Some aspects of the non-linear finite element method

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Abstract

The paper discusses four separate aspects of the non-linear finite element method: (i) An alternative formulation for the static co-rotational technique in conjunction with a simple faceted shell idealisation. (ii) Solution procedures for non-linear dynamics with emphasis on energy conserving techniques (iii) The use of interface elements and fracture energy related softening "stress–strain" curves for modelling mixed mode delamination in "composites" (iv) Hybrid static/dynamic solution procedures. While the topics are separate, there are links and these are explored. © 1997 Elsevier Science B.V.

1. Introduction

The co-rotational technique is a well-known formulation for the non-linear analysis of structures. It is usually applied in a static context, but in one of the following sections, it will also be applied to dynamics. However, we will firstly consider the application of the co-rotational technique to the simplest possible faceted shell idealisation. To this end, we introduce some ideas that were firstly developed in applying the co-rotational technique to continua [1, 2]. In particular, these involve the neglect of certain terms in the derivation of the tangent stiffness matrix. The idea is to simplify the formulation without impairing the convergence characteristics. Some of the concepts can be traced back to the work of Nour-Omid and Rankin [3].

For the first of the topics on dynamics, we consider the development of “energy conserving algorithms”. The primary objective is to develop techniques that are stable in a geometrically non-linear environment.

While the topic on composites would at first appear totally separate from the others; this turns out not to be so. In tracing some of the complex equilibrium paths associated with the modelling of delamination, we find that there is an urgent need for sophisticated path-following techniques. In practice, however some of these paths will be dynamic (although this may be an artifact of the mesh). In these situations it might be better to switch from statics to dynamics and back to statics. Such ideas are explored in the last section of the paper.

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2. An alternative co-rotational formulation for the faceted, Morley shell

A non-linear formulation for this shell idealisation was described by Peng and Crisfield [4]. However, while the idealisation was very simple (Fig. 1), unfortunately the formulation was not. Recent work with Moita [1] on co-rotational solids has led to a simpler general framework [2, 5] (with links to earlier work by Nour-Omid and Rankin [3]) which can be applied to the faceted shell idealisation and leads to a much neater formulation. In the following, we will not attempt to give the full detail but rather will aim to highlight the main features.

In the first instance, the local element frame, is chosen so that the vector \( e_1 \) coincides with a particular side (see Fig. 1). However, to avoid any dependence on the node numbering and with a view to later extensions to a large-strain formulation, this initial choice is later modified to make the in-plane “spin” with respect to the local axes (subscript \( \ell \)) zero so that

\[
\Omega_\ell = \begin{bmatrix} \frac{\partial u}{\partial Y} - \frac{\partial v}{\partial X} \end{bmatrix}_\ell = 0. \tag{1}
\]

The new element base vectors will be referred to as \( E = [e_1, e_2, e_3] \). We will later require the relationship:

\[
\delta E = S(\delta \beta)E, \tag{2}
\]

where \( S \) is a skew symmetric matrix containing the three components of the vector \( \delta \beta \). To find a relationship between the vector \( \delta \beta \) and the translational displacement changes, we differentiate (1)

![Fig. 1. Facet idealisation for curved shell with initial choice of \([e_1, e_2, e_3]\) frame.](image)
Fig. 2. Rotational variables and triads for use with Morley’s triangle: (a) initial mid-side triad \([R_1, R_2, R_3]\) for side \(ji\) (initial configuration); (b) intermediate mid-side triad \([r'_1, r'_2, r'_3]\) in current configuration; (c) final mid-side triad in current configuration; (d) global mid-side rotation; (e) local mid-side rotation, \(\theta_i\).

as well as the conditions:

\[
\begin{align*}
\omega_{r2} &= e_2^T[x_2 - x_1] = 0, \\
\omega_{r3} &= e_3^T[x_3 - x_1] = 0,
\end{align*}
\]

where \(x_1\), for example, is the current position vector of node 1. Eqs. (3) and (4) ensure that the local element plane always passes through nodes 2 and 3 (as well as the origin at node 1). The three variations lead to the relationship:

\[
\delta \beta = V^T \delta p_t,
\]

where \(\delta p_t\) contains the nine global translational displacement changes (three at each corner node).

As in the earlier work [4], the local rotations \(\theta_i\) are defined (Fig. 2) via:

\[
\sin \theta_i = -e_3^T r_1
\]

where the current mid-side “nodal triad”, \([r_1, r_2, r_3]\) is found via a two-stage updating whereby, for the first stage, the initial “nodal triad” \([R_1, R_2, R_3]\) is rotated in the plane defined by \(R_2\) and \(r_2\) onto \(r_2\) where the latter is known because it passes along the current configuration of the side. For the second stage, the intermediate triad \([r'_1, r'_2, r'_3]\) is rotated about \(r_2\) via a global rotation, \(\theta\).

