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AN IMPROVED EKMAN LAYER APPROXIMATION FOR SMOOTH EDDY DIFFUSIVITY PROFILES

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Abstract. The Ekman boundary-layer model is extended analytically for a gradually varying eddy diffusivity $K(z) \geq 0$, $z \geq 0$. A solution for the Ekman layer is provided having similar structure to the constant-K case; that is, exponentially decaying sine functions for the two horizontal wind components. The analytical asymptotic solution compares well with its numerical counterpart for various $K(z)$. The result can be useful in theoretical studies such as Ekman pumping, for efficient estimation of the Ekman layer profiles in various analyses with near-neutral stratifications, or for a rapid initialization of mesoscale models.

Keywords: Boundary-layer pumping, Lambert's W, WKB.

1. Introduction

The Ekman layer model for geophysical boundary layers balances the pressure gradient, Coriolis, and turbulent friction forces to yield a flow vector that spirals with height (or depth). Grisogono (1995, hereafter G95) employed the WKB[1] method to solve the Ekman model analytically for almost any gradually varying eddy diffusivity $K(z)$. However, for the WKB method to be valid it was required that $K(0) > 0$ and, more naturally, that $K(z \to \infty) \to 0$. In this way, the strange feature of the classic Ekman model extending constant K up into the geostrophic flow is eliminated. This way of solving the Ekman layer equations was used in more complex boundary-layer studies by Berger and Grisogono (1998), Tan (2001) and Zhang and Tan (2002). G95's WKB solutions agreed with the numerical ones so long as the basic assumption was fulfilled, i.e., that $K(z)$ varied more gradually than the

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[1] After Wentzel, Kramers and Brillouin, who popularized the method in theoretical physics. Sometimes also referred to as WKBJ, including Jeffreys who, together with Rayleigh, contributed to its early development. The original approximation was made independently by Liouville and Green in 1837.

calculated quantities. However, the way that the WKB method was used in G95 required that the value of $K(0)$ be finite ($\neq 0$). Although the requirement of $K(z)$ to vary smoothly may be met, the latter requirement of a finite, non-zero value at the ground is basically unphysical and needs to be relieved. This improvement is the focus of the present work.

In a mathematically similar problem, the Prandtl model for pure katabatic flows, Grisogono and Oerlemans (2001a, b, hereafter GOa and GOb) improved the same WKB method so that $K(0) \geq 0$ with no specific mathematical restriction at $z = 0$. That was done by a simple patching (i.e. local matching) of the zero- and first-order WKB solutions. The improved model of Prandtl was checked there against numerical solutions and a dataset, it was justified *a posteriori* in Grisogono and Oerlemans (2002) and checked against another, larger dataset by Parmhed et al. (2004). The dataset used by Parmhed et al. (2004) is also less favourable than that used in GOa and GOb in that the terrain where the observations are made is more complex. Also, that area presents both a rough surface and experiences occasional disturbances from larger scale flows such as the north Atlantic storm track.

In this study, the above mentioned WKB improvement (from GOa and GOb) is extended to the Ekman model using a gradual $K(z) \geq 0$, $z \geq 0$, i.e., allowing $K(0) = 0$. The equation used to prescribe $K(z)$ is the same as that used in GOa and GOb, a generalization of the O'Brien third-order polynomial often used in boundary-layer studies (see e.g., Stull, 1988). When applying the approach devised by GOa and GOb to the Ekman layer, however, we find it insufficient. The sharp near-surface gradients are strongly smoothed. The method is then further improved in the present study by determining the height at which to patch the zero- and first-order solutions for optimal results. In this way, an approximate solution for the Ekman layer is provided having a structure similar to that for constant-K cases and requiring as input only the maximum value of the eddy diffusivity, or equivalently the layer height, and the background geostrophic wind. This means that almost all of the simplicity and generality of the Ekman layer is retained while its significant limitation, the need for a finite non-zero eddy diffusivity, is relieved. The only limitation remaining on the eddy diffusivity profile, for the solution to exist mathematically, is that it should be smoothly varying, a requirement that is readily met in nature.

