Quantum Logic for Quantum Computers

Mladen Pavičić[†,1]

[†]Department of Mathematics, University of Zagreb, GF, Kačićeva 26, POB-217, HR-10001 Zagreb, Croatia.

Abstract. The following results obtained within a project of finding algebra of states in a general purpose quantum computer are reported: 1. all operations of an orthomodular lattice, including the identity, are five-fold defined; 2. there are non-orthomodular models for both quantum and classical logics; 3. there is a 4-variable orthoarguesian lattice condition which contains all known orthoarguesian lattice conditions including 6- and 5-variable ones.

PACS numbers: 03.65.Bz, 02.10.By, 02.10.Gd

Keywords: quantum logic, orthomodular lattices, quantum computation, classical logic, Boolean algebra, orthoarguesian properties, models

1 Introduction

A computer is a computational device in which a 2×2 unitary matrices called *logic gates* act on elementary bits $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$ and on bits obtained by such operations. A classical gate is for example a NOT gate which flips bits in the following way: $\mathsf{NOT}|0\rangle = \mathsf{NOT}(1,0) = |1\rangle$ and $\mathsf{NOT}|1\rangle = \mathsf{NOT}(0,1) = |0\rangle$ and which can be represented as

$$\mathsf{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

A quantum gate which is characteristic of the existing experimental hardware is the *controlled* NOT gate which acts on two qubits in a conditional way [as simple NOT gate on the second (target) qubit provided the first (control) qubit is 1] as follows:

$$\mathsf{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

[1]E-mail: mpavicic@faust.irb.hr

The transformation CNOT—and all other classical operations transformed to quantum gates by making them *controlled* ones—are obviously unitary, they do preserve superpositions, and they cannot be decomposed into a tensor product of two single-bit transformations, but without qubit rotations and without phase shifts

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \tag{3}$$

genuine quantum tasks cannot be processed. For example, even the simplest problem of a photon passing two successive polarizers (quantum Malus law) could not be solved. On the other hand, these non-classical rotations and phase shifts essential for quantum computers depend on classical continuous variables and this cause problems which we are going to focus on later on.

Even more essential difference between classical and quantum computers is contained in elementary information units themselves. Classical unit is always either 0 or 1 (one bit). Quantum unit—called a *qubit*—is a two state quantum system. We describe the system by a unit vector in the Hilbert space $\mathcal{H}^2$ over the field of complex numbers. We denote the two orthogonal states by $|0\rangle = (1,0)$ and $|1\rangle = (0,1)$. The states make an orthogonal basis for $\mathcal{H}^2$. In a quantum computer we deal with a big number n of qubits which build up a composite Hilbert space $\mathcal{H} = \mathcal{H}^2 \otimes \ldots \otimes \mathcal{H}^2$. The computational basis, i.e., the basis of this space, consists of the following 2^n vectors: $|00\cdots00\rangle$, $|00\cdots01\rangle, \ldots, |11\cdots11\rangle$, where, e.g., $|00\rangle$ means $|0\rangle \otimes |0\rangle$. Classical bits correspond to quantum states: $i_1i_2...i_n \longleftrightarrow |i_n\rangle \equiv |i_1....i_n\rangle$.

To compute the function $f : i_1i_2...i_n \longmapsto f(i_1, ...i_n)$. means to let the corresponding states evolve according to the time evolution unitary operator U:

$$|i_1i_2...i_n\rangle \longmapsto U|i_1i_2...i_n\rangle = |f(i_1, ...i_n)\rangle. \tag{4}$$

More explicitly

$$|\Psi_f\rangle = \exp\left(-\frac{i}{\hbar}\int \mathcal{H}dt\right)|\Psi_0\rangle = U|\Psi_0\rangle \tag{5}$$

which follows directly from the Schrödinger equation. The unitarity of U assures reversibility and therefore prevents energy dissipation. This can be achieved with classical devices as well but only at the cost of exponentially growing hardware or exponentially rising time. The reason for that is simple: n classical states describing a system in a classical computer can only be specified by ascribing values all 2^n basis states. Quantum computers on the other hand achieve the aim as well as a parallel way of computing—which is their most attractive feature—by using superposition which puts n quantum states in a superposition of all 2^n basis states in one step. Again, for a parallel computation a classical computer would need either an exponentially growing hardware or an exponentially rising time.

