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Dynamical stability of the response of oscillators with discontinuous or steep first derivative of restoring characteristic

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Abstract

The influence of factors which can lead to incorrect prediction of dynamical stability of the periodic response of oscillators which contain a non-linear restoring characteristic with discontinuous or steep first derivative is considered in this paper. For that purpose, a simple one degree-of-freedom system with a piecewise-linear force-displacement relationship subjected to a harmonic excitation is analysed. Stability of the periodic response obtained in the frequency domain by the incremental harmonic balance method is determined by using the Floquet–Liapounov theorem. Responses in the time domain are obtained by digital simulation. The accuracy of determining the eigenvalues of the monodromy matrix (in the considered example) significantly depend on the corrective vector norm $\|\{r\}\|$, the accuracy ε of numerical determination of the times when the system undergoes a stiffness change, and on the number of step functions M (used in the Hsu's procedure), only for $\|\{r\}\| > 1 \times 10^{-5}$, $\varepsilon > 1 \times 10^{-5}$ and $M < 2000$. Otherwise, except if the maximum modulus of the eigenvalues of the monodromy matrix is very close to unity, their influence on estimation of dynamical stability is minor. On the contrary, neglecting very small harmonic terms of the actual time domain response can cause a very large error in the evaluation of the eigenvalues of the monodromy matrix, and so they can lead to incorrect prediction of the dynamical stability of the solution, regardless of whether the maximum modulus of the eigenvalues of the monodromy matrix is close to unity or not.

Keywords: Dynamical stability; Floquet–Liapounov theorem; Non-linear oscillator

1. Introduction

Among the great number of various types of non-linear dynamic systems a very specific group constitutes non-linear systems described by differential equations which contain nonlinearities with discontinuous or steep first derivative (for example, systems with clearance). Responses (both periodic and aperiodic) of these systems can be relatively easily determined in the time domain by using digital simulation. But procedures of that kind can be exceptionally time consuming, particularly inside the frequency ranges of co-existence of multiple stable solutions (where many combinations of initial conditions have to be examined for obtaining all possible steady-state solutions), for lightly damped systems (since a great number of excitation periods must be simulated to obtain a steady-state response), and when the state of the system is near to bifurcation. Moreover, these methods are not suitable for obtaining unstable solutions and for bifurcation analysis.

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On the other side, very efficient methods for solving this type of non-linear differential equations in the frequency domain are multi-harmonic balance methods (Lau et al., 1983; Pierre et al., 1985; Choi and Noah, 1988; Wong et al., 1991; Lau and Yuen, 1993; Leung and Chui, 1995; Kahraman and Blankenship, 1996; Lerusse et al., 1996). These methods become exceptionally efficient in combination with path following techniques (Narayanan and Sekar, 1998; Cardona et al., 1998; Raghothama and Narayanan, 1999) and can be successfully applied to a wide range of non-linear problems. The latter are very well suited for parametric studies because a new solution can be sought by these methods, with the previous solution used as a very good approximation. Since these methods enable obtaining both dynamically stable and unstable solutions, as well as bifurcation analysis, the determination of the dynamical stability of these solutions should be reliable and numerically efficient. As the estimation of dynamical stability by Floquet–Liapounov theorem could be a sensitive procedure (Szemplinska-Stupnicka, 1990; Awrejcewicz et al., 1998; Wolf and Stegić, 1999; Awrejcewicz and Lamarque, 2003; Wolf et al., 2004), the influence of the factors which can lead to incorrect prediction of the dynamical stability of the response is considered in this paper. For that purpose, a simple one degree-of-freedom system with piecewise-linear force-displacement relationship subjected to a harmonic excitation is analysed. The considerable advantage of using this piecewise-linear model is the possibility to determine elements of Jacobian matrix $[k]$ and the corrector $\{r\}$ in explicit form, as well as to express the monodromy matrix exactly as a product of matrix exponentials, what is not possible for a model with a general non-linear function. The stability of the periodic solutions obtained in the frequency domain by the incremental harmonic balance method (IHBM) is estimated by the Floquet–Liapounov theorem (Minorsky, 1962). Responses in the time domain are obtained by digital simulation.

2. Model of a mechanical system with a clearance

A model of a simple mechanical system with clearance is shown in Fig. 1. It consists of an inertia element m, a linear viscous damper c, and a non-linear elastic element $kg(x)$. The non-linear elastic element is defined by a piecewise-linear function $g(x)$ and a coefficient k. When the system is excited by a periodic harmonic force $F(t)$, the motion of the system can be described by the non-linear differential equation:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + c\frac{\mathrm{d}x}{\mathrm{d}t} + kg(x) = F(t) = f_0 + f_C\cos(\Omega t) + f_S\sin(\Omega t), \tag{1}$$

where f_0 represents mean transmitted force, f_C and f_S are force component amplitudes of the corresponding harmonic terms, and Ω is the excitation frequency.

