SECOND-METACYCLIC FINITE 2-GROUPS

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Introduction

The authors wish to express their gratitude to Professor Zvonimir Janko for suggesting the investigation of the above problem. The starting point is the following result of N.Blackburn.

Theorem 1 *(see Janko 1, Th.7.1) Let G be a minimal non-metacyclic 2-group. Then G is one of the following groups:*
(a) The elementary abelian group E_8 of order 8
(b) The direct product $Q_8 \times Z_2$
(c) The central product $Q_8 * Z_4$ of order 2^4
(d) $G = \langle a, b, c \mid a^4 = b^4 = [a, b] = 1,\ c^2 = a^2b^2,\ a^c = a^{-1},\ b^c = a^2b^3 \rangle$,
where G is special of order 2^5 with $\exp(G) = 4$,
$\Omega_1(G) = G' = Z(G) = \Phi(G) = \langle a^2, b^2 \rangle \cong E_4$,
$M = \langle a \rangle \times \langle b \rangle \cong Z_4 \times Z_4$ is the unique abelian maximal subgroup of G.

For brevity, we call the groups from the title *second-metacyclic groups* and denote them as $MC(2)$-groups. It is clear that their non-metacyclic maximal subgroups are minimal non-metacyclic groups and thus each $MC(2)$-group contains some group from Th.1 as maximal subgroup. Especially, an $MC(2)$-group G is of order 16, 32 or 64. Our main result is stated in the following theorem.

Theorem 2 *Let G be a second-metacyclic group. Then G is one of the following 17 groups:*

(a) four groups of order 16:

$E_{16},\ Z_4 \times E_4,\ D_8 \times Z_2,$ *or*

$G = \langle a, b, c \mid a^2 = b^2 = c^4 = 1,\ [a, b] = [a, c] = 1, b^c = ab\rangle \cong E_4 \cdot Z_4$

(b) ten groups of order 32:

1) $G > H \cong Q_8 \times Z_2,$

$H = \langle a, b, c \mid a^4 = 1, b^2 = a^2, c^2 = 1, a^b = a^{-1}, [a, c] = [b, c] = 1, \rangle :$

$G_1 = \langle H, d \mid d^2 = 1, [a, d] = [b, d] = 1, c^d = a^2c\rangle = \langle a, b\rangle * \langle c, d\rangle \cong Q_8 * D_8$

$G_2 = \langle H, d \mid d^2 = 1, a^d = a, b^d = abc, c^d = a^2c\rangle$

$G_3 = \langle H, d \mid d^2 = a^2, a^d = a^{-1}, b^d = ab, c^d = c\rangle$

$G_4 = \langle H, d \mid d^2 = a^2, a^d = a^{-1},\ b^d = bc, c^d = c\rangle$

$G_5 = \langle H, d \mid d^2 = c, a^d = a^{-1}, b^d = ab, c^d = c\rangle$

$G_6 = \langle H, d \mid d^2 = c, [a, d] = [b, d] = [c, d] = 1\rangle = \langle a, b\rangle \times \langle d\rangle \cong Q_8 \times Z_4$

$G_7 = \langle H, d \mid d^2 = a, a^d = a, b^d = bc, c^d = a^2c\rangle$

2) $G > H \cong Q_8 * Z_4,$

$H = \langle a, b, c \mid a^4 = 1, b^2 = c^2 = a^2, a^b = a^{-1}, [a, c] = [b, c] = 1\rangle$ *and if* $G > L$*,*
then $L \ncong Q_8 \times Z_2$:

$G_8 = \langle H, d \mid d^2 = c, [a, d] = [b, d] = [c, d] = 1\rangle$

$G_9 = \langle H, d \mid d^2 = 1, a^d = a^{-1}, b^d = ab, c^d = c\rangle$

$G_{10} = \langle H, d \mid d^2 = ac, a^d = a, b^d = ab, c^d = c\rangle$

(c) three groups of order 64:

$G > H = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, a^b = a, a^c = a^{-1}, b^c = a^2b^3\rangle$

$G_1 = \langle H, d \mid d^2 = a^2, a^d = a^3b^2, b^d = b^{-1}, c^d = c\rangle$

$G_2 = \langle H, d \mid d^2 = b^2,\ a^d = a^{-1},\ b^d = a^2b^3,\ c^d = ac\rangle$

$G_3 = \langle H, d \mid d^2 = b, a^d = ab^2, b^d = b, c^d = ac\rangle$

We begin by recalling some basic definitions and facts.

Definition 1 A group G is *metacyclic* if there exists a cyclic normal subgroup N of G with cyclic factor group G/N.

Theorem 3 *Let G be a metacyclic group, H a subgroup of G, and K a normal subgroup of G. Then H and G/K are also metacyclic.*

Proof: By Def.1 there exists N, $N \leq G$, such that N and G/N are both cyclic. By a known theorem $NH/N \cong H/N \cap H$. Since $NH/N \leq G/N$ and $N \cap H \leq N$, the groups $H/N \cap H$ and $N \cap H$ are both cyclic, and H is metacyclic by Def.1.
For $K \trianglelefteq G$, the groups $G/K\big/NK/K \cong G\big/NK \cong G/N\big/NK/N$ and $NK/K \cong N/N \cap K$ are both cyclic, as NK/N and $N \cap K$ are subgroups of cyclic groups G/N and N, respectively. So G/K is metacyclic too.

Theorem 4 *Let G be a group. If $G/Z(G)$ is cyclic, then $G = Z(G)$, that is, G is abelian.*

Theorem 5 *Let G be an abelian p-group, $\mid G \mid = p^{\alpha}$, for some prime p. Then $G \cong Z_{p^{\alpha_1}} \times Z_{p^{\alpha_2}} \times \cdots \times Z_{p^{\alpha_k}}$, $\sum\limits_{i=1}^{k} \alpha_i = \alpha$. The k-tuple $(\alpha_1,\ \alpha_2, \ldots,\ \alpha_k)$ is uniquely determined up to order and so we can assume $\alpha_1 \geq \cdots \geq \alpha_i \geq \alpha_{i+1} \geq \cdots \geq \alpha_k$.*

Definition 2 An abelian p-group $G \cong \underbrace{Z_p \times \cdots \times Z_p}_{n\ times} \equiv E_{p^n}$ is called *elementary abelian* p-group of order p^n.

