Uncertainty of sequences of events

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Abstract. The paper investigates uncertainties of occurrences of sequences of events employing the entropy concept. Some aspects of probabilistic engineering event oriented system analysis are summarized. Joint and successive sequences of events are considered with particular concern with uncertainties that could be associated with them. Numerical examples attempt to demonstrate the procedure and helpfulness to engineering modeling problems for member selection optimization.

Keywords: system analysis, system of events, probability, reliability, entropy, uncertainty, sequence of occurrence, engineering, modeling.

1. Introduction

The probabilistic engineering modeling is normally dealing with probabilities and uncertainties related to occurrences of events. However, some technical problems evoke interest not only in occurrences of single events but also in sequences of occurrences of events. The analysis of sequences of events normally provides the ordering with respect to probability of their occurrences. Since the uncertainties arise not only due to occurrence of a single event out of many other, the paper attempts to investigate the uncertainties related to occurrences of a number of sequences of events. The article reveals the assessments of the uncertainty of joint and successive sequences of occurrences based on the traditional system analysis relying on algebra of events, probability theory and entropy concept [1, 2, 3, 4]. The aim of the paper is to demonstrate how the event oriented system analysis [5, 6, 7] accompanied with uncertainty analysis of sequences of events based on entropy concept presented in the paper may lead to more apt system evaluation in engineering [8]. The numerical example of a reliability analysis of a simple system demonstrates the intention of the paper.

2. Systems of events

A system \( \mathcal{S} \) constituted of disjoint events \( E_i \) with associated probabilities \( p_i = p(E_i) \), \( i = 1, 2, \ldots, n \), is characterized by several statements based on algebra of events [1, 2] and can be presented as a finite scheme:

\[
\mathcal{S} = \left\{ \begin{array}{c}
E_1 \\
E_2 \\
\vdots \\
E_n
\end{array} \right\}
\]

The probability of the system of events \( \mathcal{S} \) is \( p(\mathcal{S}) = \sum_{i=1}^{n} p(E_i) \). The system \( \mathcal{S} \) which satisfies the relation of completeness \( p(\mathcal{S}) = 1 \) is denoted as a complete otherwise it is an incomplete system of events [2]. The Shannon’s entropy [3] is appropriate for uncertainty assessment of complete systems:

\[
H(\mathcal{S}) = -\sum_{i=1}^{n} p(E_i) \log p(E_i) \quad (1)
\]

For incomplete system \( \mathcal{S} \) the Renyi’s entropy of order one [2] is more appropriate:

\[
H^1(\mathcal{S}) = -\frac{1}{l} \sum_{i=1}^{n} p(E_i) \log \frac{p(E_i)}{p(\mathcal{S})} \quad (2)
\]

Other measures of uncertainty are available [4].

3. Sequences of occurrences of events

Any group of events out of the system of all events \( \mathcal{S} \) regardless of their ordering can be viewed as a possible sequence of events. Sequences of different number and permutation of events might be of interest. The sequence can comprise all the events in the system \( \mathcal{S} \) or only some. By simply selection of \( r \) mutually exclusive events of interest out of all \( n \) into a particular set or subsystem \( \mathcal{S}_k \) the probabilities can be summed \( p(\mathcal{S}_{(k)}) = \sum_{j=1}^{r} p(E_{k(j)}) \).
But such a selection is not a sequence. However, the entropy of such a selection of events \( \mathcal{S}_{K(k)} \) can be defined generally as for incomplete system of events by the Renyi’s entropy [2].