The piecewise linear function $g(x)$ and its derivative are shown in Fig. 2(a) and Fig. 2(b) respectively. b denotes one-half of the clearance space. Since the procedure of prediction of the dynamical stability is based on the derivative of a non-linear function, the expressions for the non-linear function and its derivative are given:

$$g(x) = h^*(x - b^*), \tag{2}$$

$$\frac{\partial g(x)}{\partial x} = h^*, \tag{3}$$

where:

$$h^* = \begin{cases} 1, & b < x, \\ 0, & -b \leqslant x \leqslant b, \\ 1, & x < -b, \end{cases} \qquad b^* = \begin{cases} b, & b < x, \\ 0, & -b \leqslant x \leqslant b, \\ -b, & x < -b. \end{cases} \tag{4}$$

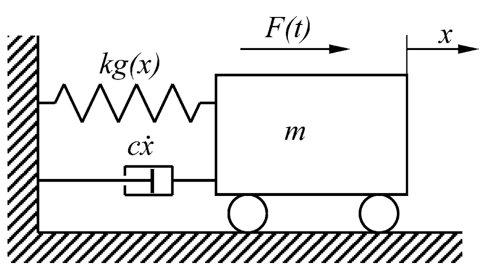


Fig. 1. Model of the dynamic system.

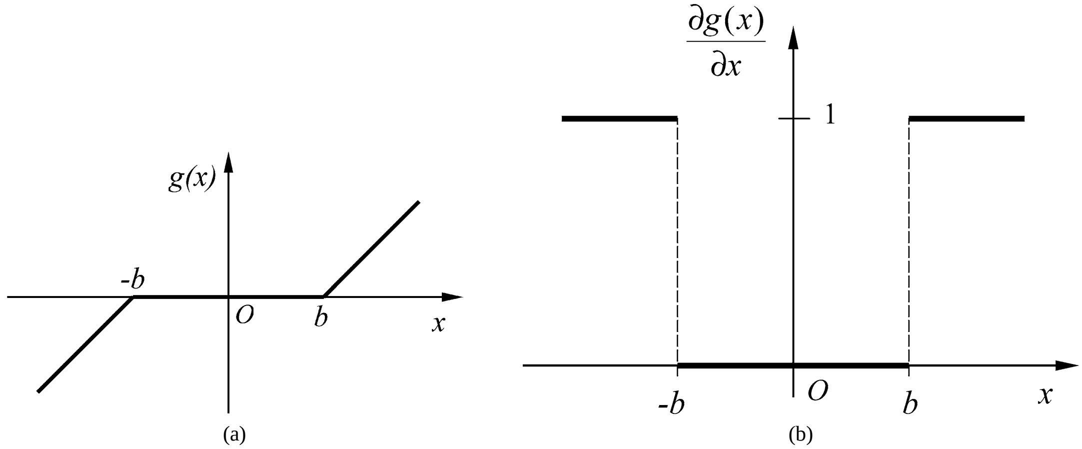


Fig. 2. Non-linear function $g(x)$ (a) and its derivative (b).

3. Short description of the incremental harmonic balance method (IHBM)

By introducing a non-dimensional time θ as a new independent variable, the differential equation (1) can be rewritten in the non-dimensional form:

$$\frac{\eta^2}{\nu^2}\frac{\mathrm{d}^2\bar{x}}{\mathrm{d}\theta^2} + \frac{2\zeta\eta}{\nu}\frac{\mathrm{d}\bar{x}}{\mathrm{d}\theta} + g(\bar{x}) = \bar{f}_0 + \bar{f}_C\cos(\nu\theta) + \bar{f}_S\sin(\nu\theta), \tag{5}$$

where:

$$\bar{x} = \frac{x}{l}, \quad \bar{b} = \frac{b}{l}, \quad \omega_0 = \sqrt{\frac{c}{m}}, \quad \zeta = \frac{k}{2m\omega_0}, \quad \bar{f}_0 = \frac{f_0}{ml\omega_0^2}, \quad \bar{f}_C = \frac{f_C}{ml\omega_0^2},$$
$$\bar{f}_S = \frac{f_S}{ml\omega_0^2}, \quad \eta = \frac{\Omega}{\omega_0}, \quad \tau = \omega_0 \cdot t, \quad \theta = \frac{\Omega t}{\nu} = \frac{\eta\tau}{\nu}.$$

In this way, the period of the response is always 2π, regardless of the number of subharmonics ν included in the supposed approximate solution. The non-dimensional time θ differs from the usually used non-dimensional time $\tau = \omega_0 \cdot t$, scaled in the way that a period of the free oscillation (of corresponding linear system) is 2π. Any characteristic dimension of the system is denoted by l here.