Definition 3 Let G be a p-group. The *Frattini-subgroup of G*,
$\Phi(G) = \bigcap\limits_{\substack{M<G \\ max}} M$, is the intersection of all maximal subgroups of G.

Theorem 6 *Let G be a p-group and $\Phi(G)$ its Frattini-subgroup. Then:*
1) $\Phi(G) \trianglelefteq G$ and $G/\Phi(G)$ is elementary abelian
2) If $\mid G/\Phi(G) \mid = p^n$, then G is generated by (at least) n elements
3) For a 2-group, $\Phi(G) = \mho_1(G) := \langle x^2 \mid x \in G \rangle$

Theorem 7 *Let $g \in G$. The conjugation by g, $\varphi_g : x \mapsto x^g = g^{-1}xg$ is an automorphism of G. For $S \subseteq G$, $S^g := \{s^g | s \in S\}$*

Theorem 8 *Let $S \subseteq G$, $H \leq G$. Denoting $N_H(S) = \{h \in H | S^h = S\}$and $C_H(S) = \{h \in H | \forall s \in S, s^h = s\}$, we have $C_H(S) \trianglelefteq N_H(S)$. For $K \leq G$, the factor group $N_H(K)/C_H(K)$ is isomorphic to some subgroup of $AutK$ - the group of all automorphisms of K.*

Remark :
$Z(G) \equiv C_G(G)$.

Definition 4 The chain $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_{i-1} \triangleright G_i \triangleright \cdots \triangleright G_k = 1$ is a *composition series* of G, if each G_i is a maximal normal subgroup of G_{i-1}. If, moreover, each G_i should be a maximal subgroup of G_{i-1} normal in G, such a series is called a *chief series* of G.

Theorem 9 *Let G be a p-group. Then*
1) $Z(G) > 1$
2) It is $N_G(P) > P$ for any $P < G$. Especially, if $|G : P| = p$, then $P \triangleleft G$.
3) All factor groups G_{i-1}/G_i of a composition series or a chief series are of order p, $|G_{i-1}/G_i| = p$.
4) Each subgroup P of G belongs to some composition series, and each normal subgroup N of G to some chief series.

Theorem 10 *Let E be an elementary abelian subgroup of a 2-group G, and $g \in G$, $g^2 \in E$. Then $|C_E(g)|^2 \geq |E|$.*

Proof: Because of $g^2 \in E$ and E abelian, $x^{g^2} = x$ for any $x \in G$. Thus $(xx^g)^g = x^g x^{g^2} = x^g x = xx^g$, for all $x \in G$, and so $xx^g \in C_E(g)$. Now $xx^g = yy^g \Leftrightarrow xy = x^g y^g = (xy)^g \Leftrightarrow xy \in C_E(g) \Leftrightarrow C_E(g)x = C_E(g)y$. Therefore $xx^g \neq yy^g \Leftrightarrow C_E(g)x \neq C_E(g)y$, and so $|C_E(g)| \geq |E : C_E(g)| \Rightarrow |C_E(g)|^2 \geq |E|$.

Theorem 11 *(see Janko[1, Proposition 1.9]) Let G be a p-group with a non-abelian subgroup P of order p^3. If $C_G(P) \leq P$, then G is of maximal class. Especially, if $p = 2$, G is metacyclic.*

Theorem 12 *(see Janko [1, Proposition 1.13]) A 2-group G is metacyclic if and only if G and all its subgroups are generated by two elements.*

OVERVIEW OF 2-GROUPS OF ORDER $\leq 2^4$

Theorem 13 *Let G be a 2-group of order $\leq 2^4$. Then one of the following holds:*
(a) For $|G| = 2 : G \cong Z_2$

(b) For$|G| = 4 : G_1 \cong Z_4$ or $G_2 \cong E_4$

(c) For $|G| = 8$:
$G_1 \cong Z_8,\ G_2 \cong Z_4 \times Z_2,\ G_3 \cong E_8,$
$G_4 = \langle a, b|\ a^4 = 1, b^2 = 1, a^b = a^{-1}\rangle \cong D_8$ *- dihedral group,*
$G_5 = \langle a, b|\ a^4 = 1, b^2 = a^2, a^b = a^{-1}\rangle \cong Q_8$ *- quaternion group*

(d) For $|G| = 16$:

(α) abelian groups:
$G_1 \cong Z_{16}$, $G_2 \cong Z_8 \times Z_2$, $G_3 \cong Z_4 \times Z_4$, $G_4 \cong Z_4 \times E_4$,
$G_5 \cong E_{16}$

(β) nonabelian groups, $\exp G = 8$ - containing a cyclic maximal subgroup:
$G_6 = \langle a, b|\ a^8 = 1, b^2 = 1, a^b = a^{-1}\rangle \cong D_{16}$ *- dihedral group,*
$G_7 = \langle a, b|\ a^8 = 1, b^2 = a^4, a^b = a^{-1}\rangle \cong Q_{16}$ *- quaternion group,*
$G_8 = \langle a, b|\ a^8 = 1, b^2 = 1, a^b = a^3\rangle \cong SD_{16}$ *- semidihedral group,*
$G_9 = \langle a, b|\ a^8 = 1, b^2 = 1, a^b = a^5\rangle \cong M_{16}$ *- M-group.*