2. Model

2.1. The governing equations

The Ekman layer is governed by the equations (see e.g., Stull, 1988; Holton, 1992; Kundu and Cohen, 2002):

$$\frac{Du}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv - \frac{\partial \overline{u'w'}}{\partial z}, \quad (1)$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu - \frac{\partial \overline{v'w'}}{\partial z}. \quad (2)$$

Assuming a steady and non-advective state $\left(\frac{D}{Dt} \equiv 0\right)$, geostrophy $\left(v_g = \frac{1}{f\rho_0}\frac{\partial p}{\partial x}, u_g = -\frac{1}{f\rho_o}\frac{\partial p}{\partial y}\right)$, and using the flux-gradient theory $\left(\overline{u'w'} = -K\frac{\partial u}{\partial z}, \overline{v'w'} = -K\frac{\partial v}{\partial z}\right)$, these equations can be summed (after (2) is multiplied by $i = \sqrt{-1}$) to yield:

$$K\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial K}{\partial z}\frac{\partial \Phi}{\partial z} - if\,\Phi = 0, \quad (3)$$

which is the single, complex, governing equation for the Ekman layer. Here, u, v and w are the components of the wind in the x, y and z directions respectively, p is the atmospheric pressure, ρ_0 is the reference density, f is the Coriolis parameter, t is time and K is the eddy diffusivity for momentum; Φ is a complex variable, $\Phi = (u - u_g) + i(v - v_g)$. The boundary conditions are: $u(0) = v(0) = 0$ and $u(\infty) = u_g, v(\infty) = v_g$.

If $K = constant$, (3) is analytically solvable exactly and, assuming the coordinate system is aligned such that $v_g \equiv 0$, the real and imaginary parts of Φ yield:

$$u = u_g(1 - e^{-\gamma z}\cos \gamma z), \quad (4)$$

$$v = u_g e^{-\gamma z}\sin \gamma z, \quad (5)$$

$$\gamma = \sqrt{\frac{f}{2K}}, \quad (6)$$

which relates to the inverse of the layer height. G95 used the WKB method (see e.g. Bender and Orszag, 1999) to allow $K(z)$ to vary with height. The basis of the WKB method is the assumption that the solution can be expressed as

$$\Phi \sim e^{\frac{1}{\delta}(S_0 + \delta S_1 + \delta^2 S_2 + \delta^3 S_3 + \cdots)}, \quad (7)$$

where δ is a presumably small parameter included to facilitate a determination of S_j ($j = 1, 2, 3,\ldots$) from a balance of terms in power of δ; it is at a later stage equated here to unity.

By again taking the real and imaginary parts of Φ, the WKB method supplies us with the following solution:

$$u(z) = u_g\left(1 - A^j e^{-F(z)}\cos F(z)\right), \quad (8)$$

$$v(z) = u_g A^j e^{-F(z)} \sin F(z), \tag{9}$$

$$F(z) = \sqrt{\frac{f}{2}} \int_0^z \frac{1}{\sqrt{K(\xi)}} d\xi, \tag{10}$$

$$A^j = \left(\frac{K_p}{K(z)} \right)^{\frac{j}{4}}, \tag{11}$$

where now $j = 0$ or 1. For the zero-order WKB solution, $A^0 = 1$, while Equation (11) gives the first-order correction. Here K_p is the value of $K(z)$ at the level of patching.

In G95 the first-order solution (using A^1) is used. The first-order solution is not valid from the surface unless $K(z)$ is larger than some value at the surface. The work by G95 was extended for the Prandtl model for katabatic flow by GOa and GOb. They used a more realistic K profile that may reach zero at the surface, it is a 'exponential' generalization to that used by O'Brien (1970):

$$K(z) = \left(\frac{K_{\max} e^{\frac{1}{2}}}{h} z e^{-\frac{1}{2}\left(\frac{z}{h}\right)^2} \right), \tag{12}$$

where h is the height of the maximum in K and $K_{\max}$ is the maximum value of K. The relevance for atmospheric conditions of the K profile given in (12) is presented in e.g., Stull (1988) and is rigorously discussed in GOa and GOb.