The supposed approximate solution is given by:

$$\bar{x} = \sum_{i=0}^{N} a_i \cos i\theta + b_i \sin i\theta = [T]\{a\}, \tag{6}$$

where:

$$[T] = [1, \cos\theta, \cos 2\theta, \ldots, \cos N\theta, \sin\theta, \sin 2\theta, \ldots, \sin N\theta],$$
$$\{a\} = [a_0, a_1, \ldots, a_N, b_1, b_2, \ldots, b_N]^{\mathrm{T}}.$$

$N = \nu K$ represents the number of all harmonics included in the supposed solution, ν is the number of subharmonics and K is the number of superharmonics. By applying this method, which consists of two basic steps: incrementation and Galerkin's procedure, the non-linear differential equation (5) is transformed into the system of $2N + 1$ linearized incremental algebraic equations:

$$[k]^j\{\Delta a\}^{j+1} = \{r\}^j, \tag{7}$$
$$\{a\}^{j+1} = \{a^j\} + \{\Delta a\}^{j+1}, \tag{8}$$

with Fourier coefficients ($a_0, a_i, b_i,\ i = 1, \ldots, N$) as unknowns. In Eqs. (7) and (8) j is the number of iterations. In each incremental step, only linear (i.e. linearized) algebraic equations have to be formed and solved. A solution is obtained from the iteration process when the corrective vector norm $\|\{r\}\|$ is smaller than a certain (arbitrary) convergence criterion. The

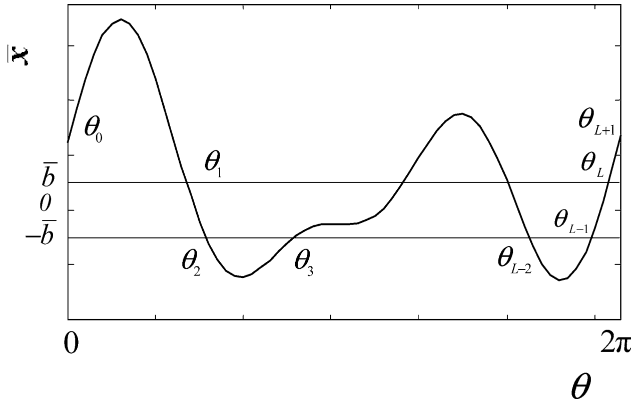


Fig. 3. Solutions to the equations $\bar{x} = \bar{b}$ and $\bar{x} = -\bar{b}$.

comprehensive description of the method, its application to piecewise-linear systems and the way of determining elements of Jacobian matrix $[k]$ and the corrector $\{r\}$ in explicit form is given by Wong et al. (1991). In the case of continuous functions (for example, a hyperbolic tangent or a sigmoid function (Lok and Wiercigroch, 1996)) some elements of $[k]$ and $\{r\}$ can be determined only by numerical integration.

Generally, the accuracy of the approximate solution obtained by using IHBM depends on the number of harmonics included in the solution, the accuracy of procedures used for determining elements of $[k]$ and $\{r\}$, and the value of the convergence criterion. The procedures of determining $[k]$ and $\{r\}$ depend on the system considered, as well as on the multi-harmonic balance method used. Since the IHBM described by Wong et al. (1991) is used in this work, the accuracy of the procedure of determining elements of $[k]$ and $\{r\}$ depends only on the precision of the numerical determination of times θ_i in which the system undergoes a stiffness change (Fig. 3).

4. The stability of the steady state solution

When the periodic solution is obtained, the stability of the given solution can be determined by examining the perturbed solution $\bar{x}^*$:

$$\bar{x}^* = \bar{x} + \Delta\bar{x}^*, \tag{9}$$

where $\Delta\bar{x}^*$ is a small perturbation of a periodic solution $\bar{x}$. By substitution of Eqs. (9) in (5), and after expanding the non-linear function $g(\bar{x})$ in Taylor's series about the periodic solution while neglecting non-linear incremental terms, one obtains a linear homogeneous differential equation with time changing periodic coefficients $\partial g(\bar{x})/\partial\bar{x}$:

$$\frac{\eta^2}{\nu^2}\frac{\mathrm{d}^2\Delta\bar{x}}{\mathrm{d}\theta^2} + \frac{2\zeta\eta}{\nu}\frac{\mathrm{d}\Delta\bar{x}}{\mathrm{d}\theta} + \frac{\partial g(\bar{x})}{\partial\bar{x}}\Delta\bar{x}^* = 0. \tag{10}$$