(γ) nonabelian groups, $\exp G = 4$*:*

1) $G > L \cong E_8$*:*

$G_{10} = \langle a,b,c |\ a^4 = 1, b^2 = c^2 = 1, a^b = a^{-1}, [a,c] = [b,c] = 1\rangle \cong D_8 \times Z_2$,
$G_{11} = \langle a,b,c |\ a^2 = b^2 = 1, c^4 = 1, [a,b] = [a,c] = 1, b^c = ab\rangle = \langle a,b\rangle \cdot \langle c\rangle \cong E_4 \cdot Z_4$*, the semidirect product of* E_4 *by* Z_4*.*

2) $G > L \Rightarrow L \ncong E_8$*. Then, by Th.8 and Th.9, there is some* $K \lhd G$, $|K| = 4$*. If* $K \cong Z_4$*, the order* $|AutK| = 2$ *and if* $K \cong E_4$*, the order* $|AutK| = 6$*. Thus* $|N_G(K)/C_G(K)| \leq 2$*, and there exists some abelian subgroup* $H < G$, $|H| = 8$*, such that* $K < H$*. As* $H \ncong Z_8, E_8$*, it is* $H \cong Z_4 \times Z_2$*. There are two possibilities:*

2a) $G > H \cong Z_4 \times Z_2$ *and* $\exists x \in G\backslash H,\ x^2 = 1$*:*

$G_{12} = \langle a,b,c |\ a^4 = 1,\ b^2 = c^2 = a^2,\ a^b = a^{-1}, [a,c] = [b,c] = 1\rangle = \langle a,b\rangle * \langle c\rangle \cong Q_8 * Z_4$*, the central product of* Q_8 *and* Z_4*.*

2b) $G > H \cong Z_4 \times Z_2$ *and* $x \in G\backslash H \Rightarrow x^2 \neq 1$*, that is* $|x| = 4$*:*

$G_{13} = \langle a,b,c |\ a^4 = 1, b^2 = a^2, c^2 = 1, a^b = a^{-1}, [a,c] = [b,c] = 1\rangle = \langle a,b\rangle \times \langle c\rangle \cong Q_8 \times Z_2$,
$G_{14} = \langle a,b \mid a^4 = b^4 = 1,\ a^b = a^{-1}\rangle = \langle a\rangle \cdot \langle b\rangle \cong Z_4 \cdot Z_4$*, the semidirect product of* Z_4 *by* Z_4*.*

PROOF OF THE THEOREM 2

A. Groups of order 16.

As group E_8 is the only (minimal) nonmetacyclic group of order 8, the second metacyclic groups of order 16 are those among them which contain E_8. According to Th.11(d), we have four such groups: $Z_4 \times E_4$, E_{16}, $D_8 \times Z_2$ and $E_4 \cdot Z_4$, the semidirect product of E_4 by Z_4.

B. Groups of order 32.

According to Th.1, such a group contains a subgroup isomorphic to $Q_8 \times Z_2$, or to $Q_8 * Z_4$, the central product of Q_8 by Z_4.

B1. $G > H \cong Q_8 \times Z_2$.

Let $H = \langle a, b, c \mid a^4 = 1, b^2 = a^2, c^2 = 1, a^b = a^{-1}, [a,c] = [b,c] = 1, \rangle = \langle a, b\rangle \times \langle c\rangle \cong Q_8 \times Z_2$. Now, $|G : H| = 2$, $G = \langle H, d\rangle$, $d^2 \in H$. Since $\Phi(H) = \mho_1(H) = \langle x^2 \mid x \in H\rangle = \langle a^2\rangle$ and $\Omega_1(H) = \langle x \in H \mid x^2 = 1\rangle = \langle a^2, c\rangle$, thus $\langle a^2\rangle$, $\langle a^2, c\rangle$ char H and so $\langle a^2\rangle$, $\langle a^2, c\rangle \lhd G$. The maximal subgroups of H are the following ones: $\langle a, b\rangle \cong \langle a, bc\rangle \cong \langle ac, b\rangle \cong \langle ac, bc\rangle \cong Q_8$, and $\langle a, c\rangle \cong \langle b, c\rangle \cong \langle ab, c\rangle \cong Z_4 \times Z_2$.

We use the bar convention for subgroups and elements of factor groups. For $\overline{G} = G/\langle a^2\rangle$, we have $\overline{H} = H/\langle a^2\rangle = \langle \overline{a}, \overline{b}, \overline{c}\rangle \cong E_8$, and $\overline{G} = \langle \overline{H}, \overline{d}\rangle$, $\overline{d}^2 \in \overline{H}$. By Th.10, it is $|C_{\overline{H}}(\overline{d})|^2 \geq |H| = 8$, and so $|C_{\overline{H}}(\overline{d})| \geq 4$.

On the other hand, $\langle a^2, c\rangle \lhd G$ implies $\langle \overline{c}\rangle \lhd \overline{G}$ and $\overline{c} \in C_{\overline{H}}(\overline{d})$. As $|C_{\overline{E}}(\overline{d}) \cap \langle \overline{a}, \overline{b}\rangle| \geq 2$, some of the elements $\overline{a}$, $\overline{b}$, or $\overline{a}\overline{b}$ is contained in $C_{\overline{E}}(\overline{d})$, and we can assume without loss that $\overline{a} \in C_{\overline{E}}(\overline{d})$.

Now, $\overline{b}^{\overline{d}} \in \overline{H} \backslash \langle \overline{a}, \overline{c}\rangle = \langle \overline{a}, \overline{c}\rangle \cdot \overline{b}$, and so:

$G = \langle H, d \mid d^2 \in H,\ a^d = az_1,\ b^d = a^\varepsilon c^\eta b z_2,\ c^d = cz_3\rangle$,

$\varepsilon, \eta \in \{0, 1\},\ z_1, z_2, z_3 \in \{1, a^2\}$.

There are 3 cases with respect to the element d.

1) $\exists d \in G\backslash H$, s.th. $d^2 = 1$.

Since $\langle a^2, c, d\rangle \not\cong E_8$ it must be $c^d \neq c$, and so $c^d = a^2 c$. If $a^d = a^3$, then $(ac)^d = a^d c^d = a^3 a^2 c = ac$, and replacing a with ac, we have without loss $a^d = a$, $c^d = a^2 c$.