The first major difference from the earlier formulation is that in applying variations to (6) (as required for virtual work), we use the relationship:

\[
\delta e_3 = -S(e_3) \delta \beta = -S(e_3)V^T \delta p_t,
\]
so that the variation of (6) is found as

$$\delta \theta = \delta \theta - \frac{1}{s} (R_2 \times r_2)^T \delta d_j - \delta d_i - r_2^T V^T \delta p_i, \quad (8)$$

where $s$ is the current length of the side under consideration and $d_j$ are the three translational displacements at the node $j$ ahead of the current mid-side and $d_i$ are the three translational displacements at the node $i$ behind the current mid-side. Eq. (8) can be combined with equivalent equations relating the local translational variables to the global translational variables so that we have a combined relationship:

$$\delta \mathbf{p} = T \delta \mathbf{p}, \quad (9)$$

where, if there is no specific subscript, the quantity is “global”.

By equating the virtual work in the local and global systems, we arrive at a relationship for the global internal nodal forces which takes the form

$$\mathbf{q}_i = T^T \mathbf{q}_{li}, \quad (10)$$

The nine ‘translational’ internal forces can be expressed as [2, 5]

$$\mathbf{q}_{li} = \text{Diag}[E] \mathbf{q}_{li} - V \mathbf{a}(\mathbf{q}_{li, ki}) + \mathbf{q}_{li}(\mathbf{q}_{li}), \quad (11)$$

where $\mathbf{q}_{li}$ are the nine local translational internal forces and $\mathbf{q}_{li}$ are the three local mid-side rotational forces. With the current formulation, we work with all of the local variables and, in these circumstance, $\mathbf{q}_{li}$ includes a contribution from the local bending behaviour.

From (8), we find that the relationship between the rotational forces in (10) take the trivial form:

$$\mathbf{q}_{li} = \mathbf{q}_{li}, \quad (12)$$

which equates the local and global rotational internal “forces” at each of the mid-side nodes.

The initial stress matrix stems from the variation of (10) or (11) with the local internal forces being kept fixed. In the current formulation, we use the fact that certain terms will vanish at equilibrium to avoid taking some of these variations (see also [3]). In particular, it can be shown [2, 5] that, at the element level, the vector $a$ in (11) represents three rotational equilibrium equations so that this vector will vanish at equilibrium. We use this observation to omit the variations of the “spin matrix”, $V$, when we obtain the “initial stress matrix” (a very similar procedure was earlier adopted for a co-rotational continuum formulation [1, 2]). The $\mathbf{q}_{li}$ terms in (11) can be shown [2, 5] to represent equilibrium equations which will become zero at the structural level once contributions from a particular side are summed and account is taken of (12). Hence, at the structural level, these terms involve local quantities (involving $\mathbf{q}_{li}$) which will be zero at equilibrium multiplied by terms that we would normally vary. This observation again allows us to avoid considering variations of $\mathbf{q}_{li}$ for the initial stress matrix. It is possible to include all of the terms [4] but the resulting equations are rather complex and we will now show that the current formulations gives as good a convergence rate.

Both the present and earlier formulations have been used to analyse the hemispherical shell of Fig. 3. The shell has a cut out which subtends a total angle of $36^\circ$. The properties were those
adopted by Buechter and Ramm [6] who specified:

\[ E = 6.825 \times 10^7, \quad v = 0.3, \quad \text{thickness } 0.04, \quad \text{radius } = 10. \]

A symmetric quarter was analysed and the load/deflection response is compared with that of Buechter and Ramm [6] in Fig. 4. These authors used a 16 x 16 mesh of four-noded enhanced strain shell elements. The present results coincide with those obtained using the earlier formulation [4]. Also, the convergence characteristics for the two co-rotational formulations were very similar and the decay of the scaled residual norm on a typical increment was as follows:

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Decay of the scaled residual norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iteration</td>
<td>0</td>
</tr>
<tr>
<td>New method</td>
<td>291</td>
</tr>
<tr>
<td>Old method</td>
<td>291</td>
</tr>
</tbody>
</table>

In this particular case, the current convergence characteristics are marginally worse than those obtained with the earlier formulation. However, the present formulation is much simpler.

3. Solution procedures for non-linear dynamics

Both Simo and co-workers [7, 8] and Crisfield and co-workers [9, 10] have demonstrated the potential pitfalls in the use of the trapezoidal form of the Newmark method when it is applied to non-linear dynamics. In the following, we will show that difficulties can also be encountered with the "\( \alpha \) method" [11].