To achieve validity in the WKB method, GOa and GOb used the zero-order solution from the surface. At the height z_p the zero-order solution is patched to the first-order solution. In GOa the patching was done at the height h, corresponding to $K_{\max}$.

2.2. Determining h and z_p

For the WKB method to be valid, the K profile must be varying only gradually with respect to the solution variations. This means that h must be larger than the height of the maximum wind speeds in u and $v \left(\frac{K_{\max}}{h} \ll |u_g| \right)$. Since the WKB solution always will place the jets in u and v *below* the same heights calculated through the constant K solutions ((4) to (6)), assuming that $K \to 0$ at the surface, the maximum of the jet heights from the constant K solution can be used to find h for a given $K_{\max}$. Thus, letting h be the height of the maximum in u (the higher jet) in the constant K solution ((4) to (6)), it is given by:

$$h = \frac{3}{4} \frac{\pi}{\sqrt{\frac{f}{2K_{\text{const.}}}}}. \tag{13}$$

Also, this can be written $h = \frac{3}{4} \frac{\pi}{\gamma}$.

The height (z_p) where the first-order solution is patched to the zero-order solution is important. It will facilitate a reasonably accurate, global asymptotic solution to Ekman layers for any gradual $K(z) \geq 0, z \geq 0$.

The validity of the WKB solution relies on the inequality $|S_0| > |S_1|$ for the terms in (7). Expanding $K(z)$ for small $z, K(z) \approx az$, where $a = \frac{K_{\max} e^{\frac{1}{2}}}{h} = ku_*$ (depending on $K_{\max}$ this is true within 1% to at least a few tens of metres for a K profile of the type (12)), with k and u_* being the von Karman constant and turbulent friction velocity. Given this expansion of $K(z)$, the necessary requirement on z can be found in terms of $S_0 \approx \sqrt{\frac{z}{a}}$ and $S_1 \approx -\frac{1}{4}\ln(z)$, leading to $z^{\frac{1}{\sqrt{z}}} = e^{-4\frac{1}{\sqrt{a}}}$. The lowest height for a given value of $K_{\max}$ where the first-order solution may be valid is:

$$z_{\mathrm{p}} = \frac{1}{4}\left[W\left(\frac{2}{\sqrt{a}}\right)\right]^2, \tag{14}$$

where W is Lambert's W function.[2] Although the Lambert W is difficult to compute exactly, in this work the approximate series solution given by Corless et al. (1997) is used:

$$W(c) \approx 1 + \frac{1}{2}\ln\left(\frac{c}{e}\right) + \frac{1}{16}\ln\left(\frac{c}{e}\right)^2 - \frac{1}{192}\ln\left(\frac{c}{e}\right)^3, \tag{15}$$

where c is the independent variable (here equal to $\frac{2}{\sqrt{a}}$). Note that (15) is not globally valid.

In summary, Equation (14) gives us a way of estimating z_p, the lowest height at which the first-order WKB solution can faithfully be applied. This is the main improvement over G95, GOa and GOb and makes the core of this study.

3. Results and Discussion

Model solutions have been computed for three cases. One similar to the one used by G95, his Case 1, with a K profile decreasing continuously from a finite value at the surface to zero at about 1500 m height. The other two cases, one of which is shown for reference in Figure 1, both have K profiles of the type (12).

The solutions are calculated in four ways:

(i) A numerical solution using an implicit, second-order, trapezoidal scheme with Gaussian elimination. The numerical computation is time-dependent and is terminated when steady state is obtained (the most accurate solution).

(ii) The classic Ekman solution using a constant K, Equations (4)–(6). The value for K is $K_{\max}/3$, where $K_{\max}$ for each case is that used for (iii) and (iv).

[2] Lambert's W(c) function is the solution to the transcendental equation $We^W = c$.