Now $b^{d^2} = b^1 = b = (b^d)^d = (a^\varepsilon c^\eta b z_2)^d = a^\varepsilon a^{2\eta} c^\eta a^\varepsilon c^\eta b z_2 z_2 = a^{2(\varepsilon+\eta)} b$. Therefore $\varepsilon = \eta = 0$ or $\varepsilon = \eta = 1$.

If $\varepsilon = \eta = 0$, then $b^d = bz_2$. For $z_2 = a^2$ it is $(bc)^d = ba^2a^2c = bc$, and replacing bc with b, we have without loss $b^d = b$. Thus

$$G_1 = \langle H, d \mid d^2 = 1, [a,d] = [b,d] = 1, c^d = a^2 c\rangle = \langle a, b\rangle * \langle c, d\rangle \cong Q_8 * D_8.$$

If $\varepsilon = \eta = 1$, then $b^d = acbz_2 = az_2bc$. Now, replacing a with az_2, we get without loss $b^d = abc$, and the group:

$$G_2 = \langle H, d \mid d^2 = 1, a^d = a, b^d = abc, c^d = a^2c \rangle$$

.

2) $x \in G \backslash H \Rightarrow x^2 \neq 1$, $\exists d \in G \backslash H$ s.th. $d^4 = 1$.
Now $d^2 \in H$ and d^2 is an involution, $d^2 = \{a^2, c, a^2c\}$. As a^2c and c are interchangeable, we may assume that $d^2 \in \{a^2, c\}$. If $d^2 = c$, then $c^d = (d^2)^d = d^2 = c$. If $d^2 = a^2$, then $(cd)^2 = cd^2c^d = ca^2cz_2 = a^2z_3 \neq 1$, by our assumption. Thus $z_3 = 1$, and so $c^d = c$ in both cases.

2a) Case $d^2 = a^2$:
Now, $(ad)^2 = ad^2a^d = a^3az_1 = z_1 \neq 1$, by our assumption. Thus $z_1 = a^2$ and so $a^d = a^3$, $c^d = c$, $d^2 = a^2$.
For $\varepsilon = 1$, $b^d = ac^\eta bz_2$. Replacing a with az_2, we may assume $b^d = ac^\eta b$. If $\eta = 1$, $b^d = acb$, and replacing a with ac, we get $b^d = ab$, as in the case $\eta = 0$, and so

$$G_3 = \langle H, d \mid d^2 = a^2, a^d = a^{-1}, b^d = ab, c^d = c \rangle$$

. For $\varepsilon = 0$, it is $b^d = c^\eta bz_2$. If $z_2 = a^2$, then $b^{ad} = (a^2b)^d = a^2c^\eta ba^2 = c^\eta b$. Replacing d with ad, we get $b^d = c^\eta b$. For $\eta = 0$, $b^d = b$ and $(bd)^2 = bd^2b^d = ba^2b = 1$, a contradiction. Thus, $\eta = 1$, $b^d = bc$, and we have:

$$G_4 = \langle H, d \mid d^2 = a^2, a^d = a^{-1}, b^d = bc, c^d = c \rangle$$

.

2b) Case $d^2 = c$:
For $\varepsilon = 1$, $b^d = ac^\eta bz_2$, and replacing a with az_2, $b^d = ac^\eta b$. Again if $\eta = 1$, replacing a with ac, we get $b^d = ab$. Now, $b^{d^2} = b = (b^d)^d = (ab)^d = a^db^d = az_1ab \Rightarrow z_1 = a^2$, and so:

$$G_5 = \langle H, d \mid d^2 = c, a^d = a^{-1}, b^d = ab, c^d = c \rangle.$$

For $\varepsilon = 0$, $b^d = c^\eta b z_2$. If $\eta = 1$, then $b^d = cbz_2$, and $(bd)^2 = bd^2b^d = bccbz_2 = b^2z_2 = a^2z_2 \neq 1$, by our assumption. Thus $z_2 = 1$ and $(bd)^2 = a^2$, which leads to the case 2a). Thus we may assume that $\eta = 0$, and so $b^d = bz_2,\ a^d = az_1,\ c^d = c$. Thus $(a^d, b^d) \in \{(a,b),(a,b^3),(a^3,b),(a^3,b^3)\}$. As a, b and ab are interchangeable here, and for $a^d = a^3,\ b^d = b^3 \Rightarrow (ab)^d = a^3b^3 = ab$, there remain, without loss, only two cases: $a^d = a,\ b^d = b$ and $a^d = a,\ b^d = b^3$.

In the latter case $(ad)^2 = ad^2a^d = aca = a^2c,\ a^{ad} = a^d = a,\ b^{ad} = (b^3)^d = b^9 = b,\ (a^2c)^{ad} = (a^2c)^d = a^2c$, and replacing c with a^2c, and d with ad, we get without loss, that $a^d = a,\ b^d = b,\ c^d = c$, and thus

$$G_6 = \langle H, d \mid d^2 = c, [a,d] = [b,d] = [c,d] = 1\rangle$$

.

3) $d \in G\backslash H \Rightarrow |d| = 8$

Now, $d^2 \in H$ and $|d^2| = 4$. As all elements of order 4 in H are interchangeable, we may assume that $d^2 = a$, and so $a^d = a$. Now $d^2 = a,\ a^d = a,\ b^d = a^\varepsilon c^\eta b z_2,\ c^d = cz_3$. If $\varepsilon = 1$, then $(bd)^2 = bd^2b^d = baac^\eta bz_2 = c^\eta z_2$, an involution, against our assumption. Therefore $\varepsilon = 0$, and $b^d = c^\eta bz_2$. Now $b^{d^2} = b^a = b^3 = (b^d)^d = (c^\eta bz_2)^d = c^\eta z_3^\eta c^\eta bz_2z_2 = z_3^\eta b \Rightarrow z_3^\eta = b^2 = a^2 \Rightarrow z_3 = a^2,\ \eta = 1$. Thus $b^d = bcz_2,\ c^d = a^2c,\ a^d = a$. If $z_2 = a^2$, replacing c with a^2c, we get $b^d = bc$ and finally:

$$G_7 = \langle H, d \mid d^2 = a, a^d = ab^d = bc, c^d = a^2c\rangle.$$

B2. $G > H \cong Q_8 * Z_4$, and if $G > L$ then $L \ncong Q_8 \times Z_2$.