The key to the problem is that, while these methods can be shown to be unconditionally stable in the linear regime, the methods can be unstable in the non-linear regime. One way to overcome these difficulties is to introduce some form of "energy conserving algorithm" [7–10].
Although it is true that for most structural systems, damping or dissipation will be involved, it nonetheless makes sense to start with an algorithm that is satisfactory in the absence of such damping. In this context, the authors believe that the main advantage of the “energy conserving algorithms” is not that they preserve energy, but rather, that they are stable in the non-linear regime.

In the following, we will be primarily concerned with beams for which a three-dimensional energy conserving formulation has been described by Simo et al. [8]. This formulation is related to the static isoparametric formulation of Simo and Vu Quoc [12] and subsequent Newmark-related dynamic formulation in [13]. An alternative energy conserving formulation has been described by Galvanetto and Crisfield [10] and is based on the co-rotational technique. We will briefly outline this method, which is related to two-dimensional beams, before moving on to a three-dimensional approach.

Conventional implicit Newmark procedures satisfy the dynamic equilibrium equations at the end of a time-step. In contrast, the energy conserving methods can be considered as satisfying the dynamic equilibrium equations at the mid-point of the step. Using the co-rotational approach, we can then write the dynamic equilibrium equations as

\[
\ddot{g}_m = \left[ \frac{T_n + T_{n+1}}{2} \right]^T \left[ q_{\ell,n} + q_{\ell,n+1} \right] - q_{em} + M \left[ \frac{\dot{p}_{n+1} - \dot{p}_n}{\Delta t} \right]
\]

\[
=q_{im} - q_{em} + M \left[ \frac{\dot{p}_{n+1} - \dot{p}_n}{\Delta t} \right] = 0.
\]

Here \( q_{\ell,n+1} \) are the local internal forces at step \( n + 1 \) and the “transformation matrix”, \( T_{n+1} \), is such that, at step \( n + 1 \), the local nodal ‘displacement changes’ (here including rotations), \( \delta p_{\ell,n+1} \) are
related to the global nodal ‘displacement changes’, \( \delta \mathbf{p} \), via:

\[
\delta \mathbf{p}_{i,n+1} = \mathbf{T}_{n+1} \delta \mathbf{p}
\]  

(14)

(see also (9)). The local internal forces in (13) are computed from the local displacements using a conventional linear stiffness matrix. The vector \( \mathbf{q}_{em} \) is the mid-point external load vector, \( \dot{\mathbf{p}} \) are the velocities and \( \mathbf{M} \) is the mass matrix. (The latter is best related to a Timoshenko formulation even if an Euler–Bernoulli formulation is used for the other terms [10]. In the examples that will follow, we either use either a Timoshenko formulation or an “engineering beam theory approach” due to Przemieniecki [14] which degenerates to an Euler–Bernoulli formulation as the shear modulus tends to infinity). It should be noted that the “mid-point” local internal forces in (13) are the mean of the values at the beginning and end of the steps and not the values computed from the mean displacements.

In relation to energy conservation, we observe that, as a good approximation [5, 10], we can write:

\[
\Delta \mathbf{p}_r \simeq \left[ \frac{T_n + T_{n+1}}{2} \right] \Delta \mathbf{p}.
\]  

(15)

Assuming, for the present an equality in (15), we can now write the change in strain energy over the step as

\[
\Delta \varphi = \Delta \mathbf{p}_r^T \left[ \frac{\mathbf{q}_{i,n} + \mathbf{q}_{i,n+1}}{2} \right] = \Delta \mathbf{p}_r^T \left[ \frac{T_n + T_{n+1}}{2} \right]^T \left[ \frac{\mathbf{q}_{i,n} + \mathbf{q}_{i,n+1}}{2} \right] = \Delta \mathbf{p}_r^T \mathbf{q}_{im}
\]  

(16)

where we have used (13). In addition, the change of kinetic energy over the step can be expressed as

\[
\Delta K = \frac{1}{2} \dot{\mathbf{p}}_{n+1}^T \mathbf{M} \dot{\mathbf{p}}_{n+1} - \frac{1}{2} \dot{\mathbf{p}}_n^T \mathbf{M} \dot{\mathbf{p}}_n = \frac{1}{2} [\dot{\mathbf{p}}_{n+1}^T - \dot{\mathbf{p}}_n]^T \mathbf{M} [\dot{\mathbf{p}}_{n+1} + \dot{\mathbf{p}}_n] = \frac{1}{\Delta t} [\dot{\mathbf{p}}_{n+1} - \dot{\mathbf{p}}_n]^T \mathbf{M} \Delta \mathbf{p},
\]  