(iii) The WKB method, Equations (8)–(11). First-order solution starts at the maximum in the K profile as in GOa and GOb, i.e., $z_{\mathrm{p}} = h$ using (13). This solution will henceforth be denoted 'WKB(I)'.

(iv) Like (iii), but the start of the first-order solution is now from (14). This solution will henceforth be denoted 'WKB(II)'.

The difference between case 2 and 3 is in the chosen strength of the maximum in the K profile. The solution profiles of these two cases are, however, close to self-similar. The difference (from self-similarity) depends only on the difference in height where the first-order solution starts. For this reason only case 2 is illustrated, in Figure 1. Case 3 uses a maximum value of the K profile that is one fifth of that in case 2.

Figure 1 indicates that the constant K solution provides a far too deep boundary layer. Too much cross-isobaric mass flux (v component) and insufficient low-level mixing (too weak near-surface gradients) along the flow are also clear features of the constant K solutions. This is partly remedied by the WKB(I) solution. However, the WKB(I) solution gravely overestimates the v maximum. In fact, the WKB(I) solution is only a minor alteration in phase from the constant K solution in the v components and can only be considered a minor improvement. In both components, the WKB(I) solution

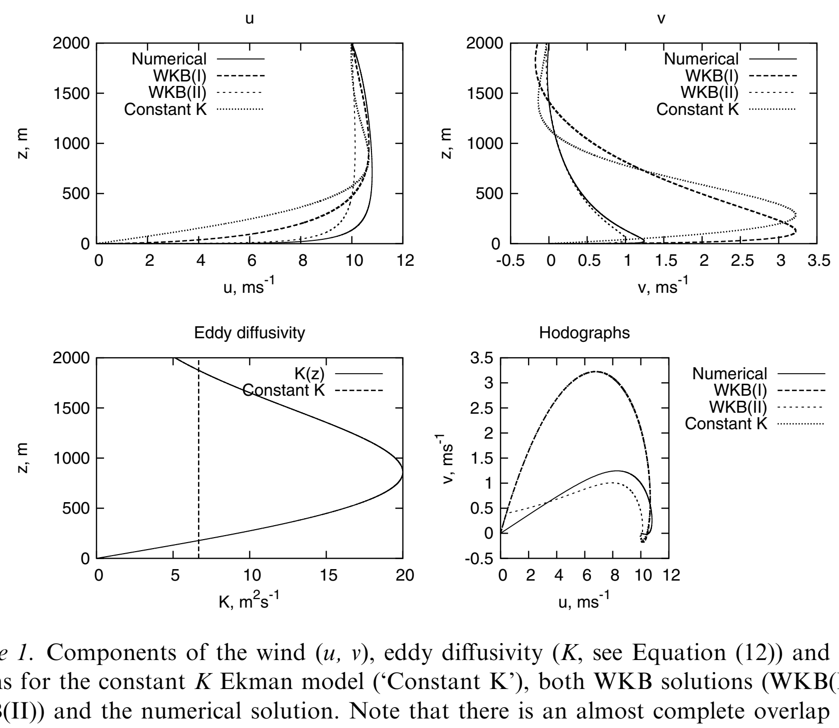

Figure 1. Components of the wind (*u, v*), eddy diffusivity (K, see Equation (12)) and hodographs for the constant K Ekman model ('Constant K'), both WKB solutions (WKB(I) and WKB(II)) and the numerical solution. Note that there is an almost complete overlap of the constant K and WKB(I) solutions in the hodograph.

provides sharper near-surface gradients than those provided by the constant K solution, but without any major change in the magnitudes. The WKB(II) solution gives both sharper near-surface gradients and acceptable wind magnitudes, close to those in the numerical solution. Furthermore, the WKB(II) solution to the v components shows a close resemblance to the numerical solution in both phase and amplitude, exhibiting less than half the maximum wind speed present in the constant K and WKB(I) solutions. A minor drawback of the WKB(II) solution can be seen in the u component. Although the WKB(II) solution displays a close resemblance in phase to the whole numerical solution, its u component gives a somewhat lower magnitude (10% or so).This minor underestimate in the u maximum is, however, less obvious and important than the improvement in the v component.