Let $H = \langle a, b, c \mid a^4 = 1, b^2 = c^2 = a^2, a^b = a^{-1}, [a, c] = [b, c] = 1\rangle$. Again, $G = \langle H, d\rangle,\ d^2 \in H$. Now $\mho_1(H) = \Phi(H) = \langle a^2\rangle,\ Z(H) = \langle c\rangle$.

There are 8 elements of order 4: $a,\ a^3,\ b,\ a^2b,\ ab,\ a^3b,\ c,\ a^2c$, and 7 involutions: $a^2,\ ac,\ a^3c,\ bc,\ a^2bc,\ abc,\ a^3bc$. The maximal subgroups of H are: $\langle a, b\rangle \cong Q_8,\ \langle a, c\rangle \cong \langle b, c\rangle \cong \langle ab, c\rangle \cong Z_4 \times Z_2$ and $\langle a, bc\rangle \cong \langle b, ac\rangle \cong \langle ab, ac\rangle \cong D_8$. Obviously, $\langle c\rangle,\ \langle a, b\rangle$ char H and so $\langle c\rangle,\ \langle a, b\rangle \lhd G$.

Again, for $\overline{G} = G/\langle a^2\rangle$, we have $\overline{H} \cong H/\langle a^2\rangle = \langle \overline{a}, \overline{b}, \overline{c}\rangle \cong E_8$, and $|C_{\overline{H}}(\overline{d})| \geq 4$ by Th.10. We may assume, without loss, that $C_{\overline{H}}(\overline{d}) \geq \langle \overline{a}, \overline{c}\rangle$, which implies $\overline{b}^{\overline{d}} \in \{\overline{b}, \overline{a}\overline{b}\}$. Returning to the originals, it means that either

1) $G = \langle H, d \mid d^2 \in H,\ a^d = az_1,\ b^d = bz_2,\ c^d = cz_3\rangle$, or

2) $G = \langle H, d \mid d^2 \in H,\ a^d = az_1,\ b^d = abz_2,\ c^d = cz_3\rangle$,

where $z_1, z_2, z_3 \in \langle a^2\rangle$.

Case 1):

1a) $\exists d \in G\backslash H,\ |d| = 2$:

$\langle a^2, ac, d\rangle,\ \langle a^2, bc, d\rangle,\ \langle a^2, abc, d\rangle \ncong E_8$, thus $(ac)^d = acz_1z_3 \neq ac$, $(bc)^d = bcz_2z_3 \neq bc,\ (abc)^d = abcz_1z_2z_3 \neq abc \Rightarrow z_1z_3, z_2z_3, z_1z_2z_3 \neq 1 \Rightarrow z_1 = z_2 = 1,\ z_3 = a^2$, and so:

$G = \langle H, d \mid d^2 = 1, [a, d] = [b, d] = 1, c^d = a^2c\rangle = \langle a, b\rangle * \langle c, d\rangle \cong Q_8 * D_8$.

But now $G > \langle a, b, d\rangle = \langle a, b\rangle \times \langle d\rangle \cong Q_8 \times Z_2$, against the assumption. G is isomorphic to G_1.

1b) $x \in G\backslash H \Rightarrow x^2 \neq 1,\ \exists d \in G\backslash H,\ |d| = 4$:

Now, d^2 is an involution on H. We may assume, without loss, that $d^2 = a^2$ or $d^2 = ac$.

If $d^2 = a^2$, then $(ad)^2 = ad^2a^d = aa^2az_1 = z_1 \neq 1,\ (bd)^2 = z_2 \neq 1$, and $(abd)^2 = z_1z_2 \neq 1$, a contradiction.

If $d^2 = ac$, then $b^{d^2} = b^{ac} = b^3 = (b^d)^d = (bz_2)^d = bz_2z_2 = b$, a contradiction again.

1c) $x \in G\backslash H \Rightarrow |x| = 8,\ d \in G\backslash H,\ d^2 \in H$:

We may assume, without loss, that $d^2 = a$, or $d^2 = c$.

If $d^2 = a,\ b^{d^2} = (bz_2)^d = b = b^a = b^3$, a contradiction.
Thus $d^2 = c,\ a^d = az_1,\ b^d = bz_2,\ c^d = c$. Now $(a^d, b^d) \in \{(a,b),(a,b^3),(a^3,b),(a^3,b^3)\}$
In the latter case $(ab)^d = a^3b^3 = ab$, and since a, b and ab may be replaced with each other, we may assume that: $a^d = a,\ b^d = b$ or $a^d = a,\ b^d = b^3$.
In the latter case $(ad)^2 = a^2c,\ b^{ad} = b$, and thus replacing c with a^2c and d with ad, the second case is reduced to the first, and we get

$$G_8 = \langle H, d \mid d^2 = c, [a,d] = [b,d] = [c,d] = 1\rangle = \langle a, b\rangle * \langle d\rangle \cong Q_8 * Z_8.$$