Consider for example the following state of 2 particles, known as the *entangled* state (Pavičić & Summhammer, 1994; Pavičić, 1995) of the particles which can then also be used for a teleportation of states or Bell experiments or quantum cryptography (we omit the normalization factors throughout):

$$|00\rangle + |11\rangle \tag{6}$$

Here no one of the two qubits has a definite state: the state of the system is not a tensor product of the states, and we cannot find a_1, a_2, b_1, b_2 such that

$$(a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle) = |00\rangle + |11\rangle$$

since

$$(a_1|0\rangle + b_1|1\rangle) \otimes (a_2|0\rangle + b_2|1\rangle) = a_1 a_2|00\rangle + a_1 b_2|01\rangle + b_1 a_2|10\rangle + b_1 b_2|11\rangle$$

and $a_1 b_2 = 0$ implies that either $a_1 a_2 = 0$ or $b_1 b_2 = 0$. These states represent situations that have no classical counterpart. These are also the states that provide the exponential growth of quantum state spaces with the number of particles.

To see this let us consider the following superposition of n qubits:

$$\sum_{i_1 i_2 \ldots i_n = 0}^{1} |i_1 i_2 \ldots i_n\rangle \tag{7}$$

Applying the linear unitary operation which computes f, from Eq. (4), to this state, yields:

$$\sum_{i_1, i_2, \ldots, i_n = 0}^{1} |f(i_1 i_2 \ldots i_n)\rangle. \tag{8}$$

Hence, U computes f *parallelly* on all the 2^n possible inputs i.

To achieve such a parallel computing in an assumed realistic computer, we start with an initial state $|i\rangle$ which corresponds to an "input" to the computation. We then perform elementary operations on the system using the quantum gates defined above. The operations correspond to the computational steps in the computation, just like logic gates are the elementary steps in classical computers, and are performed on an isolated system, so the evolution can always be described by a unitary matrix operating on the state of the system.

Therefore we can always implement a unitary operator which is given by the Hamiltonian of a given process or state of a system as a set of instructions on how to transform input states in time. But the crucial problem are initial states themselves. Can we write down a general input state:

$$|\Psi_0\rangle = |i_1 i_2 \ldots i_n\rangle \tag{9}$$

by means of quantum gate operations over elementary input propositions so as to correspond to a general wave function of the Hilbert space which describes the input state? The answer is currently in the negative. There is no known finite and definite receipt for such a correspondence. But in the recent years a lot has been achieved to narrow the gap between an algebra of elementary propositions (corresponding to pure states) and the Hilbert space description. Let us consider some most important details in a possible construction of a quantum machine language which could mimic quantum system and therefore directly correspond to its Hilbert space representation.

In Sec. 3 we show (using only lattice theory) that both a proper quantum logic of propositions and a proper classical logic of propositions have models that are not even orthomodular. Therefore in both classical and quantum computers one must use algebras instead. Such an algebra—orthomodular lattice—which is usually considered to be an algebra underlying quantum measurement and a Hilbert space representation we analyze in Sec. 2 and show that all its binary operations are ambiguous and that bare orthomodular lattices cannot be employed in quantum computers. In Sec. 4 we show how one can construct Hilbertian lattices which enable a direct representation in a Hilbert space of quantum computational simulation and provide a lattice for the purpose, which at the same time eliminates the so-called 4-dim postulate. We also show that quantum theory is at least so incompatible with the strong form of the Church-Turing principle as any classical theory contrary to the Deutsch's claim. (Deutsch, 1985)

2 All Operations in Orthomodular Lattices Are Ambiguous

The Birkhoff-von Neumann requirement (Kalmbach, 1983):

$$a \to_i b \qquad \Rightarrow \qquad a \le b, \qquad\qquad i = 1, \dots, 5, \tag{10}$$

where $a \to_1 b \stackrel{\text{def}}{=} a' \cup (a \cap b)$, $a \to_2 b \stackrel{\text{def}}{=} b' \to_1 a'$, $a \to_3 b \stackrel{\text{def}}{=} (a' \cap b) \cup (a' \cap b') \cup (a \to_1 b)$, $a \to_4 b \stackrel{\text{def}}{=} b' \to_3 a'$, and $a \to_5 b \stackrel{\text{def}}{=} (a \cap b) \cup (a' \cap b) \cup (a' \cap b')$, not only holds in every orthomodular lattice but also amounts to the orthomodularity itself in the sense that condition (10) added to an ortho-lattice makes it orthomodular. (Pavičić, 1987)