Here it is important to remember that the improvements shown are entirely due to the chosen K profile, compared to the constant K profile. All of the presented results are hidden within the governing equations given by Ekman (1905). However, his simple analytic solution required the use of a constant value for K leading to the deficiencies mentioned above.

The improvement in the v component motivates a comparison of the cross-isobaric mass flux, important for Ekman pumping, cyclonic spin-down and parameterization of the lower boundary in models of large-scale geophysical circulations. Table I shows the cross-isobaric mass flux normalized by ρ_0, for the four solutions for each of the K profiles discussed. From Table I summarizing the main finding in this study, it is obvious that the WKB(II) solution constitutes a clear improvement in the description of the Ekman layer. Here, it is interesting to note the difference in the mass flux between the three K profiles considered. In the first case, relating to G95, the mass flux is underestimated by the constant K solution, because $K(z)$ has a maximum at the surface. In the second and third case, the $K(z)$ profiles have zero values at the surface, meaning that the constant K solution overestimates the mass flux. The relative difference between the constant K, WKB and numerical solutions is also not so large in the first, least realistic, case as in the second and third cases.

The hodographs in Figure 1 illustrate the effect of the WKB solutions on the mass flux calculations. Horizontal transports of mass and momentum toward

TABLE I

Cross-isobaric wind integrated in vertical $(\int_0^\infty v dz)[\mathrm{m}^{-2}\mathrm{s}^{-1}]$ for the three cases considered. Also given is the difference relative to the numerical solution, in percent.

K (case number)	Constant K	WKB (I)	WKB (II)	Numerical
1	1826 (−33%)	2548 (−0.7%)	2548 (−0.7%)	2730
2	1826 (+243%)	1979 (+272%)	477 (−10%)	532
3	816 (+224%)	883 (+250%)	279 (+11%)	252

the lower pressure calculated numerically and via the WKB(II) generally agree. A kink seen in the hodograph in Figure 1 is another small drawback of the WKB(II) solution. The kink appears after the lowest wind values. It is caused by transition from the zero-order (where the v component is still increasing) to the first-order (where the v component is already decreasing) solutions. Again, while the WKB (II) solutions align reasonably closely to the numerical solutions, the constant K and WKB(I) solutions fail primarily in determining viable magnitudes of the v component. Needless to say, the vertical momentum fluxes $(K\frac{\partial u}{\partial z}, K\frac{\partial v}{\partial z})$ are overestimated by the constant K and WKB(I) solutions, while in the WKB(II) they are very close to the numerical solutions.

The analysis presented here can be extended back to GOa and GOb, including local katabatic/anabatic flows. There, for given slopes and other input parameters, surface fluxes can be estimated. Improving the surface-flux calculations is essential for numerical modelling – to avoid artificial low-level decoupling and to provide triggering for deep convection.

4. Conclusion

In this study, the Ekman boundary-layer model is extended analytically for a gradually varying eddy diffusivity, $K(z) \geq 0$, $z \geq 0$, by using the WKB method, following the work by G95, GOa and GOb. The improvements in this study, the WKB (II), are:

(i) The extension of the model from $K(z) > 0$, $z \geq 0$ to $K(z) \geq 0$, $z \geq 0$.

(ii) The determination of the height (z_p) from where the first-order WKB solution (WKB(II)) should be applied (cf. Equation (14)).

The corresponding improvement in the description of the Ekman layer is seen in Figure 1 and is also summarized in Table I. Namely, the much improved cross-isobaric mass flux, determining the boundary-layer pumping, is quantified in Table I. Moreover, the hodographs illustrate that the WKB(II) spiral is close to that obtained numerically. The new solution is always within approximately 10% of the corresponding numerical solution. We find that the description of the Ekman boundary layer, and the associated boundary-layer pumping, is considerably improved using the presented method. With more realistic $K(z)$, such as a generalized O'Brien (1970) profile (e.g., Pielke, 1984; Stull, 1988) from (12), the Ekman layer mass transport toward the lower pressure becomes two to three (or even up to four) times smaller than in the constant K case. This is lacking in the classic description of the Ekman layer. The solutions obtained here, in particular, the hodographs, also compare favourably with the evidence of an Ekman spiral presented by Chereskin (1995).

