3_0	4_1	7_0	10_1	13_1	14_0	16_0	16_1	23_1	25_0	27_0
3_0	5_1	7_1	9_0	14_0	14_1	15_0	17_1	19_0	22_1	28_0
3_0	5_1	8_1	10_0	12_0	15_1	16_1	17_0	20_1	26_0	26_1
3_0	6_0	9_1	10_1	11_1	15_0	15_1	18_0	21_1	25_1	27_1
4_0	5_0	6_1	10_1	12_0	13_0	13_1	15_1	22_1	24_0	28_0
4_0	5_1	8_0	9_1	12_0	12_1	14_0	18_1	21_1	23_0	25_0
4_0	6_0	7_1	9_1	11_0	13_1	17_0	17_1	21_0	23_1	26_0
5_0	6_0	7_1	8_1	11_1	13_0	14_0	18_1	20_0	20_1	27_0
7_0	8_1	9_1	10_0	11_0	11_1	12_1	16_0	19_0	24_0	28_0

BIPLANE B_{24}

1_0	1_1	2_0	3_0	4_0	5_0	6_0	7_0	8_0	9_0	10_0
1_0	2_0	2_1	11_0	12_0	13_0	14_0	15_0	16_0	17_0	18_0
1_0	3_0	3_1	11_1	12_0	13_1	19_0	20_0	21_0	22_0	23_0
1_0	4_0	4_1	11_1	14_1	15_0	19_1	20_1	24_0	25_0	26_0
1_0	5_0	5_1	11_0	16_1	17_1	21_1	22_0	24_0	25_1	27_0
1_0	6_0	6_1	12_1	14_0	16_1	19_1	21_0	26_1	27_1	28_0
1_0	7_0	7_1	12_1	15_1	18_0	22_1	23_0	24_1	26_0	27_0
1_0	8_0	8_1	13_0	15_1	17_1	19_0	23_1	25_0	27_1	28_1
1_0	9_0	9_1	13_1	16_0	18_1	20_1	22_1	25_1	26_1	28_1
1_0	10_0	10_1	14_1	17_0	18_1	20_0	21_1	23_1	24_1	28_0
2_0	3_1	4_0	5_1	11_1	18_0	23_1	26_1	27_0	28_0	28_1
2_0	3_0	6_1	7_1	12_0	17_1	20_1	24_1	25_0	25_1	28_0
2_0	3_1	8_1	9_0	13_1	14_0	21_1	24_0	24_1	26_0	27_1
2_0	4_1	7_1	8_0	15_0	16_1	20_0	21_0	21_1	22_1	28_1
2_0	4_1	9_1	10_0	12_1	17_0	19_0	20_1	22_0	27_0	27_1
2_0	5_0	6_1	9_1	15_1	16_0	19_1	20_0	23_0	23_1	24_0
2_0	5_1	7_0	10_1	13_0	18_1	19_0	19_1	21_0	25_1	26_0
2_0	6_0	8_1	10_1	11_0	14_1	22_0	22_1	23_0	25_0	26_1
3_0	4_0	6_1	8_1	16_1	17_0	18_0	18_1	19_0	22_1	24_0
3_0	4_1	7_0	10_1	13_1	14_0	16_0	16_1	23_1	25_0	27_0
3_0	5_1	7_1	9_0	14_0	14_1	15_1	17_0	19_1	22_0	28_1
3_0	5_1	8_1	10_0	12_1	15_0	16_0	17_1	20_0	26_0	26_1
3_0	6_0	9_1	10_1	11_1	15_0	15_1	18_0	21_1	25_1	27_1
4_0	5_0	6_1	10_1	12_1	13_0	13_1	15_0	22_0	24_1	28_1
4_0	5_1	8_0	9_1	12_0	12_1	14_0	18_1	21_1	23_0	25_0
4_0	6_0	7_1	9_1	11_0	13_1	17_0	17_1	21_0	23_1	26_0
5_0	6_0	7_1	8_1	11_1	13_0	14_0	18_1	20_0	20_1	27_0
7_0	8_1	9_1	10_0	11_0	11_1	12_0	16_1	19_1	24_1	28_1