Let us consider next for example \( s \) sequences each of \( r_k \) events out of all \( n \) events of the system \( \mathcal{S} \) denoted by \( \omega \) as follows:

\[
\omega_{K(i)} = \begin{pmatrix} E_{K(i),1} & E_{K(i),2} & \ldots & E_{K(i),n} \\ p(E_{K(i),1}) & p(E_{K(i),2}) & \ldots & p(E_{K(i),n}) \end{pmatrix}
\]

\[
\omega_{K(k)} = \begin{pmatrix} E_{K(k),1} & E_{K(k),2} & \ldots & E_{K(k),n} \\ p(E_{K(k),1}) & p(E_{K(k),2}) & \ldots & p(E_{K(k),n}) \end{pmatrix}
\]

\[
\vdots
\]

\[
\omega_{K(s)} = \begin{pmatrix} E_{K(s),1} & E_{K(s),2} & \ldots & E_{K(s),n} \\ p(E_{K(s),1}) & p(E_{K(s),2}) & \ldots & p(E_{K(s),n}) \end{pmatrix}
\]

\( K(k) \) is the \( k \)-th different selection of \( r_k \) elements; \( k=1,\ldots,s \) where \( s \) is the total number of considered sequences.

All \( s \) different sequences comprised of events out of all \( n \) events of basic system \( \mathcal{S} \) when collected together represent a set, i.e. a system of sequences \( \sigma(\mathcal{S}) \) as follows:

\[
\sigma(\mathcal{S}) = \left\{ \omega_{K(1)}, \omega_{K(2)}, \ldots, \omega_{K(s)} \right\}
\]

For complete system of occurrences \( p(\sigma(\mathcal{S})) = \sum_{i=1}^{s} p(\omega_i) = 1 \) otherwise, if all sequences are not accounted for, it is denoted as incomplete. The entropy of the sequence of events \( \sigma(\mathcal{S}) \) can be defined generally as for incomplete system by the Renyi’s entropy [2]:

\[
H[\sigma(\mathcal{S})] = -\sum_{i=1}^{s} \frac{p(\omega_i)}{p(\sigma(\mathcal{S}))} \log \frac{p(\omega_i)}{p(\sigma(\mathcal{S}))}
\]  

### 3.1. Sequential occurrence of events

The probability of a sequence \( p(\omega_{K(k)}) \) in the paper is apprehended as a probability of an event of the sequence occurring. Occurrence of \( r \) independent repeatable events may be viewed either as simultaneous or sequential. To each of such a sequence the probability can be assigned:

\[
p(\omega_{K(k)}) = \prod_{j=1}^{r} p(E_j)
\]  

For example, the complete system of all sequences of joint occurrences of all \( n \) (not necessarily equi-probable) repeatable, always same but differently permuted events can be presented as:

\[
\sigma_{r=n!}^{\omega=n!}(\mathcal{S}) = \left\{ \frac{1}{n!}, \ldots, \frac{1}{n!}, \ldots, \frac{1}{n!} \right\}
\]

Since in such a case all the sequences have the same probability regardless to the probability distribution of the system \( \mathcal{S} \) the entropy (3) of the system of sequences attains its maximum:

\[
H[\sigma_{r=n!}^{\omega=n!}(\mathcal{S})] = \log n!
\]

### 3.2. Successive occurrence of events

The sequences of occurrences when an event that already occurred may not occur again are discussed next. To each of the sequences of non-repeatable events appropriate probability can be assigned under the condition that not any of the events that already occurred can occur again, as presented:

\[
p(\omega_{K(k)}) = \prod_{j=1}^{r} \frac{p(E_j)}{\sum_{m=j}^{n} p(E_m)}
\]  

Particularly, a complete system of all the sequences of all events \( \sigma_{r=n!}^{\omega=n!}(\mathcal{S}) \) fulfills the completeness condition:

\[
p[\sigma_{r=n!}^{\omega=n!}(\mathcal{S})] = \sum_{k=1}^{s} p(\omega_{K(k)}) = \sum_{k=1}^{s} \left[ \prod_{j=1}^{r} \frac{p(E_j)}{\sum_{m=j}^{n} p(E_m)} \right] = 1
\]

Since the probabilities of \( n \) elementary events comprising system \( \mathcal{S} \) are in general different, so the probabilities of \( s \) sequences of occurrences \( \sigma(\mathcal{S}) \) are different too. Therefore, there might exist the most and least probable sequence, as well as an appropriate ranking of sequences by their probabilities. The entropy of the system of sequences of non-repeatable events viewed as a particular set, depends on the probability distribution of elementary events, and therefore it must be calculated according to (3).