(17)

where we have introduced the relationship that is used for up-dating the velocities. If we now assume a fixed external load vector (as with gravity loading), the total energy change over the step is

\[
\Delta K + \Delta \varphi + \Delta P = \Delta \mathbf{p}_r^T \left[ \mathbf{q}_{im} - \mathbf{q}_{em} + \mathbf{M} \left[ \frac{\dot{\mathbf{p}}_{n+1} - \dot{\mathbf{p}}_n}{\Delta t} \right] \right] = \Delta \mathbf{p}_r^T \tilde{\mathbf{g}}_m,
\]  

(18)

where \( \tilde{\mathbf{g}}_m \) is the residual of (13). Once mid-point dynamic equilibrium is obtained, the latter term is zero and hence the energy change over the step is zero.

The previous developments have assumed the validity of (15) which is, in fact, only an approximation. It can be shown [10] that, for small strains, an alternative, exact expression takes the form:

\[
\Delta \mathbf{p}_r = \Delta \mathbf{p}_r + P \left[ \frac{T_n + T_{n+1}}{2} \right] \Delta \mathbf{p}_t,
\]  

(19)

where \( \Delta \mathbf{p}_t \) contains the incremental global rotations with all of the other variables zero while \( \Delta \mathbf{p}_r \) contains the incremental translations with the rotations being set to zero. If the variables are ordered
as \( u_1, v_1, \theta_1, u_2, v_2, \theta_2 \), then \( P \) in (19) is a diagonal matrix with:

\[
\text{Diag}(P) = \left( *, *, \frac{\Delta x}{\sin \Delta \alpha}, 2, *, \frac{\Delta x}{\cos \Delta \alpha}, *, \frac{\Delta x}{\sin \Delta \alpha} \right)
\]

(20)

and \( \Delta x \) is the rigid-body rotation between steps \( n \) and \( n+1 \) and is easily computed [10]. (The co-rotational formulation being adopted in [10] was such that only the three strain-producing local variables were considered as non-zero and hence the values of the *'s were immaterial). Eq. (13) is easily modified to account for (19) and (20) so that, for small strains, energy is exactly conserved.

In relation to three-dimensional beams, we can modify (13) to take the form:

\[
\bar{g}_m = \left[ \frac{T_n + T_{n+1}}{2} \right]^T \left[ \frac{q_{ri,n} + q_{ri,n+1}}{2} \right] - q_{em} + q_{mas} = 0,
\]

(21)

where

\[
q_{mas} = \frac{1}{\Delta t} M_t \{ \dot{p}_{i,n+1} - \dot{p}_{i,n} \} + \frac{1}{\Delta t} \bar{E}_{n+1} M_t \ddot{p}_{r,n+1} - \frac{1}{\Delta t} \bar{E}_n M_t \dot{p}_{r,n},
\]

(22)

with \( M_t \) containing the translational parts of the conventional fixed mass matrix and \( M_r \) the rotational parts. Also \( \dot{p}_i \) contains the translational nodal velocities (with the rotational terms set to zero) while \( \ddot{p}_i \) contains the (body attached [5]) rotational nodal velocities (with the translational terms set to zero). The ordering of the global velocities is

\[
\dot{p}^T = (d_1^T, \omega_1^T, d_2^T, \omega_2^T)
\]

(23)

with \( d_i \) having the translational velocities at node \( i \) and \( \omega_i \) the (body attached) rotational velocities at node \( i \). The \( \bar{E} \) matrix in (22) takes the form

\[
\bar{E} = \begin{bmatrix}
I & E \\
E & I \\
\end{bmatrix}
\]

(24)

where \( E \) is the 3 \( \times \) 3 matrix containing base vectors of the local element frame. (Since the original draft of this paper, we have found that it is better to replace the element, \( E \), frames in (24) with “nodal frames”).

Applying variations to (13) or (21) leads to a non-symmetric effective tangent stiffness matrix. In the three-dimensional case, this is not a problem because, even with a conventional end-point, equilibrium, using the Newmark method, the equivalent matrix is non-symmetric. We have not yet extended the three-dimensional formulation to include the equivalent sophistications to those that were applied in (19) and (20) for the two-dimensional case. Hence, the current three-dimensional mid-point procedure is only approximately energy conserving. Nonetheless, we will show that it leads to a significantly improved numerical performance in comparison to that of the basic Newmark method.
The first example is two-dimensional and involves a shallow arch (Fig. 5(a)) which was subjected to an initial velocity. The arch was of span 20 and rise 0.1 with elastic constants such that $EA = 2.8 \times 10^7$, $EI = 1.4 \times 10^4$, $GA = 1.0 \times 10^7$, $\rho = 1.2$. It was modelled using ten co-rotational Timoshenko elements. The initial velocity (and hence kinetic energy) was such that the arch should respond with low amplitude motion (no snap-through). Such a response is predicted when the time step is 0.03 s but when it is increased to 0.065 s, the use of the Newmark algorithm leads to the response changing to high amplitude motion (with snap-through, Fig. 5(b)). Such a response should only occur if a higher initial velocity was specified. For this example, the solution jumps to a new energy state but equilibrium convergence is still achieved. When the energy-conserving procedure is applied, the response (labelled average in Fig. 5(c)) correctly remains of a low amplitude.