This result can be useful in both theoretical and practical applications offering an elegant and accurate, but still simple and analytic, description of the near-neutral, horizontally homogeneous geophysical boundary layer. For example, improving numerical model parameterizations almost exclusively evolves from analytical reasoning.

References

Bender, C. M. and Orszag, S. A.: 1999, *Advanced Mathematical Methods for Scientists and Engineers,* Springer, New York, 593 pp.

Berger, B. W. and Grisogono, B.: 1998, 'The Baroclinic, Variable Eddy Viscosity Ekman Layer. An Approximate Analytical Solution', *Boundary-Layer Meteorol.* **87**, 363–380.

Chereskin, T. K.: 1995, 'Evidence for an Ekman Balance in the California Current', *J. Geophys. Res.* **100**(C9), 12727–12748.

Corless, R. M., Jeffrey, D. J., and Knuth, D. E.: 1997, 'A Sequence of Series for the Lambert W Function', Proceedings ISSAC '97, Maui, Ed. Wolfgang W. Kuechlin, pp. 197–204. http://www.apmaths.uwo.ca/~rcorless/frames/PAPERS/LambertW / [2003-07-23].

Ekman, V. W.: 1905, 'On the Influence of the earth's Rotation on Ocean-Currents',*Arkiv. Matem., Astr. Fysik, Stockholm* **2**(11).

Grisogono, B.: 1995, 'A Generalized Ekman Layer Profile with Gradually-Varying Eddy Diffusivities', *Quart. J. Roy. Meteorol. Soc.* **121B**, 445–453.

Grisogono, B. and Oerlemans, J.: 2001a, 'Katabatic Flow: Analytic Solution for Gradually Varying Eddy Diffusivities', *J. Atmos. Sci.* **58**, 3349–3354.

Grisogono, B. and Oerlemans, J.: 2001b, 'A Theory for the Estimation of Surface Fluxes in Simple Katabatic Flows', *Quart. J. Roy. Meteorol. Soc.* **127B**, 2125–2739.

Grisogono, B. and Oerlemans, J.: 2002, 'Justifying the WKB Approximation in Pure Katabatic Flows', *Tellus* **54A**, 453–462.

Holton, J. R.: 1992, *An Introduction to Dynamic Meteorology*, 3rd edn., Academic Press, San Diego, Calif., London, 511 pp.

Kundu, P. K. and Cohen, I. M.: 2002, *Fluid Mechanics,* 2nd edn., Academic Press, San Diego, Calif., London 730 pp.

O'Brien, J. J.: 1970, 'A Note on the Vertical Structure of the Eddy Exchange Coefficient in the Planetary Boundary Layer', *J. Atmos. Sci.* **27**, 1213–1215.

Parmhed, O., Oerlemans, J., and Grisogono, B.: 2004, 'Describing Surface-Fluxes in Katabatic Flow on Breidamerkurjökull, Iceland', *Quart. J. Roy. Meteorol. Soc.* **130**, 1137–1151.

Pielke, R. A.: 1984., *Mesoscale Meteorological Modeling,* Academic Press, Orlando, Florida, London, 611 pp.

Stull, R. B.: 1988, An *Introduction to Boundary Layer Meteorology,* Kluwer Academic Publishers, Dordrecht, 666 pp.

Tan, Z.-M.: 2001, 'An Approximate Analytical Solution for the Baroclinic and Variable Eddy Diffusivity Semi-Geostrophic Ekman Boundary Layer', *Boundary-Layer Meteorol.* **98**, 361–385.

Zhang, Y. and Tan, Z.-M.: 2002, 'The Diurnal Wind Variation in a Variable Eddy Viscosity Semi-Geostrophic Ekman Boundary-Layer model: Analytical Study', *Meterol. Atmos. Phys.* **81**, 207–217.