When the steady state solution $\bar{x}(\theta)$ is determined, the values of $\partial g(\bar{x})/\partial\bar{x}$ are known inside the period of the response. A very efficient and very often used method for determining the stability of the periodic solution is based on the Floquet–Liapounov theorem (Minorsky, 1962; Nayfeh and Balachandram, 1995). For that purpose Eq. (10) can be rewritten in the state variable form as:

$$\left\{\frac{\mathrm{d}\overline{X}^*}{\mathrm{d}\theta}\right\} = \big[A(\theta)\big]\{\overline{X}^*\}, \tag{11}$$

where

$$\{\overline{X}^*\} = \begin{Bmatrix}\Delta\bar{x}^* \\ \mathrm{d}\Delta\bar{x}^*/\mathrm{d}\theta\end{Bmatrix}, \quad \left\{\frac{\mathrm{d}\overline{X}^*}{\mathrm{d}\theta}\right\} = \begin{Bmatrix}\mathrm{d}\Delta\bar{x}^*/\mathrm{d}\theta \\ \mathrm{d}^2\Delta\bar{x}^*/\mathrm{d}\theta^2\end{Bmatrix}, \quad \big[A(\theta)\big] = \begin{bmatrix} 0 & 1 \\ -\frac{\nu^2}{\eta^2}\left(\frac{\partial g(\bar{x})}{\partial\bar{x}}\right) & -\frac{2\nu\zeta}{\eta}\end{bmatrix}. \tag{12}$$

Since the matrix $[A(\theta)]$ is a periodic function of θ with period 2π, the stability criteria are related to the eigenvalues of the monodromy matrix, which is defined as the state transition matrix at the end of one period. According to Floquet–Liapounov theorem, the solution is stable if all the moduli of the eigenvalues of the monodromy matrix are less than unity. Otherwise the solution is unstable. Bifurcation occurs when one of the eigenvalues of the monodromy matrix crosses the unit circle in the complex plane (i.e. when one of the moduli of the monodromy matrix reaches unity). Generally, it is not possible to derive an analytic expression for the transition matrix. But, if the non-linear force-displacement relationship is piecewise-linear, its derivative $(\partial g(x)/\partial x = h^*)$ is, according to (4), constant inside each of the intervals $[\theta_i, \theta_{i+1}]$ (Fig. 3). Fig. 3 shows a period of the response where $\theta_0 = 0$ and $\theta_{L+1} = 2\pi$. There are L times denoted as $\theta_1, \theta_2, \ldots, \theta_L$, in which the system undergoes a stiffness change. Consequently, $[A(\theta_i, \theta_{i+1})]$ is also a constant matrix inside that interval. According to D'Souza and Garg (1984), for the constant $[A(\theta_i, \theta_{i+1})]$ (inside the interval $[\theta_i, \theta_{i+1}]$), transition matrix $[\Phi(\theta_{i+1}, \theta_i)]$ can be expressed as:

$$\left[\Phi(\theta_{i+1}, \theta_i)\right] = \mathrm{e}^{[A(\theta_i, \theta_{i+1})](\theta_{i+1} - \theta_i)}; \tag{13}$$

and for the whole interval $[0, 2\pi]$ according to Wong et al. (1991) one obtains:

$$\left[\Phi(2\pi, 0)\right] = \prod_{i=0}^{L} \mathrm{e}^{[A(\theta_i, \theta_{i+1})](\theta_{i+1} - \theta_i)}. \tag{14}$$

Beside the precision of numerical determination of times θ_i in which the system changes stage stiffness region ($\bar{x} = \bar{b}, \bar{x} = -\bar{b}$), the only approximation occurring in this procedure is the accuracy of computation of the matrix exponential $\mathrm{e}^{[A(\theta)](\theta_{i+1} - \theta_i)}$ and the product of matrix exponentials $\prod_{i=0}^{L} \mathrm{e}^{[A(\theta_i, \theta_{i+1})](\theta_{i+1} - \theta_i)}$. To evaluate the matrix exponential and the product of matrix exponentials as accurately as possible, the algorithms recommended by Cardona et al. (1998) are used. The influence of using other procedures is not considered in this paper. If a non-linear force-displacement relationship $g(x)$ is a continuous non-linear function, its derivative is a time changing function, and, consequently, $[A(\theta)]$ is then a time-changing matrix. So, the monodromy matrix cannot be obtained in the previously described way, i.e. by using (13) and (14). Among the various methods of approximating the monodromy matrix, Friedman et al. (1977) concluded that the most efficient procedure is the one proposed by Hsu and Cheng (1974), i.e. by approximating the periodic matrix $[A(\theta)]$ by a series of step functions. For that purpose the period of the response (2π) is divided into M equal intervals $\Delta\theta = 2\pi/M$. Inside each of the intervals, the time changing matrix $[A(\theta)]$ is replaced by its average value, i.e. by a constant matrix $[A_j]$, $j = 1, 2, \ldots, M$. For the j-th interval, the transition matrix can be expressed as:

$$[\Phi_j] = \mathrm{e}^{[\mathrm{A}_j]\Delta\theta}, \tag{15}$$

and for the whole period of the response $[0, 2\pi]$ as:

$$[\Phi(2\pi, 0)] = \prod_{j=1}^{M} \mathrm{e}^{[A_j]\Delta\theta}. \tag{16}$$

For numerical evaluation of (15) and (16), the algorithms recommended by Cardona et al. (1998) are used, i.e. the same ones as for the evaluation of (13) and (14).