We next consider a three-dimensional example (Fig. 6) originally described by Simo and Vu Quoc [13]. The example involves a clamped right-angled frame subject to an initial point loading (Fig. 6(b)) at the elbow in the “3” direction (Fig. 6(a)). Both legs of the frame have the same properties. In relation to the local 123 co-ordinate system shown in Fig. 6(a), the “material properties” are

$$EA = GA_2 = GA_3 = 10^6; \quad EI_2 = EI_3 = GJ = 10^3,$$

where $A_2$ and $I_2$ relate to the 13 plane while $A_3$ and $I_3$ relate to the 12 plane. Using a similar convention, the “mass properties” are

$$\rho A = 1; \quad \rho I_2 = \rho I_3 = 10; \quad \rho(I_2 + I_3) = 20.$$ 

For the analyses, each leg was idealised using five elements.
Simo and Vu Quoc [13] used the Newmark method to analyse this example with a time step of $\Delta t = 0.25\,\text{s}$ and gave results that were effectively the same as those obtained by the present authors (using quadratic isoparametric elements and following the method in [13]) which are plotted in Fig. 7. The results of Simo and Vu Quoc were only given for a total time of 30\,s. In the present work, we have increased the total time and the solution procedure has failed soon after a total time of 56\,s. This failure, which took the form of non-convergence, was accompanied by a sudden growth in energy (Fig. 8). This behaviour contrasted with that of the earlier two-dimensional example for which the jump in energy was not accompanied by a convergence failure.
We have also applied a Newmark formulation to the co-rotational procedure [5] using five linear Timoshenko elements per leg. The results are illustrated in Fig. 7 which shows very similar results to those obtained with the earlier quadratic isoparametric elements. As with these isoparametric elements, failure again followed (now after about 60 s). Once the co-rotational mid-point scheme was adopted (Fig. 8), satisfactory results were obtained with no energy growth up to an elapsed time of 7200 s. It is worth recalling that, in contrast to the two-dimensional formulation, the current three-dimensional mid-point scheme is only approximately energy conserving.

In a further attempt to avoid the “energy blow-ups”, we have applied the $\alpha$ method [11], with $\alpha = -0.05$. One can observe from Fig. 8 that, with the co-rotational Timoshenko formulation, the method leads to an initial energy decay but eventually an “energy blow-up” is again experienced (now after about 1270 s). With the isoparametric procedure, no “energy blow-up” was experienced within a time period of 7200 s. However, the final energy was then reduced to 25% of its original value. We have also solved the problem using the “engineering” co-rotational beams. The results of all the analyses are summarised in Table 2.

In the table, T refers to “Timoshenko” and E to “engineering”. The figures in paranthesis refer to the latest modified formulation with nodal triads in place of the element triads, E, in (24). For the co-rotational engineering solution using $\alpha = -0.05$, although no “blow-up” was observed for a total time of 7200 s, at this time the energy had decayed so that it was only 9% of its maximum value.
Total energy

Co-rotational elements

--- Co-rotational Timoshenko element
--- Co-rotational engineering element

Newmark method
HHT method, alpha = -0.05
Midpoint method

Time

Fig. 8. Total energy against time (co-rotational formulation).

Table 2
Time to “energy blow-up”

<table>
<thead>
<tr>
<th>Element type</th>
<th>Integration scheme</th>
<th>Time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isoparametric</td>
<td></td>
<td>62</td>
</tr>
<tr>
<td>Co-rotational-T</td>
<td>Newmark</td>
<td>60</td>
</tr>
<tr>
<td>Co-rotational-E</td>
<td></td>
<td>25</td>
</tr>
<tr>
<td>Isoparametric</td>
<td></td>
<td>&gt; 7200</td>
</tr>
<tr>
<td>Co-rotational-T</td>
<td>( \alpha = -0.05 )</td>
<td>1270</td>
</tr>
<tr>
<td>Co-rotational-E</td>
<td></td>
<td>&gt; 5400 (&gt; 7200)</td>
</tr>
<tr>
<td>Isoparametric</td>
<td></td>
<td>N/A</td>
</tr>
<tr>
<td>Co-rotational-T</td>
<td>Mid-point</td>
<td>500 (&gt; 7200)</td>
</tr>
<tr>
<td>Co-rotational-E</td>
<td></td>
<td>401 (1310)</td>
</tr>
</tbody>
</table>

