SECOND-METACYCLIC p-GROUPS $(p > 2)$

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Let G be a p-group that contains a proper nonmetacyclic subgroup and such that all its subgroups of index p^2 are metacyclic. Such group is called an $MC(2)$-group. The 2-groups which are $MC(2)$-groups were classified in [1]. The aim of this investigation is to classify the $MC(2)$-groups whose order is a power of a prime p, $p > 2$.

There are two well-known Blackburn's Theorems

Theorem 1. (Blackburn[2]) Let $p > 2$ and $|G| = p^n$ where $n \geq 5$. If all subgroups of G of order p^r for every integer r with $3 \leq r \leq n-2$ are generated by two elements, then one of the following assertions holds:

a) G is metacyclic;

b) G is a 3-group of maximal class;

c) $r = 3$ and the elements of order p of G (with e) form a nonabelian normal subgroup $\Omega_1(G)$ of G of order p^3 with a cyclic factor group $G/\Omega_1(G)$.

Any $MC(2)$ group contains some nonmetacyclic maximal subgroup N whose all proper subgroups are metacyclic. The following Blackburn's Theorem determines such groups.

Theorem 2. (Blackburn [2]) Let G be a p-group, all whose proper subgroups are metacyclic, but G itself is not. Then one of the following assertions holds:

a) G is elementary abelian of order p^3.

b) It is $p > 2$ and G is a nonabelian group of order p^3 and exponent p.

c)G is a 3-group of class 3 and of order 3^4.

d) G is a 2-group with $|G| \leq 2^5$.

According these theorems each $MC(2)$-group is either of order p^4 for $p \geq 3$, or of order 3^5.

There is a full description of groups of order p^n where $p \geq 3$, $n \leq 6$ in the lists obtained by R. James ([3]) and O.Pylyavska ([4]). However, the authors preferred to give a direct proof based on structural properties of the investigated groups.

Authors have proved that there are ten $MC(2)$ groups of order p^4, $p > 2$, more precisely

1) 7 groups, which contain an elementary abelian subgroup of order p^3:

a) two abelian : the elementary abelian and the group of type (2,1,1);

b) three groups with center $Z(G)$ of order p^2 and commutator subgroup G' of order p :

$G = \langle a, b, c \mid [a, b] = d, [a, c] = [b, c] = [a, d] = [b, d] = [c, d] = 1, a^p = d, b^p = c^p = d^p = 1\rangle$

$G = \langle a, b \mid [a, b] = d, [a, d] = [b, d] = 1, a^{p^2} = b^p = d^p = 1\rangle$

$G = \langle a, b, c \mid [a, b] = d, [a, c] = [b, c] = [a, d] = [b, d] = [c, d] = 1, a^p = b^p = c^p = d^p = 1\rangle$

c) two groups G of maximal class with center $Z(G)$ of order p and commutator subgroup G' of order p^2 :

$G = \langle a, b \mid [a, b] = c, [a, c] = d, [b, c] = [a, d] = [b, d] = [c, d] = 1, a^p = d, b^p = c^p = d^p = 1\rangle$

$G = \langle a, b \mid [a,b] = c, [a,c] = d, [b,c] = [a,d] = [b,d] = [c,d] = 1, a^p = b^p = c^p = d^p = 1\rangle$

2) 3 groups, which have none elementary abelian subgroup of order p^3, but have a nonabelian subgroup of order p^3 and exponent p :

$G = \langle a, b, c \mid [a,b] = d, [a,c] = [b,c] = [a,d] = [b,d] = [c,d] = 1, c^p = d, a^p = b^p = d^p = 1\rangle$

$G = \langle a, b \mid [a,b] = c, [a,c] = d, [b,c] = [a,d] = [b,d] = [c,d] = 1, b^p = d, a^p = c^p = d^p = 1\rangle$

$G = \langle a, b \mid [a,b] = c, [a,c] = d, [b,c] = [a,d] = [b,d] = [c,d] = 1, b^p = d^\delta, a^p = c^p = d^p = 1\rangle$

where δ is a nonquadratic residue modulo p.

For $p = 3$ there are two groups of order 3^5 which are MC(2)-groups:

1) the group of maximal class with an abelian maximal subgroup:

$G = \langle d, f \mid [d,f] = c, [c,d] = b, [b,d] = a^2, [a,b] = [a,c] = [a,d] = [a,f] = [b,c] = [b,f] = [c,f] = 1, a^3 = b^3 = 1, c^3 = d^3 = a, f^3 = ab\rangle$

2) the group of maximal class without any abelian maximal subgroup

$G = \langle d, f \mid [d,f] = c, [c,d] = b, [b,d] = a^2, [c,f] = a, [a,b] = [a,c] = [a,d] = [a,f] = [b,c] = [b,f] = 1, a^3 = b^3 = 1, c^3 = d^3 = a, f^3 = ab\rangle$

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[2] N. Blackburn, *Generalizations of certain elementary theorems on p-groups*, Proc.London Math.Soc.-1961.- 11, P.1-22

[3] R. James, *The groups of order* p^6 *(p - an odd prime)*, Math.Comp.-1980.-34, V.150-P.613-637.

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