Since in any orthomodular lattice (Pavičić & Megill, 1998b)

$$a \cup b = (a \to_i b) \to_i (((a \to_i b) \to_i (b \to_i a)) \to_i a) \qquad i = 1, \dots, 5, \tag{11}$$

this means that all operations (five-fold negation follows trivially) in an orthomodular lattice are fivefold definable.

At the first sight it still seems that one can prove a conjecture that the relation of equation in the lattice is uniquely definable. The reasons for such a conjecture are the following ones. All five quantum implications $a \to_i b$ collapse to the classical one $a \to_0 b \stackrel{\text{def}}{=} a' \cup b$ in a distributive lattice. Even more

$$a \to_i b = a \to_j b, \qquad\qquad i \ne j; \qquad i, j = 0, \dots, 5, \tag{12}$$

makes an ortho-lattice distributive.(Pavičić, 1987) On the other hand

$$a \to_0 b \qquad \Rightarrow \qquad a \le b, \tag{13}$$

also makes an ortholattice distributive. (Pavičić, 1998) In any orthomodular lattice we have:

$$a \leftrightarrow_i b = a \equiv b, \qquad\qquad i = 1, \dots, 5 \tag{14}$$

where $a \leftrightarrow_i b \stackrel{\text{def}}{=} (a \to_i b) \cap (b \to_i a)$ and $a \equiv b \stackrel{\text{def}}{=} (a \cap b) \cup (a' \cap b')$. The identity operation $a \equiv b$ reduces to $a \equiv_0 b \stackrel{\text{def}}{=} (a' \cup b) \cap (b' \cup a)$ in a distributive theory and since $a \equiv b$ is an equivalence relation in an othomodular lattice and $a \equiv_0$ is an equivalence relation in a distributive lattice, we could hope that the conjecture does hold, i.e., that the relation of equation '=' in orthomodular lattices can be uniquely defined and connected to the operation of identity by the following rule

$$a \equiv b = 1 \qquad \Leftrightarrow \qquad a = b, \tag{15}$$

which is known to make an ortholattice orthomodular (Pavičić, 1993) and which can be compared to the rule

$$a \equiv_0 b = 1 \qquad \Leftrightarrow \qquad a = b, \tag{16}$$

which makes an ortholattice distributive (Pavičić, 1998).

Unfortunately, the conjecture does not hold. The reason is simple. In a distributive lattice $a \to_i b$, $i = 1, \ldots, 5$ all merge to $a \to_0 b$ and therefore $(a \to_i b) \cap (b \to_j a)$, $i \neq j$, $i, j = 1, \ldots, 5$ must merge to $a \equiv_0 b \stackrel{\text{def}}{=} (a \to_0 b) \cap (b \to_0 a)$. But in an orthomodular lattice the mixed biimplications $(a \to_i b) \cap (b \to_j a)$ are equal—depending on the values of i and j—to the following *five* identities $a \equiv b$, $a \equiv_1 b \stackrel{\text{def}}{=} (a \cup b') \cap (a' \cup (a \cap b))$, $a \equiv_2 b \stackrel{\text{def}}{=} (a \cup b') \cap (b \cup (a' \cap b'))$, $a \equiv_3 b \stackrel{\text{def}}{=} (a' \cup b) \cap (a \cup (a' \cap b'))$, and $a \equiv_4 b \stackrel{\text{def}}{=} (a' \cup b) \cap (b' \cup (a \cap b))$ as given in the Table 1. (Pavičić & Megill, 1999)