In case of non-repeatable but equi-probable successive events all the sequences has the same probability of occurrence \( p(\omega_i) = \frac{1}{n!} \) and the entropy (3) of the system of sequences also
attains its maximum in amount of
\[ H \left[ \sigma_{r=3}^{\infty} (\mathcal{F}) \right] = \log n! . \]

Let us consider the example of three non-repeatable events with different probabilities comprising the system \( \mathcal{F} = \left( E_1 / \frac{3}{6}, E_2 / \frac{2}{6}, E_3 / \frac{1}{6} \right) . \)

The different probabilities of sequences of events are calculated according to (5), Table 1.

The system of all possible sequences \( \sigma_{r=3}^{\infty} (\mathcal{F}) \) (a set) can be presented in the following scheme:

\[
\sigma_{r=3}^{\infty} (\mathcal{F}) = \left( \begin{array}{cccc}
\frac{20}{60} & \frac{10}{60} & \frac{15}{60} & \frac{5}{60} \\
\frac{1}{60} & \frac{2}{60} & \frac{6}{60} & \frac{4}{60}
\end{array} \right).
\]

Obviously it is a complete system due to

\[
p \left[ \sigma_{r=3}^{\infty} (\mathcal{F}) \right] = \sum_{k=1}^{6} p(\sigma_k) = 1.
\]

The ranking by probabilities is given in Table 1.

<table>
<thead>
<tr>
<th>( \sigma_k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( p(\sigma_k) )</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_1 )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{20}{60} )</td>
<td>1</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{10}{60} )</td>
<td>2</td>
</tr>
<tr>
<td>( \sigma_3 )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{15}{60} )</td>
<td>3</td>
</tr>
<tr>
<td>( \sigma_4 )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{5}{60} )</td>
<td>4</td>
</tr>
<tr>
<td>( \sigma_5 )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{6}{60} )</td>
<td>5</td>
</tr>
<tr>
<td>( \sigma_6 )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{2}{6} )</td>
<td>( \frac{3}{6} )</td>
<td>( \frac{4}{60} )</td>
<td>6</td>
</tr>
</tbody>
</table>

The system of all possible sequences \( \sigma_{r=3}^{\infty} (\mathcal{F}) \) is uncertain due to different sequence probabilities, Table 1, as a consequence of different event probabilities and can be assessed by entropy (1) of the system of sequences as:

\[
H \left[ \sigma_{r=3}^{\infty} (\mathcal{F}) \right] = -\sum_{k=1}^{6} p(\sigma_k) \log p(\sigma_k) = 2.35 \text{ bits}
\]

the maximal entropy is \( \log 6 = 2.58 \) bits.

It can be proved that the most and least probable sequences of successive events are those in which the probabilities of non-repeatable events are in ascending or descending order.

### 4. Subsystems of events

Exclusive events in system of events \( \mathcal{F} = \left( \mathcal{F}_1 + \mathcal{F}_2 + \cdots + \mathcal{F}_r \right) \) can be grouped into subsystems of events \( \mathcal{F}_i \), \( i = 1, 2, \ldots, r \) with \( E_j \), \( j = 1, 2, \ldots, m_i \) as it is shown:

\[
\mathcal{F}_i = \left( \begin{array}{ccc}
E_{i1} & \ldots & E_{ij} & \ldots & E_{im_i}
\end{array} \right).
\]

The probability \( p(\mathcal{F}_i) \) associated with each of the subsystems \( \mathcal{F}_i \), \( i = 1, 2, \ldots, r \) is as follows:

\[
p(\mathcal{F}_i) = \sum_{j=1}^{m} p(E_j) \quad (7)
\]

The conditional entropy [2] (2) is as follows:

\[
H_{m_i} (\mathcal{F} / \mathcal{F}_i) = -\sum_{j=1}^{m} \frac{p(E_j)}{p(\mathcal{F}_i)} \log \frac{p(E_j)}{p(\mathcal{F}_i)} \quad (8)
\]