4. Interface elements and softening traction/relative-displacement relationships for mixed-mode delamination in composites

Hilleborg [15] was the first to use a softening “traction/relative-displacement relationship” in which the area under the curve was related to the fracture energy. This work was applied to the cracking of concrete and, much subsequent work, on “smeared cracking” has followed these ideas [16]. The concepts were first applied to delamination in composites by Schellekens and de Borst
In the following, we will briefly describe some related work that the authors have conducted on this topic, which is an extension of the work by Crisfield and Hellweg [16]. Emphasis will be placed on mixed mode delamination under plane-strain conditions, with a combination of modes 1 and 2 fracture being assumed to occur along an interface element (Fig. 9). For such mixed mode behaviour, a safe lower-bound to the fracture relationships can be taken as

$$\frac{G_1}{G_{1c}} + \frac{G_2}{G_{2c}} = 1$$

with $G$ being the energy release rate and with the subscript $c$ meaning “critical”. For the finite element formulation, we adopt a form of damage model [5] whereby:

$$\bar{\sigma} = \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \end{bmatrix} = \begin{bmatrix} I - \frac{\kappa}{1 + \kappa} F \end{bmatrix} E_0 \bar{\epsilon} = [I - D] E_0 \bar{\epsilon}.$$  

(26)

In (26), the bars on the $\sigma$ and $\epsilon$ terms, imply that these are not strictly stresses and strains but rather are tractions and relative displacements with the $\bar{\sigma}_2$ being a shearing traction. The matrix $E_0$ in (26) contains the diagonal stiffnesses of the vanishingly thin interface medium, which are highly penalty-like terms. The final form in (26) shows the damage nature of the relationship with the matrix $D$ replacing the scalar $d$ that is present in usual scalar damage models [C5]. In (26), the matrix $F$ is

$$F = \text{Diag} \left[ \frac{\bar{\epsilon}_{\text{max}}}{\bar{\epsilon}_{\text{max}} - \bar{\epsilon}_o} \right].$$

(27)
Fig. 10. Softening traction/relative displacement relationship.

where $\bar{e}_{\text{max}}$ and $\bar{e}_o$ are illustrated in Fig. 10. Relationships of this form are assumed to be input for the two modes with the areas under the curves giving the input critical fracture energies.

We have yet to define the scalar $\kappa$ in (26) which is given by

$$\kappa = \left[ \frac{\bar{e}_1}{\bar{e}_{o1}} \right]^2 \left[ \frac{\bar{e}_2}{\bar{e}_{o2}} \right]^2 - 1 \tag{28}$$

where $\bar{e}_{o1}$ is $\bar{e}_o$ (Fig. 10) for mode 1 and $\bar{e}_{o2}$ is $\bar{e}_o$ for mode 2. Damaging (or fracturing) is assumed to occur for $\kappa > 0$ and the formulation is embedded within a damage algorithm via a “damage surface”:

$$f = \kappa(\bar{e}) - \kappa_{\text{id}} \tag{29}$$

where $f > 0$ for damaging.

If we apply the formulation to an individual mode (1 or 2), it is easy to show that

$$\int_0^{\bar{e}_{\text{max}}} \bar{\sigma} \, d\bar{e} = G_c. \tag{30}$$

This merely reflects the fact that the input area under the curve is, by definition, equal to the critical fracture energy. It can also be shown (but it is less straightforward) that, if we apply proportional “straining” with the ratio between the modes 1 and 2 “relative displacements” being kept fixed and, in addition, we set $\bar{e}_{o1} = \bar{e}_{o2}$, then once all of the softening has occurred and we integrate the individual consumed energies in a manner analogous to that in (30), we obtain (25). The adopted mixed-mode formulation has some links with the work of Corigliano [18].

To test the method, we have used the mixed mode bending test described by Reeder et al. [19] (Fig. 11). We will first consider a pure mode 1 formulation for which we could set $c = -L$ and $P = -P$. However, in reality, we simply introduce an opening displacement at the right-hand end in the manner of a Double Cantilever Beam (DCB) test. Fig. 12 shows finite element results, experimental results [20] and the results from a closed-form, beam theory solution [21] for such a DCB test. The adopted orthotropic properties were:

$$E_x = 126,000 \text{ N/mm}^2, \quad E_y = E_z = 7500 \text{ N/mm}^2, \quad G_{xy} = 4980 \text{ N/mm}^2,$$  

$$\nu_{xy} = \nu_{yz} = \nu_{xz} = 0.263, \quad G_{1c} = 0.281 \text{ N/mm}, \quad \bar{\sigma}_1 = 57 \text{ N/mm}^2, \quad \bar{e}_o = 1 \times 10^{-6} \text{ mm}, \quad a = 30 \text{ mm},$$
while the total depth of the beam was 3 mm, the total length 100 mm and the adopted mesh involved
4 × 200 eight-noded, quadrilateral, plane-strain elements.