$i \downarrow \ \backslash \ j \to$	$b \to_0 a$	$b \to_1 a$	$b \to_2 a$	$b \to_3 a$	$b \to_4 a$	$b \to_5 a$
$a \to_0 b$	$a \equiv_0 b$	$a \equiv_4 b$	$a \equiv_3 b$	$a \equiv_2 b$	$a \equiv_1 b$	$a \equiv b$
$a \to_1 b$	$a \equiv_1 b$	$a \equiv b$	$a \equiv b$	$a \equiv b$	$a \equiv_1 b$	$a \equiv b$
$a \to_2 b$	$a \equiv_2 b$	$a \equiv b$	$a \equiv b$	$a \equiv_2 b$	$a \equiv b$	$a \equiv b$
$a \to_3 b$	$a \equiv_3 b$	$a \equiv b$	$a \equiv_3 b$	$a \equiv b$	$a \equiv b$	$a \equiv b$
$a \to_4 b$	$a \equiv_4 b$	$a \equiv_4 b$	$a \equiv b$	$a \equiv b$	$a \equiv b$	$a \equiv b$
$a \to_5 b$	$a \equiv b$	$a \equiv b$	$a \equiv b$	$a \equiv b$	$a \equiv b$	$a \equiv b$

Table 1: Products $(a \to_i b) \cap (b \to_j a)$, $i = 0, \ldots, 5$ (rows), $j = 0, \ldots, 5$ (columns). Identities $a \equiv_i b$, $i = 1, \ldots, 4$ are asymmetrical.

The expressions $a \equiv_i b$, $i = 1, \ldots, 4$ are all asymmetrical and at first we would think it would be inappropriate to name them identities. And also because $a \equiv_i b = a \equiv_j b$ when added to an ortho-lattice does not make it even orthomodular but apparently weakly distributive (see next section), as opposed to Eq. (12). Nevertheless, we are able to prove the following theorem. (Pavičić & Megill, 1999)

Theorem 2.1. *An ortholattice in which*

$$a \equiv_i b = 1 \qquad \Rightarrow \qquad a = b, \qquad\qquad i = 2, \ldots, 5 \tag{17}$$

holds is an orthomodular lattice and vice versa.

Hence, putting together Eq. (15) and Eq. (17) we have an indication that the relation of equivalence which establishes a connection between quantum logic and its models might turn out to be based on several different operations of identity at the same time thus making a direct evaluation of elementary logical propositions impossible. However, as we will see in the next section, there is an even more important reason why we cannot use proper quantum logic to evaluate quantum propositions and this is that a proper quantum logic is not orthomodular. As even bigger surprise comes the result that even standard classical logic need not be orthomodular.

3 Non-Orthomodular Models for Both Quantum and Classical Logics

A crucial difference between logics and lattices as their models is that properties that play a decisive role in lattices do not play such a role in logics at all. To explain this difference let us consider the orthomodularity and distributivity properties. When we add the orthomodularity (distributivity) property to an ortholattice it becomes an orthomodular (distributive) lattice. We can compare what happens in a logic by looking at a lattice we obtain by mapping logical axioms $\vdash A$ to an ortholattice where they take over the form $a = 1$; here $a = v(A)$ and v is a morphism from the logic to the lattice. As we have shown in (Pavičić & Megill, 1998a) the property $(a \cup (a' \cap (a \cup b))) \equiv (a \cup b) = 1$, we obtain by mapping the logical formula for "orthomodularity" $\vdash (A \vee (\neg A \wedge (A \vee B)) \equiv (A \vee B)$ into an ortholattice, is true in all ortholattices. On the other hand, as we have shown in (Pavičić & Megill, 1999), $(a \cap (b \cup c)) \equiv_0 ((a \cap b) \cup (a \cap c)) = 1$ which we obtain by an analogous mapping of the distributivity, is true in all weakly distributive lattices which are not even orthomodular.

The reason for such different structures of logics as opposed to their models lies in completely different syntax of $a = 1$ lattice equations (which correspond to logical wwf's) and $a = b$ lattice equations. For example, another way of expressing orthomodularity is $\vdash A \vee (B \wedge (\neg A \vee \neg B)) \equiv A \vee B$ whose lattice mapping is $((a \to_1 b) \to_0 b) \equiv (a \cup b) = 1$ which, when added to an ortholattice, makes it weakly orthomodular. This means that the "orthomodularity" from $\mathcal{QL}$ sometimes maps to an ortho-property and sometimes to a weakly orthomodular property but never to an orthomodular property. The reader can find details on weakly orthomodular logics and lattices in (Pavičić & Megill, 1998a, 1999).