Consequently, if a non-linear function is continuous, besides the factors mentioned in Section 3, the stability estimation depends additionally on the number M of intervals $\Delta\theta$, and the way of determining the constant matrix $[A_j]$.

5. Numerical examples

Fig. 4 shows an amplitude-frequency plot for the parameter values: $\bar{b} = 1$, $\zeta = 0.03$, $\bar{f}_0 = 0.25$, $\bar{f}_C = 0.25$, $\bar{f}_S = 0$, obtained by IHBM for the value of corrective vector norm $\|\{r\}\| \leqslant 1 \times 10^{-6}$, and the absolute accuracy of numerical determination of times θ_i in which the system undergoes stiffness change $\varepsilon \leqslant 2.22 \times 10^{-16}$ (since the non-dimensional time θ is used, the period of the response is always 2π). The point considered in further examples (obtained at $\eta = 0.5243$) is indicated in Fig. 4 by "$*$". Since the governing equation (5) is written in the non-dimensional form, all values in Figs. 4–10 are also non-dimensional. Fig. 5 shows the dependence of accuracy of determining effective amplitude $\bar{x}_p$ (obtained by IHBM) and maximum modulus of the eigenvalues of the corresponding monodromy matrix $|\lambda_{\max}|$, on the value of corrective vector norm $\|\{r\}\|$ and the number of harmonics N included in the supposed solution ($N = 12$, 16 and 20). Fig. 5(a) shows that in the considered range of $\|\{r\}\|$, the effective amplitude $\bar{x}_p$ does not significantly depend on $\|\{r\}\|$ and N, whereas $|\lambda_{\max}|$ can significantly depend both on $\|\{r\}\|$ and N (Fig. 5(b)). But, for sufficiently small values of $\|\{r\}\|$, its influence (in this example $\|\{r\}\| \leqslant 10^{-5}$) on the accuracy of determining $|\lambda_{\max}|$ is minor. Selection of sufficiently small $\|\{r\}\|$ is of great importance only if $|\lambda_{\max}|$ is close to unity, because then a very small inaccuracy in the procedure of evaluating λ can lead to incorrect prediction of the dynamical stability of the solution. Fig. 6 shows the dependence of $|\lambda_{\max}|$ on the accuracy ε of numerical determination of times in which the system undergoes stiffness change, for several values of $\|\{r\}\|$, and $N = 20$. One can see that in this example, $|\lambda_{\max}|$ depends significantly on ε only for $\varepsilon \geqslant 1.00 \times 10^{-5}$. For $\varepsilon < 1.00 \times 10^{-5}$, its influence on prediction of dynamical stability is not negligible only if $|\lambda_{\max}|$ is close to unity. Fig. 7 shows the dependence of $|\lambda_{\max}|$ on the number M of intervals $\Delta\theta$ (the width

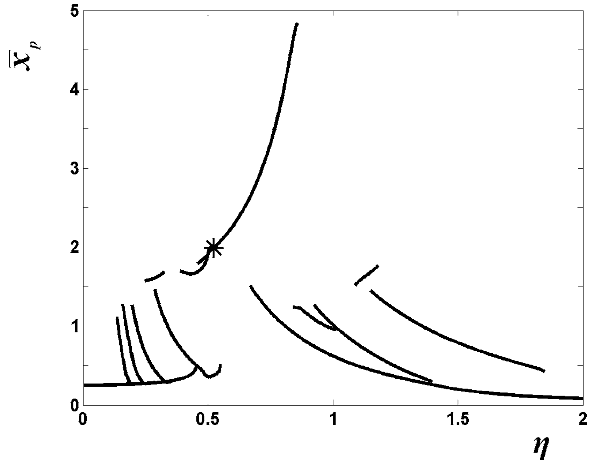
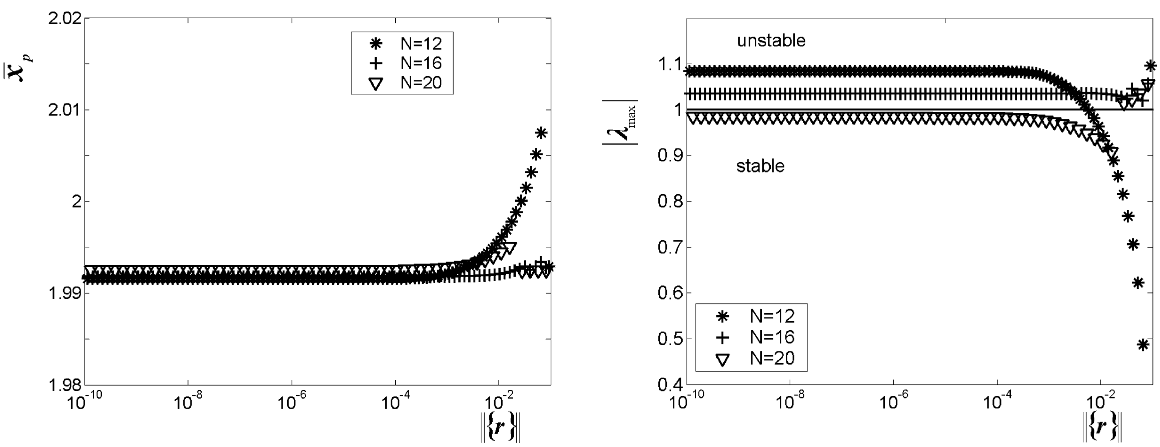