Case 2):
Replacing a with az_2, we may assume that $b^d = ab$.
2a) $\exists d \in G\backslash H,\ |d| = 2:$
$\langle a^2, ac, d\rangle \not\cong E_8 \Rightarrow (ac)^d = acz_1z_3 \neq ac \Rightarrow z_3 \neq z_1$. If $a^d = a$, then $b^{d^2} = b = (b^d)^d = (ab)^d = aab = b^3$, a contradiction. Therefore $a^d = a^3,\ c^d = c$, and

$$G_9 = \langle H, d \mid d^2 = 1, a^d = a^{-1}, b^d = ab, c^d = c\rangle.$$

2b) $x \in G\backslash H \Rightarrow x^2 \neq 1,\ \exists d \in G\backslash H,\ |d| = 4:$
Now, without loss $d^2 = a^2$ or $d^2 = ac$ or $d^2 = bc$.
If $d^2 = a^2$, then $(ad)^2 = z_1 \neq 1$ and $(cd)^2 = z_3 \neq 1$, thus $z_1 = z_3 = a^2$. So $G = \langle H, d \mid d^2 = a^2,\ a^d = a^3,\ b^d = ab,\ c^d = c^3\rangle$.
Here $(ac)^d = a^3c^3 = ac$, and $G > \langle a, d, ac\rangle = \langle a, d\rangle \times \langle ac\rangle \cong Q_8 \times Z_2$, against the assumption. Actually, $G \cong G_2$.
If $d^2 = ac$, then $b^{d^2} = (ab)^d = az_1ab = b^{ac} = b^3$, and so $z_1 = 1$. Now, $(ac)^d = ac = a^dc^d = acz_3$, thus also $z_3 = 1$. Therefore:

$$G_{10} = \langle H, d \mid d^2 = ac, a^d = a, b^d = ab, c^d = c\rangle.$$

If $d^2 = bc$, then $(bc)^d = bc = abcz_3 = az_3bc$, implying $az_3 = 1$, a contradiction.
2c) $x \in G\backslash H \Rightarrow |x| = 8,\ d \in G\backslash H,\ d^2 \in H:$
We may assume, without loss, that $d^2 = a$ or $d^2 = b$ or $d^2 = c$.
For $d^2 = a$, we get $(bd)^2 = bd^2b^d = baab = 1$, a contradiction. As $b^d = ab$, it

cannot be $d^2 = b$. If $d^2 = c$, then $(bd)^2 = bcab = b^2ca^b = a^2ca^{-1} = ac$, and so $|bd| = 4$, a contradiction again.

C. Groups of order 64.

According a previous remark and by Th.1(d), such a group G contains a subgroup $H = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2b^2, a^b = a, a^c = a^{-1}, b^c = a^2b^3 \rangle$, where $\Omega_1(H) = \langle x \in H \mid x^2 = 1 \rangle = Z(H) = \Phi(H) = \langle a^2, b^2 \rangle \equiv K \cong E_4$. One can easily check that there are only 4 square roots for a^2 (that is, such $x \in H$, that $x^2 = a^2$), and 12 square roots for b^2 and a^2b^2 each. Thus $A = \langle a^2 \rangle \mathrm{char} H$. The square roots of a^2 generate the subgroup $N = \langle a, b^2 \rangle \cong Z_4 \times Z_2$. The group $L = \langle a, b \rangle$ is the unique subgroup of H isomorphic to $Z_4 \times Z_4$. Thus, A, K, N, L are all characteristic in H and consequently normal in G, as $H \lhd G$.

It can easily be seen that

$$(*) \begin{cases} AutH = \Phi \sqcup \Psi, \text{where} \\ \Phi = \{\varphi | \varphi : a \mapsto a\zeta_1,\ b \mapsto b\zeta_2,\ c \mapsto a^{\alpha} b^{2\beta} c\}, \\ \Psi = \{\psi | \psi : a \mapsto a\zeta_1,\ b \mapsto ab\zeta_2,\ c \mapsto a^{\alpha} b^{1+2\beta} c\}, \\ \text{and } \zeta_1, \zeta_2 \in K,\ \alpha \in \{0,1,2,3\},\ \beta \in \{0,1\} \end{cases}$$

As $A \lhd K \lhd N \lhd L \lhd H \lhd G$ is a normal chain G, we have:

$$(**)\ G = \langle H, d \mid (H),\ d^2 \in H,\ a^d = az_1,\ b^d = a^{\varepsilon} b z_2,\ c^d = a^{\gamma} b^{\delta} z_3 c \rangle,$$

where $z_1, z_2, z_3 \in K$ and $\gamma, \delta, \varepsilon \in \{0, 1\}$.

We split our proof into several steps.

(i) If $T < G,\ |T| = 8$, than T is abelian and $T \ncong E_8$:

Let T be a nonabelian subgroup of order 8 in G, thus $T \cong D_8$ or $T \cong Q_8$ and $|Z(T)| = 2$. If $C_G(T) \leq T$, then G is metacyclic by Th.11, a contradiction. Hence $C_G(T) \nleq T$ and take in $C_G(T)$ a subgroup U of order 4 containing

$Z(T)$. Now $\langle T, U\rangle = T * U$, the central product of T and U, is isomorfic to some of the groups $D_8 \times Z_2$, $Q_8 \times Z_2$, or $Q_8 * Z_4 \cong D_8 * Z_4$. Thus $\langle T, U\rangle$ would be a non-metacyclic subgroup of order 16 in G, a contradiction. Therefore every subgroup T of order 8 in G is abelian and, being metacyclic, it must be $T \ncong E_8$.

(*ii*) If $g \in G \backslash H$, then $g^2 \neq 1$:
If $g^2 = 1$, then $T = \langle K, g\rangle = \langle a^2, b^2, g\rangle$ is abelian, by (*i*), and isomorphic to E_8, a contradiction.

(*iii*) $G/L \cong E_4$
Else $G/L \cong Z_4$ and $G = \langle L, d \mid d^2 \in H \setminus L = Lc\rangle$. By (*i*) we see that we can assume without loss that $d^2 = c$, and so $d^4 = c^2 = a^2b^2$. Now $a^{d^2} = a^c = a^3 = (a^d)^d = (az_1)^d = az_1z_1^d$ implying $z_1^d = a^2z_1$ and thus $z_1 \in \{b^2, a^2b^2\}$ and $(b^2)^d = a^2b^2 = c^2 = (c^2)^d$, and so $b^2 = c^2$, a contradiction.