It can be seen that the overall load/displacement solution obtained by the finite element method,
is in close agreement with the other results. However, when a coarse mesh is adopted, we find that,
superimposed on the overall shape, we have many local “bumps” which sometimes involve sharp
“snap-backs” [16, 22]. Crisfield and Peng [23] have recently pointed out that such “false instabilities”
can be induced by coarse meshes in the rather different context of the non-linear analysis of elastic
shells.

Fig. 11. Mixed-mode bending test: (a) apparatus; (b) applied loads.
Fig. 12. Load/displacement relationships for a pure mode 1 test.

Fig. 13. Mesh sensitivity for a pure mode 1 test.
To obtain a robust solution, one must include many sophistications in the solution algorithms. Using a conventional “cylindrical arc-length method” [24], we have sometimes found severe difficulties. Consequently, we have introduced an alternative procedure [22] in which a new technique is used to choose between the alternative roots to the “arc-length constraint”. In this method, both alternatives are tried and the chosen root is that involving the “minimum residual” [22]. Other sophistications are also required. Occasionally, we reach a situation in which the equilibrium path is so convoluted that the solution enters a cycle and oscillates between two points. This must be automatically detected and “increment cutting” applied.

Fig. 14 shows the solutions obtained for a mixed-mode test with a high component of mode 2. The length $c$ of Fig. 11 was chosen as so that $c/L = 0.425$ (so that $G_2/G_1 = 20$ [19]). The adopted properties were:

$$E_x = 135\,300\,\text{N/mm}^2,\quad E_y = E_z = 9000\,\text{N/mm}^2,\quad G_{xy} = 5200\,\text{N/mm}^2,$$

$$
u_{xy} = 
u_{zx} = 0.24,\quad
\nu_{yz} = 0.46,$$

$$G_{1c} = G_{2c} = 4.0\,\text{N/mm},\quad \bar{\sigma}_{11} = \bar{\sigma}_{12} = 57\,\text{N/mm}^2,\quad \bar{\epsilon}_0 = 1 \times 10^{-7}\,\text{mm},\quad a = 30\,\text{mm}.$$  

Here the finite element results are in close agreement with closed-form, beam theory solutions [21]. They also show that while, with a long initial crack, the response is stable (under displacement-controlled loading), with a shorter initial crack, it is unstable so that, in reality, there would be a dynamic snap between point A and B in Fig. 14.

5. Hybrid static/dynamic solution procedures

An example of a static solution involving a snap-back was given in the previous section and some of the difficulties associated with the path-following were discussed. Instead of developing
more sophisticated static path-following procedures such as that in [22], one could instead switch to a dynamic algorithm for those parts of the solution curves involving “snap-throughs” (in the case of load control) or “snap-backs” (in the case of displacement control). Methods for implementing such procedures have recently been described by Riks et al. [25] and Skeie et al. [26]. The following formulation incorporates some of the ideas contained in these papers.

We will be largely concerned with “limit-points” although “bifurcation points” can also be considered. Assuming load-control, beyond the limit point, the pseudo-static response is unstable (Fig. 15(a)) but can be followed using some form of arc-length method. In reality, the structure would instead experience the dynamic snap illustrated in Fig. 15(a).

The hybrid procedure starts with a conventional static path-following technique using some form of arc-length method. As soon as the “current stiffness parameter” [5] indicates that a limit point has been passed, we simply add a perturbation load $\Delta \lambda q_{ef}$ where $q_{ef}$ is the fixed loading vector that is being monotonically increased. At the same time we automatically switch from statics to

---

Fig. 15. Static/dynamic solution procedure: (a) Static and dynamic paths; (b) Solution strategy.
dynamics with the initial velocity being set to zero. (Riks et al. [25] note that we could speed up the procedure by introducing an initial velocity but they also point out some potential dangers). Riks et al. [25] suggest firstly "homing in" on the limit point using, for example, some "bracketing technique" [5]. In relation to Fig. 15(b), this would involve starting the dynamic procedure from point A'. Instead, the authors have simply added the perturbation load to the first converged load configuration beyond the limit-point (point B, Fig. 15(b)) and have then added a perturbation load and applied the dynamic procedure from point B' (Fig. 15(b)).