Fig. 4. Amplitude-frequency plot.



(a) (b)

Fig. 5. The effective amplitude $\bar{x}_p$ obtained by IHBM (a) and maximum modulus of the eigenvalues of the corresponding monodromy matrix (b) in dependence on the value of corrective vector norm and number of harmonics included in the supposed solution.

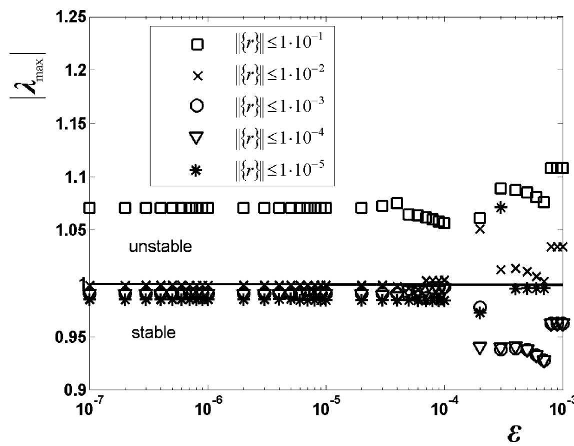


Fig. 6. Maximum modulus of the eigenvalues of the monodromy matrix in dependence on numerical precision of determining θ_i and the value of corrective vector norm.

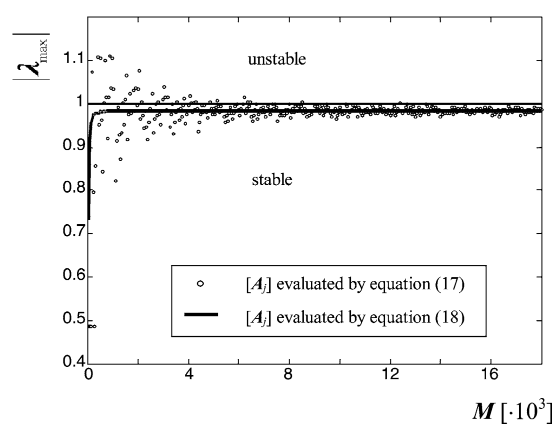
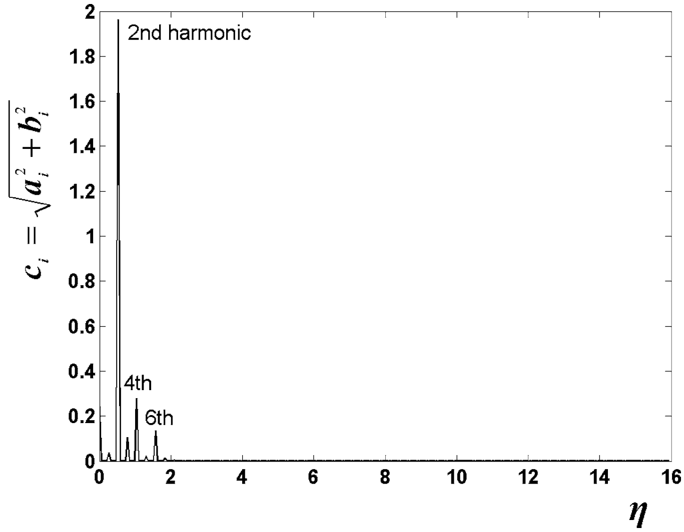
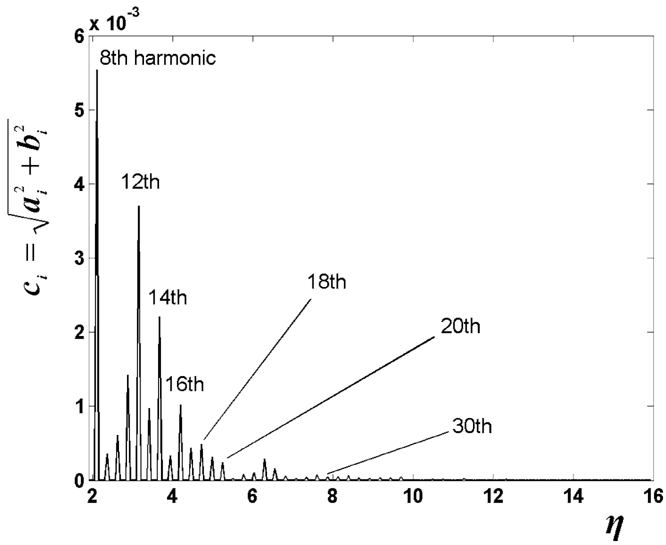