(*iv*) If there exists some $d \in G \backslash H$, s.th. $|d| = 4$, then either:
$G \cong G_1 = \langle a, b, c, d \mid (H),\ d^2 = a^2,\ a^d = a^3b^2,\ b^d = b^3,\ c^d = c\rangle$, or $G \cong G_2 = \langle a, b, c, d \mid (H),\ d^2 = b^2,\ a^d = a^3,\ b^d = a^2b^3,\ c^d = ac\rangle$.
As $d^2 \in L$, and d^2 is an involution, we have $d^2 \in \Omega_1(L)^\sharp = \{a^2, b^2, a^2b^2\} = K^\sharp$. By (*i*), $\langle a^2, b^2, d\rangle$ is abelian, and so $b^d = bz_2$. Now $(c^2)^d = c^2 = (c^d)^2 = (a^\gamma b^\delta z_3 c)^2 = a^\gamma b^\delta z_3 c^2 a^{3\gamma} a^{2\delta} b^{3\delta} z_3 = a^{2\delta} c^2$, implying $d = 0$ and $c^d = a^\gamma z_3 c$.
From $c^{ad} = (a^2c)^d = a^2c^d$, $c^{bd} = (a^2b^2c)^d = a^2b^2c^d$, and $c^{abd}(a^2c)^{bd} = (a^2a^2b^2c)^d = b^2c^d$, and, by (*ii*), $(ad)^2 = ad^2a^d = a^2d^2z_1 \neq 1$, $(bd)^2 = b^2d^2z_2 \neq 1$ and $(abd)^2 = a^2b^2d^2z_1z_2 \neq 1$, we conclude that

(1) $z_1 \neq a^2d^2$, $z_2 \neq b^2d^2$, $z_1z_2 \neq a^2b^2d^2$, and replacing, if needed, d by ad or bd or abd, we can assume, without loss, that $c^d = a^\gamma c$, and so:

(2) $c^d = c$ or $c^d = ac$.

Case 1) $c^d = c$:
Now $(cd)^2 = c^2d^2 = a^2b^2d^2 \neq 1$, by (*ii*), and so $d^2 \neq a^2b^2$. Therefore $d^2 \in \{a^2, b^2\}$. If $d^2 = b^2$, then $(cd)^2 = c^2d^2 = a^2$. Thus, replacing d by cd, we

may assume without loss that $d^2 = a^2$. Besides of (1), we also have now the following conditions on z_1, z_2: $(acd)^2 = acd^2az_1c = ac^2d^2a^cz_1 = c^2d^2z_1 \neq 1$, and similarly $(bcd)^2 = b^2d^2z_2 \neq 1,\ (abcd)^2 = abcd^2az_1bz_2c = b^2d^2z_1z_2 \neq 1$, that is:

(3) $z_1 \neq c^2d^2,\ z_2 \neq b^2d^2,\ z_1z_2 \neq b^2d^2$.

From (1) and (3) we get:
$z_1 \neq 1, b^2,\ z_2 \neq a^2b^2,\ z_1z_2 \neq b^2, a^2b^2$. If $z_1 = a^2$, then $a^d = a^3$, and $\langle a, d \mid a^4 = 1,\ d^2 = a^2,\ a^d = a^3\rangle \cong Q_8$, a contradiction, by (i). Thus $z_1 = a^2b^2,\ z_2 \neq a^2b^2,\ a^2b^2z_2 \neq b^2, a^2b^2$, and so $z_2 = b^2$, giving:

$G_1 = \langle a, b, c, d \mid (H), d^2 = a^2,\ a^d = a^3b^2,\ b^d = b^3,\ c^d = c\rangle$.

Case 2) $c^d = ac$:

We already know that $a^d = az_1,\ b^d = bz_2$ and $d^2 \in \{a^2, b^2, a^2b^2\}$. Now $c = c^{d^2} = (c^d)^d = (ac)^d = az_1ac = a^2z_1c$, so $z_1 = a^2$ and $a^d = a^3$. If $d^2 = a^2$, then $\langle a, d \mid a^4 = 1,\ d^2 = a^2,\ a^d = a^3\rangle \cong Q_8$, against (i). Therefore $d^2 \in \{b^2, a^2b^2\}$. Since $(bd)^2 = bd^2b^d = b^2z_2d^2$, we have $z_2 \neq b_2d_2$.

Case 2.1) $d^2 = b^2$:

From $z_2 \neq b^2d^2$ it follows $z_2 \in \{a^2, b^2, a^2b^2\}$. If $z_2 = a^2$, then $b^d = a^2b$ and $\langle a, bd \mid a^4 = 1,\ (bd)^2 = a^2,\ a^{bd} = a^3\rangle \cong Q_8$, against (i). Similarly, for $z_2 = b^2$ we have $\langle b, d \mid b^4 = 1,\ d^2 = b^2,\ b^d = b^3\rangle \cong Q_8$ again. It remains as the only possibility $z_2 = a^2b^2$, and we obtain the group

$G_2 = \langle a, b, c, d \mid (H), d^2 = b^2,\ a^d = a^3,\ b^d = a^2b^3,\ c^d = ac\rangle$.

Case 2.2) $d^2 = a^2b^2$:

Because of $z_2 \neq b^2d^2 = a^2$, we have now $z_2 \in \{1, b^2, a^2b^2\}$, that is $b^d \in \{b, b^3, a^2b^3\}$. If $b^d = b$, then $(bd)^2 = bd^2b^d = ba^2b^2b = a^2$, and $\langle a, bd\rangle \cong Q_8$, against (i). If $b^d = b^3$, then $(ab)^d = (ab)^3$ and $\langle ab, d\rangle \cong Q_8$, again the same contradiction. Therefore $b^d = a^2b^3$. Replacing a by $a' = a^3b^2,\ b$ by $b' = ab,\ c$ by $c' = bc$, we get:

$d^2 = b'^2,\ a'^d = (a^3b^2)^d = ab^2 = a'^3,\ b'^d = (ab)^d = a^3a^2b^3 = ab^3 = (a^3b^2)^2 \cdot (ab)^3 = a'^2b'^3,\ c'^d = (bc)^d = a^2b^3ac = a^3b^2bc = a' \cdot c'$, that is the relations of G_2. Thus, this group is isomorphic to G_2.