Classical logic also does not necessarily map its syntactical structure to its model. More precisely, it does if valuated on {0,1} or if valuated by classical Kolmogorovian probability functions. Therefore, our results show that for classical logics there are non-orthomodular models which do not use {0,1} valuation of propositions. Since all standard applications of classical logic invoke exactly such valuation the above discovery most probably will not have serious repercussions for classical reasoning and computing. Quantum logic as well as orthomodular lattices, on the other hand, in principle cannot ascribe definite values to their propositions and cannot have {0,1} valuation at all. This might have serious repercussions to the quantum computers, though, as we will see in the next section. The reader can find detailed soundness and completeness proofs for both quantum and classical logics in (Pavičić & Megill, 1999).

4 Quantum Algebra for Quantum Computers

Computational instructions to a quantum computer for handling inputs to give desired outputs are lately simply called quantum logic. (Christianson, Knight, & Beth, 1998) The latter logic can however not be a proper logic, especially if we want it to be a general machine language capable of solving and simulating any given Hamiltonian. Recently devised algorithms such as factorization of big numbers in cryptography (Shor, 1997) or searching big data bases in networks (Grover, 1997) are certainly ingenious but do not use any general quantum algebra. They make a direct use of hardware prepared and hardware processed input states. In order to build up a general quantum algebra input states must satisfy additional conditions which do not result from qubit superposition, entanglement, and rotation and phase shift control. Algebraically these conditions amount to an extension of orhtomodular lattices which we call the Hilbertian lattice, HL and will consider in this section as a structure isomorphic to a Hilbert space description of an arbitrary quantum system.

Classical computer states obey all the conditions required by the Boolean algebra (distributivity etc.). As opposed to this, quantum computer states which appear in the known algorithms (e.g., Shor's and Grover's) do not obey all the conditions required by HL. On the other hand it is still unclear how one can implement HL conditions into a quantum computer. So, even the Schrödinger equation itself which is describing the evolution of states in a quantum computer must be simulated by a specially designed approximative algorithm.(Boghosian & Taylor, 1998) Such quantum computer is therefore still not what it was conceived to be: a quantum simulator which should mimic quantum systems by giving precise instructions on how to produce input states how to evolve them and how to read off the final states.(Feynman, 1982, 1986) Let us analyze conditions which quantum states should obey in order to enable full quantum computing, i.e., proper quantum mathematics.

In order to enable an isomorphism between an orthocomplemented orthomodular lattice and the corresponding Hilbert space we have to add further conditions to the lattice. The conditions correspond to the essential properties of any quantum system such as superposition. Combining (Holland, JR., 1995) and (Ivert & Sjödin, 1978) the conditions are:

Additional Conditions for a Hilbertian lattice HL.

- *Completeness:* The meet and join of any subset of a lattice always exist.
- *Atomicity:* Every non-zero element in HL majorizes an atom which is a non-zero element a∈HL with $0 < b \leq a$ only if $b = a$.
- *Superposition Principle:*
    1. Given two different atoms a and b, there is at least one other atom c, $c \neq a$ and $c \neq b$, that is a superposition of a and b.
    2. If the atom c is a superposition of a the distinct atoms a and b, then a is a superposition of b and c.
- *Unitary operators:* Given any two orthogonal atoms a and b in HL, there is a unitary operator U such that $U(a) = b$.

It is well-known that if the HL is of dimension ≥ 4, then there exists a field F and a vector space E over F such that HL is ortho-isomorphic to the lattice L_E of E-closed subspaces of E.

In an "orthomodular approach" the condition ≥ 4 must be postulated. (Maczyński, 1972) But if we found a condition which must be satisfied in HL and which requires at least four nonequivalent variables then the condition would be automatically satisfied. A natural idea is to look for conditions equivalent to those in Eqs. (15) and (16) with new "full quantum identities" which would solve the ambiguity problem of the relation of equality '=' and of lattice operations. The following definitions do the job.