BIPLANE B_{26}

1_0	1_1	2_0	3_0	4_0	5_0	6_0	7_0	8_0	9_0	10_0
1_0	2_0	2_1	11_0	12_0	13_0	14_0	15_0	16_0	17_0	18_0
1_0	3_0	3_1	11_0	12_1	13_1	19_0	20_0	21_1	22_0	23_0
1_0	4_0	4_1	14_1	15_0	16_1	19_1	20_1	21_0	24_0	25_0
1_0	5_1	6_0	11_1	12_0	17_1	19_0	24_0	24_1	26_1	27_0
1_0	5_0	6_1	14_0	15_1	18_1	20_1	22_0	22_1	26_1	28_0
1_0	7_1	8_0	11_1	13_1	16_0	22_1	25_0	25_1	26_0	27_1
1_0	7_0	8_1	14_1	17_0	17_1	20_0	21_1	23_1	27_1	28_0
1_0	9_1	10_0	12_1	18_0	18_1	19_1	21_1	25_1	27_0	28_1
1_0	9_0	10_1	13_0	15_1	16_1	23_0	23_1	24_1	26_0	28_1
2_0	3_0	5_1	7_1	11_0	15_1	21_1	24_0	25_0	28_0	28_1
2_0	3_1	5_0	9_1	14_0	14_1	19_0	23_1	25_0	26_0	27_0
2_0	3_1	7_1	10_0	16_1	18_0	20_0	20_1	24_1	26_1	27_1
2_0	4_1	6_0	8_1	12_0	12_1	20_1	23_0	25_1	26_0	28_0
2_0	4_0	6_1	10_1	11_1	15_0	21_1	22_0	23_0	27_0	27_1
2_0	4_1	7_0	10_1	13_1	17_0	19_0	19_1	22_1	26_1	28_1
2_0	5_1	8_0	8_1	16_0	18_1	19_1	21_0	22_0	23_1	24_1
2_0	6_1	9_0	9_1	13_0	17_1	20_0	21_0	22_1	24_0	25_1
3_0	4_1	5_0	9_1	12_0	16_0	16_1	17_1	22_0	27_1	28_1
3_0	4_1	6_1	8_0	13_0	13_1	14_1	18_0	24_1	27_0	28_0
3_0	4_0	8_1	9_1	11_1	15_1	17_0	18_0	21_0	26_0	26_1
3_0	6_0	6_1	7_1	15_0	16_1	17_0	18_1	19_0	23_1	25_1
3_0	8_1	10_0	10_1	12_1	14_0	15_0	17_1	22_1	24_1	25_0
4_0	5_0	5_1	10_1	11_0	13_1	17_1	18_0	20_1	23_1	25_1
4_0	7_0	7_1	9_1	11_0	12_0	14_1	18_1	22_1	23_0	24_1
5_0	6_0	7_1	10_1	12_1	13_0	14_1	16_0	21_0	21_1	26_1
5_0	7_1	8_1	9_0	12_0	13_1	15_0	15_1	19_1	20_0	27_0
6_0	8_0	9_1	10_1	11_0	11_1	14_0	16_1	19_1	20_0	28_0

Hence the following theorem has been proved.

THEOREM 1. *Let* $\mathcal{D}$ *be a biplane with parameters* $(56, 11, 2)$ *admitting an involutory automorphism acting fixed-point-freely. Then* $\mathcal{D}$ *is isomorphic to one of the following four known biplanes: Hall's biplane* B_{20}*, Salwach and Mezzaroba's biplane* B_{22}*, Denniston's biplane* B_{24}*, or Denniston's biplane* B_{26}.

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Received: 9.1.2004.
Revised: 14.4.2004.