Clearly, some form of damping must be introduced. Here, we follow Skeie et al. [26] and introduce a form of mass-proportional critical damping. In particular, at step \( n + 1 \), we introduce a damping matrix:

\[
C_{n+1} = 2\omega_n M, \tag{31}
\]

where \( \omega_n \) gives a measure of the lowest eigenfrequency of the structure via:

\[
\omega_n^2 = \left| \frac{\Delta p_n^T K_t \Delta p_n}{\Delta p_n^T M \Delta p_n} \right|, \tag{32}
\]

where \( \Delta p_n \) are the incremental displacements that have occurred over step \( n \) and \( K_t \) is the tangent stiffness matrix at the end of step \( n \).

We can also use (32) to provide an automatic time-stepping for the dynamic part of the algorithm. In particular, we compute the period as \( T = 2\pi/\omega_n^2 \) and then set the time step for the next increment as 0.01 \( T \) although we allow no more than a doubling of the time step. The concepts go back to those of Bergan and Mollestad [27].

Finally, we need some criterion for stopping the dynamic algorithm and, as suggested by Riks et al. [25], we use the kinetic energy. In particular, we switch back to statics (at point C in Fig. 15(b)), once the kinetic energy is less than an input tolerance times the maximum kinetic energy. At this stage, we switch off dynamics and use the displacements at the end of the dynamic stage as a predictor for a static Newton–Raphson iteration. Following convergence, the load parameter \( \lambda \) can be increased as part of a static path-following technique in the usual way.

Preliminary studies have been conducted on the introduction of the dynamic method from the last converged (unstable) point beyond a bifurcation point. It would seem that, in these circumstances, we must compute the lowest eigenmode and apply an initial velocity as a factor of this mode. (More discussion on this topic can be found in the paper by Riks et al. [25]).

For a numerical application, we have analysed one quadrant of the hinged shell of Fig. 16 using semi-loof shell elements. In relation to Fig. 16, the dimensions are given by

\[
R = 2540 \text{ mm}, \quad t = 12.7 \text{ mm}, \quad a = 508 \text{ mm}, \quad b = 508 \text{ mm}.
\]

The material properties are:

\[
E = 3103 \text{ N/mm}^2, \quad v = 0.3, \quad \text{mass density} = 8.7 \times 10^{-9} \text{ s}^2/\text{mm}^4.
\]

The computed static and static/dynamic solutions are illustrated in Fig. 16. Further details on this work can be found in Zhong and Crisfield [28, 29] (Fig. 17).
6. Conclusions

In relation to co-rotational static analysis, it has been shown that the method can be formulated using an expression for the spin of the local frame. In these circumstances, some of the terms which multiply the "spin matrix" can be shown to vanish at the element (or side) level at equilibrium. This observation allows a number of terms to be neglected in the derivation of the initial stress
matrix. For a particular, non-linear shell problem, it has been shown that this can be achieved with no significant deterioration to the convergence characteristics.

In relation to non-linear dynamics, it has been shown that the Newmark method can lead to "energy blow-up". An energy conserving formulation has been proposed for two-dimensional beams which can be applied in a co-rotational framework and eliminates these difficulties. For three-dimensional beams, an approximate energy-conserving procedure has been proposed which leads to much more stable solutions than those obtained with the Newmark method. However, with a long enough time, "energy blow-up" again occurs. It has been shown that, while these problems can be alleviated by introducing the z method (which includes numerical damping), for the co-rotational formulation at least, they cannot be eliminated. Future work will involve the extensions to three-dimensions that convert the approximately energy conserving procedure to a fully energy technique. In addition, for the isoparametric formulation, we will consider an alternative energy conserving procedure to that advocated by Simo et al. [8] (which applies a Cayley parametrisation).

It has been shown that "softening interface elements" can be used to model mixed mode delamination. Although the formulation differs from the conventional fracture-based formulations, it nonetheless introduces the same key parameters, which are the critical fracture energies. However, in contrast to conventional methods, an initial flaw is not mandatory. In addition, the method would appear to have considerable scope for extension to complex failure modelling, including multiple delaminations combined with intra-laminar failure. A draw-back is that sophisticated non-linear solution procedures are required to trace the pseudo-static equilibrium paths which often involve "snap-throughs" and "snap-backs". (These may partially be artifacts of the mesh).

Instead of introducing sophisticated "pseudo-static" path following formulations, the paper has demonstrated that one may adopt a hybrid static/dynamic solution algorithm which switches from statics to dynamics and back to statics.

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All of the examples that have been shown in this paper were run on a research version of the finite element computer program, LUSAS, that is mounted at Imperial College. The authors would like to thank FEA Ltd. for providing this code.

References