Fig. 7. Maximum modulus of the eigenvalues of the corresponding monodromy matrix in dependence on the number of intervals $\Delta\theta$.



(a)



(b)

Fig. 8. The spectrum of the time domain response.

of the intervals is $\Delta\theta = 2\pi/M$) when the procedure for general non-linear function (Eqs. (15) and (16)) is applied to the piecewise-linear function ($N = 20$, $\|\{r\}\| \leqslant 10^{-6}$). The results are shown for the cases of determining matrices $[A_j]$ by using:

$$[A_j] = \frac{[A(\theta_{j+1})] + [A(\theta_j)]}{2}, \quad j = 1, 2, \ldots, M, \tag{17}$$

and by using:

$$[A_j] = \frac{1}{\Delta\theta} \int_{\theta_j}^{\theta_{j+1}} \big[A(\theta)\big]\,\mathrm{d}\theta, \quad j = 1, 2, \ldots, M. \tag{18}$$

One can see that even a very large M does not enable sufficient accuracy of evaluating $|\lambda_{\max}|$ if $[A_j]$ is determined by using (17). $|\lambda_{\max}|$ is determined much more accurately if $[A_j]$ is determined by using (18). Wolf et al. (2004) showed that neglecting small harmonic terms of actual time-domain response can cause significant error in the evaluation of the eigenvalues of the monodromy matrix and can lead to incorrect prediction of the dynamical stability of the solution. The previously obtained

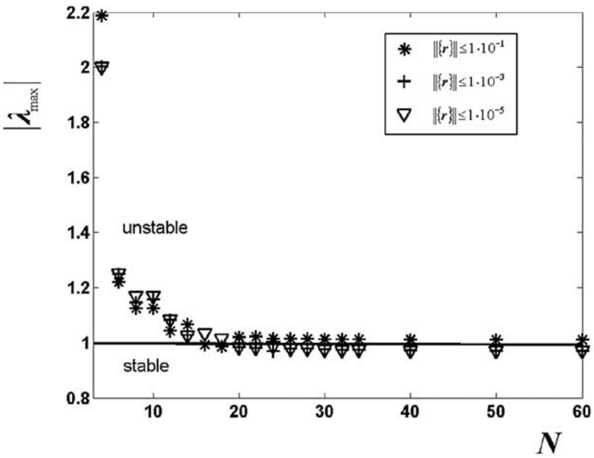


Fig. 9. Maximum modulus of the eigenvalues of the monodromy matrix in dependence on the number of harmonics included in the approximate solution (obtained by IHBM) and the value of corrective vector norm.

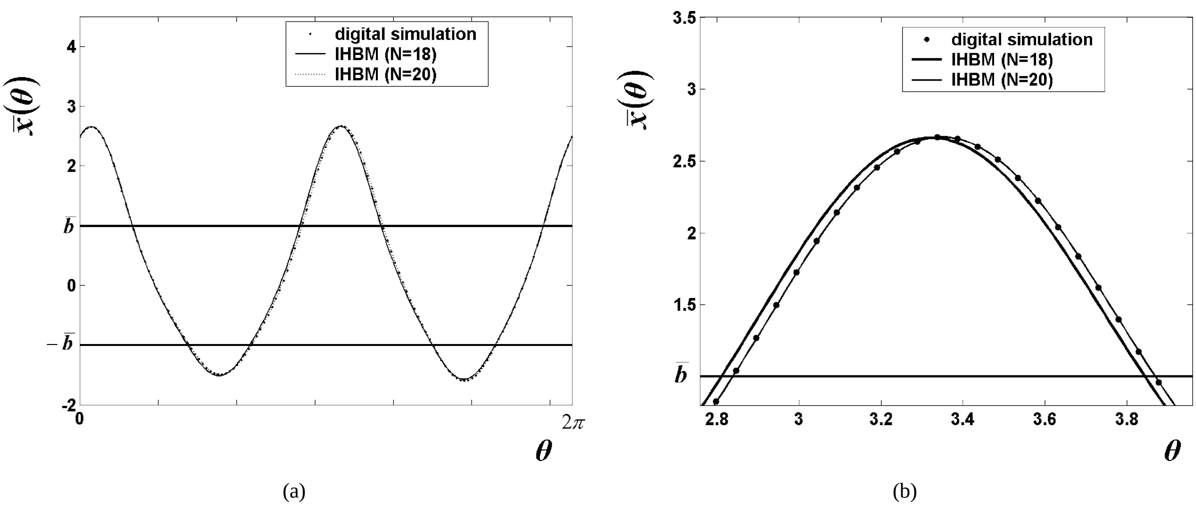


Fig. 10. Responses obtained by digital simulation and IHBM for $N = 18$ and $N = 20$.