Definition 4.1.

$$a \overset{c}{\equiv}_i b \overset{\text{def}}{=} ((a \to_i c) \cap (b \to_i c)) \cup ((a' \to_i c) \cap (b' \to_i c)); \qquad i = 1,3,5 \tag{18}$$
$$a \overset{c}{\equiv}_i b \overset{\text{def}}{=} ((c \to_i a) \cap (c \to_i b)) \cup ((c \to_i a') \cap (c \to_i b')); \qquad i = 2,4 \tag{19}$$
$$a \overset{c,d}{\equiv}_i b \overset{\text{def}}{=} a \overset{d}{\equiv}_i b \cup (a \overset{d}{\equiv}_i c \cap b \overset{d}{\equiv}_i c); \qquad i = 1,\dots,5 \tag{20}$$

Theorem 4.2. *An ortholattice to which*

$$a \overset{c}{\equiv}_i b = 1 \quad \Leftrightarrow \quad a \to_i c = b \to_i c, \qquad i = 1,3,5, \tag{21}$$
$$a \overset{c}{\equiv}_i b = 1 \quad \Leftrightarrow \quad c \to_i a = c \to_i b, \qquad i = 2,4\,, \tag{22}$$

are added is OML for i=5 and a variety of OML which fails in lattice $\widehat{\mathrm{L}}$ *(Fig. 1a) for i=1,2,3,4.*

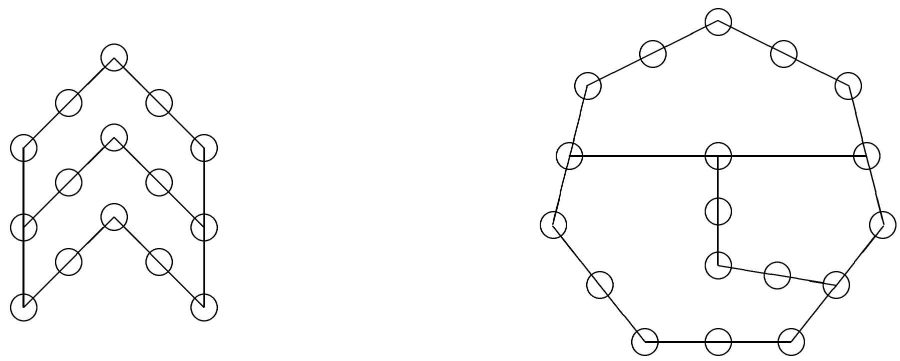

Figure 1: (a) $\widehat{\mathrm{L}}$ from(Godowski & Greechie, 1984); (b) L38 from (Pavičić & Megill, 1999).

Theorem 4.3. *An ortholattice to which*

$$a \overset{c,d}{\equiv}_i b = 1 \quad \Leftrightarrow \quad a \to_i d = b \to_i d, \qquad i = 1,3,5 \tag{23}$$

is added is a variety of OML which fails in lattice L38 (Fig. 1b) for i=1,3 and a variety of OML which fails in lattice $\widehat{\mathrm{L}}$ *(Fig. 1a) for i=5.*

The reader can check that the equations really fail in the quoted lattices (and many others from the literature) after compiling *lattice.c* written in C by Norman D. Megill.[2]

The new identities $\stackrel{c}{\equiv}_i$ and $\stackrel{c,d}{\equiv}_i$ when equal to one are relations of equivalence. The previous theorems narrow down the ambiguity of operations $\to_i$ (and therefore of relations $\le$ and $=$, as well) to two. The role of the Sasaki projection $\phi_a b = (a \to_1 b')'$ of b on a in the the covering property which is a consequence of the superposition principle then apparently resolves the ambiguity completely.

In the end we are able to prove that the previous theorems follow from the following one.(Pavičić & Megill, 1999)

Definition 4.4. *A* 3OA *is an* OML *in which the following additional condition is satisfied:*

$$(a \to_1 c) \cap (a \stackrel{c}{\equiv}_1 b) \le (b \to_1 c)\,. \tag{24}$$

A 4OA *is an* OML *in which the following additional condition is satisfied:*

$$(a \to_1 d) \cap (a \stackrel{c,d}{\equiv}_1 b) \le (b \to_1 d)\,. \tag{25}$$

Theorem 4.5. *Every* 4OA *is a* 3OA, *but there exist* 3OA*s that are not* 4OA*s.* 4OA *fails in L38 and $\hat{L}$ and* 3OA *only in $\hat{L}$. (Fig. 1)*
Eq. (22), i=1 follows from Eq. (24) and Eq. (23), i=1 follows from Eq. (25).