(v) If all elements in $G\backslash H$ are of order 8, then

$G = G_3 = \langle a, b, c, d \mid (H), d^2 = b,\ a^d = ab^2,\ b^d = b,\ c^d = ac\rangle$.

As $G/L \cong E_4,\ G = \langle H, d\rangle,\ |d| = 8$, it follows that d^2 is an element of order 4 in L. According to $(*)$, all such elements are replaceable by a or b, and so we may assume without loss that:

$G = \langle H, d \mid (H), d^2 \in \{a, b\},\ a^d = az_1,\ b^d = a^\varepsilon bz_2,\ c^d = a^\gamma b^\delta z_3 c\rangle$, where $z_1, z_2, z_3 \in K,\ \gamma, \delta, \varepsilon \in \{0, 1\}$.

Now $(c^2)^d = (c^d)^2 = (a^\gamma b^\delta z_3 c)^2 = a^\gamma b^\delta z_3 c^2 a^{3\gamma} a^{2\delta} b^{3\delta} z_3 = a^{2\delta} c^2$.

Case 1) $d^2 = a$:

Now $a^d = a$. If $\varepsilon = 0$ then $z^d = z$ for $z \in K$, and from $(c^2)^d = c^2 = a^{2\delta}c^2$ it follows $\delta = 0,\ c^d = a^\gamma z_3 c$. Similarly, from $c^{d^2} = c^a = a^2 c = (a^\gamma z_3 c)^d = a^\gamma z_3 a^\gamma z_3 c = a^{2\gamma} c$, we get $\gamma = 1$ and $c^d = az_3 c$. But now $(cd)^2 = cd^2 c^d = a^2 c^2 z_3 \in K$, and $|cd| = 4$, against our assumption.

If $\varepsilon = 1$, then $b^d = abz_2,\ (b^2)^d = c^2$ and $(c^2)^d = b^2 = a^{2\delta}c^2$. Thus $\delta = 1$, and $c^d = a^\gamma bz_3 c$. Therefore, $c^{d^2} = a^2 c = (a^\gamma bz_3 c)^d = a^\gamma \cdot abz_2 z_3^d a^\gamma bz_3 c = a \cdot a^{2\gamma} bb^2 z_2 z_3 z_3^d c = azc$, for some $z \in K$. This implies $z = a$, a contradiction because a is not in K.

Case 2) $d^2 = b$:

Now $b^d = b$, and $z^d = z$ for $z \in K$. From $(c^2)^d = c^2 = a^{2\delta}c^2$ it follows $\delta = 0,\ c^d = a^\gamma z_3 c$. Since $[b, c] = b^{-1}b^c = a^2 b^2 = [c, b]$, we have $c^{d^2} = c^b = a^2 b^2 c = (a^\gamma z_3 c)^d = a^\gamma z_1^\gamma z_3 a^\gamma z_3 c$, implying $a^{2\gamma} z_1^\gamma = a^2 b^2$. Thus $\gamma = 1,\ z_1 = b^2$ and so $a^d = ab^2,\ c^d = az_3 c$. Replacing a by az_3, we get the group G_3 as stated above.

Remarks :

It is of some interest to check the maximal subgroups of second-metacyclic groups. We present them in the following table:

(i) $|G| = 32$:

$G_1 \rightarrow\ 5 \cdot (Q_8 \times Z_2),\ 10 \cdot (Q_8 * Z_4)$

$G_2 \rightarrow\ Q_8 \times Z_2,\ Q_8 * Z_4,\ 2 \cdot SD_{16},\ 2 \cdot Q_{16},\ M_{16}$

$G_3 \rightarrow 2 \cdot (Q_8 \times Z_2),\ Z_8 \times Z_2,\ 4 \cdot Q_{16}$

$G_4 \rightarrow 2 \cdot (Q_8 \times Z_2),\ Z_4 \times Z_4,\ 4 \cdot (Z_4 \cdot Z_4)$

$G_5 \rightarrow Q_8 \times Z_2,\ Z_4 \cdot Z_4,\ Z_8 \times Z_2$

$G_6 \rightarrow Q_8 \times Z_2,\ 3 \cdot (Z_4 \times Z_4),\ 3 \cdot (Z_4 \cdot Z_4)$

$G_7 \rightarrow Q_8 \times Z_2,\ 2 \cdot M_{16}$

$G_8 \rightarrow Q_8 * Z_4,\ 3 \cdot (Z_8 \times Z_2),\ 3 \cdot M_{16}$

$G_9 \rightarrow 2 \cdot (Q_8 * Z_4),\ Z_8 \times Z_2,\ 2 \cdot SD_{16},\ Q_{16},\ D_{16}$

$G_{10} \rightarrow Q_8 * Z_4,\ Z_4 \times Z_4,\ M_{16}$

(ii) $|G| = 64$:

Denoting $H_r = \langle a, b \mid a^8 = b^4 = 1,\ a^b = a^r \rangle$, we have:

$G_1 \rightarrow 15 \cdot H$

$G_2 \rightarrow 2 \cdot H,\ 2 \cdot H_3,\ 2 \cdot H_7,\ Z_8 \times Z_4$

$G_3 \rightarrow H,\ 2 \cdot H_5.$

The factor groups $\overline{G}_i = G_i / \langle a^2 \rangle,\ i = 1, 2, 3,$ are isomorphic to the groups $G_1,\ G_4$ and G_7 of order 32 from (i), respectively.

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