results are extended here to the other factors that influence the stability prediction (Fig. 5 and Fig. 9). A spectrum of the time domain response of the considered system obtained by digital simulation is shown in Fig. 8(a) and Fig. 8(b) (η denotes non-dimensional frequency and $c_i = \sqrt{a_i^2 + b_i^2}$ denotes a non-dimensional amplitude of the i-th harmonic). The dependence of $|\lambda_{\max}|$ (obtained by IHBM) on the number of harmonics N included in the supposed approximate solution and the corrective vector norm $\|\{r\}\|$ (for $\varepsilon \leqslant 2.22 \times 10^{-16}$) is shown in Fig. 9. Fig. 8 and Fig. 9 show that neglecting small harmonic terms (11th, 12th, ...) of actual time-domain response, whose amplitudes are less than 0.11% of the amplitude of the largest (second) harmonic, can cause significant error in the evaluation of the eigenvalues of the monodromy matrix, and can lead to incorrect prediction of the dynamical stability of the solution. If the maximum modulus of the eigenvalues of the monodromy matrix is close to unity, the stability estimation can be an extremely sensitive procedure. In the considered example neglecting of 20th harmonic in the spectrum, whose amplitude is only 0.015% of the amplitude of the largest harmonic, causes an incorrect prediction of the dynamical stability of the solution. Fig. 9 also shows that if $\|\{r\}\|$ is not small enough, the stability prediction can be incorrect even if a sufficient number of harmonics is taken in consideration. The period of the response obtained both by digital simulation and IHBM (for $N = 18$ and $N = 20, \varepsilon \leqslant 2.22 \times 10^{-16}, \|\{r\}\| \leqslant 1 \times 10^{-6}$) and an area in which the system undergoes a stiffness change are shown in Fig. 10(a) and Fig. 10(b) respectively. Very good agreement of the responses obtained by digital simulation and IHBM for $N = 20$ is achieved, whereas a significant disagreement of the responses is obtained for $N = 18$. This shows that neglecting very small harmonic terms in the supposed approximate solution can cause significant disagreement of the approximate solution in consideration of the actual time-domain response, and in this way, the erroneously determining the times in which the system undergoes a stiffness change. This can significantly affect the accuracy of evaluation of the eigenvalues of the monodromy matrix and can lead to incorrect prediction of the dynamical stability of the solution.

6. Conclusions

The influence of factors which can lead to incorrect prediction of dynamical stability of the periodic response of oscillators with discontinuous or steep first derivative of the restoring characteristic is considered in this paper. It is observed that accuracy of determining the eigenvalues of the monodromy matrix (in the considered example) depend significantly on the corrective vector norm, $\|\{r\}\|$, the accuracy ε of numerical determination of the times at which the system undergoes a stiffness change, and on the number of step functions M (when the procedure for general non-linear function is applied), only if $\|\{r\}\| > 1 \times 10^{-5}$, $\varepsilon > 1 \times 10^{-5}$ and $M < 2000$. Otherwise, except if the maximum modulus of the eigenvalues of the monodromy matrix is close to unity, their influence on estimation of dynamical stability is minor. On the other hand, neglecting of very small harmonic terms of actual time domain response (which in-significantly influence the r.m.s values of the response and are very small in comparison to other terms of the spectrum) can cause very large error in evaluation of the eigenvalues of the monodromy matrix, and so they can lead to incorrect prediction of the dynamical stability of the solution, regardless of whether the maximum modulus of the eigenvalues of the monodromy matrix is close to unity or not. Although the previous analysis is performed for a simple one degree-of-freedom system with piecewise-linear restoring characteristic subjected to a harmonic excitation (the advantage of using this model is the possibility to determine elements of Jacobian matrix $[k]$ and the corrector $\{r\}$ in explicit form, as well as to express the monodromy matrix exactly as a product of matrix exponentials, what is not possible for a model with a general non-linear function), it can be extended on multiple degree-of-freedom systems with arbitrary number of general non-linear functions subjected to a general periodic excitation, i.e. to the systems on which the incremental harmonic balance method and the Hsu's procedure can be successfully applied. This analysis can be generalized and the obtained results can be interesting for stability analysis of dynamic systems with such type of discontinuities when slight differences in obtained response cause large changes in the first derivative of the restoring function.

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