The 4OA law (25) is equivalent to the orthoarguesian law discovered by A. Day [cf. (Godowski & Greechie, 1984)]. Thus the orthoarguesian law may be expressed by an equation with only 4 variables instead of 6. In addition, we are able to prove that apparently all known ortho-arguesian derivates follow from, or are identical to either 4OA or 3OA laws given above. (Pavičić & Megill, 1999)

We therefore obtained the result that HL must be of dimension ≥ 4 and that therefore—with the afore-cited additional conditions—is ortho-isomorphic to the lattice of subspaces of a Hilbert space. On the other hand, as the consequence of the afore-stated additional conditions we obtain that the number of atoms of a lattice (pure states) of any Hilbert space of dimension ≥ 3 must be infinite. (Ivert & Sjödin, 1978) This is in a direct relation to a coordinatization of Hilbert spaces. For example, if we want to have a complete description of a spin-1 system we cannot achieve this in the spin space alone. We have to include the orientations of preparations and measurements in space (otherwise we would not have even the Malus law) in our description and these are continuous variables. In a qubit preparation within a quantum computer the continuous variable is the angle α in Eq. (3). Thus the lattice is infinite although the number of input qubits and the unitary transformation needed to calculate the result of a measurement remains finite-dimensional. (Deutsch, 1985) This invalidates the following Deutsch's claim: "['Quantum' Church-Turing principle] is so strong that it is *not* satisfied by Turing's machine in classical physics. Owing to continuity of classical dynamics, the possible states of a classical system necessarily form a continuum.

[2] `ftp://ftp.shore.net/members/ndm/quantum-logic`

Yet there are only countably many ways of preparing a finite input for [a quantum Turing machine]. Consequently [it] cannot perfectly simulate any classical system."

This distinction does not hold, because, as we have seen, a continuum appears with quantum spin systems as well and on the other hand preparing a finite input is not in contradiction with the existence of a continuum of possible states within a lattice. Infinite number of states does not mean an infinite number of calculated spin projections for a quantum system or positions and momentums for a classical system. The infinity contained in the continuous variables is actually not a problem but an essential feature which actually enables the Hilbert space representation with the help of the recent M. P. Solèr's discovery: we need not *postulate* (as it was considered necessary until several years ago) a complex (or real or quaternionic) field for our Hilbert space—it *follows* from the infinity of the lattice. (Holland, JR., 1995)

5 Conclusion

States of a general purpose quantum computer must—apart from conditions imposed by the standard quantum logic gates—satisfy additional conditions given in Sec. 4 and required to yield a general algebra of the states. One of the conditions is a 4-variable orthoarguesian law given by Eq. (25) which gives all known orthoarguesian equations (including 6-variable ones) and which eliminates the so-called 4-dim lattice postulate. The obtained algebra, which is a Hilbertian lattice is then isomorphic to the subspaces of the Hilbert spaces which characterizes general computation algorithms. Propositions a, b, c of the lattice are therefore connected to probabilistic outcomes of a calculation of an observable A by means of $\mu(a) = \langle\Psi|P_A|\Psi\rangle$ where $P_{A,E}$ (E is a Borel set) is a projector of A, μ is the pure full ($\mu(a) \le \mu(b) \Rightarrow a \le b$) strongly convex ($\mu_j \in \mathcal{S}$ & $\sum c_j = 1 \ \Rightarrow\ \sum c_j\mu_j \in \mathcal{S}$) probability measure on HL: μ : HL $\mapsto [0,1]$ and Ψ we obtain from the Gleason theorem: if μ is a pure probability measure, then there exists a vector $\Psi \in \mathcal{H}$, which satisfies the above $\mu(a)$. The mean value of the operator A is then given by the spectral theorem.

In Sec. 3 we have shown that no calculation can be carried out within propositional quantum logic since the latter can be modeled with a non-orthomodular model. (In addition we show that the standard classical logic has a non-orthomodular model too and explain why this is of no consequence for classical computers.)

In Sec. 2 we have shown that an orthomodular lattice cannot be used as a satisfactory algebra of states because all operations in the lattice are five-fold defined, including the identity and the relation of equivalence.

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