The general relation representing the uncertainty conservation property relates the conditional entropy of \( \mathcal{F} \) and those of the disjoint subsystems \( \mathcal{F}_i \) and can be derived by taking the weighted summa [5] as shown:

\[
\sum_{i=1}^{r} p(\mathcal{F}_i) \cdot H_{m_i} (\mathcal{F} / \mathcal{F}_i) = H(\mathcal{F}) - H(\mathcal{F}') \quad (9)
\]

The entropy (1) of the system of subsystems [2] \( \mathcal{F}' = \left( \mathcal{F}_1', \ldots, \mathcal{F}_i', \ldots, \mathcal{F}_r' \right) \) can be calculated as:

\[
H(\mathcal{F}') = -\sum_{i=1}^{r} p(\mathcal{F}_i') \log p(\mathcal{F}_i') \quad (10)
\]

Any series of \( m_i \) events out of the subsystem of events \( \mathcal{F}_i \) can be considered as a possible sequence of events under the condition of occurrence of the subsystem itself, and presented

\[
\mathcal{F}_k(\mathcal{F}_i) = \left( \begin{array}{ccc}
E_i / \mathcal{F}_i & \ldots & E_{m_i} / \mathcal{F}_i
\end{array} \right)
\]

where \( K(k) \) is the \( k \)-th sequence of \( m_i \) elements, \( k = 1, \ldots, s \) and \( s \) is the number of sequences in consideration. The approach presented for systems of events is applicable for uncertainty assessment of sequences within subsystems of events. It is also applicable for the subsystems of sequences of events.

For example, let us consider a complete system

\[
\mathcal{F} = \left( \begin{array}{cccc}
E_1 & E_2 & E_3 & E_4 \ \\
3 & 2 & 1 & 2 \ \\
\frac{12}{12} & \frac{12}{12} & \frac{12}{12} & \frac{12}{12}
\end{array} \right)
\]
comprised of two subsystems:
\[
\mathcal{S} / \mathcal{S}_1 = \begin{bmatrix} E_{1,1} & E_{1,2} & E_{1,3} \\ \frac{3}{6} & \frac{2}{6} & \frac{1}{6} \end{bmatrix}
\quad \text{and}\quad
\mathcal{S} / \mathcal{S}_2 = \begin{bmatrix} E_{2,1} & E_{2,2} & E_{2,3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.
\]

The appropriate entropy (1) of the system is
\[H(\mathcal{S}) = 2.52 \text{ bits}\]
and of the subsystems (8) are
\[H(\mathcal{S} / \mathcal{S}_1) = 1.46 \text{ bits} \quad \text{and} \quad H(\mathcal{S} / \mathcal{S}_2) = 1.58 \text{ bits}.\]

Note that the systems can be treated under the condition that subsystems occurred and the before mentioned procedure is now applicable to systems of sequences as follows:
\[
\sigma_{r,s}^{e,f}(\mathcal{S} / \mathcal{S}_1) = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ \frac{20}{60} & \frac{10}{60} & \frac{15}{60} & \frac{5}{60} & \frac{6}{60} & \frac{4}{60} \end{bmatrix}
\quad \text{with}\quad
H\left[\sigma_{r,s}^{e,f}(\mathcal{S} / \mathcal{S}_1)\right] = 2.35 \text{ bits} \quad \text{(see Table 1.)}
\]

and
\[
\sigma_{r,s}^{e,f}(\mathcal{S} / \mathcal{S}_2) = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}
\quad \text{with}\quad
H\left[\sigma_{r,s}^{e,f}(\mathcal{S} / \mathcal{S}_2)\right] = 2.58 \text{ bits}.
\]

In addition, the system of sequences \(\sigma_{r,s}^{e,f}(\mathcal{S} / \mathcal{S}_1)\) can for example also be decomposed into two subsystems as shown:
\[
1^\text{st} \sigma_{r,s}^{e,f}(\mathcal{S} / \mathcal{S}_1) = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \frac{20}{60} & \frac{10}{60} \end{bmatrix}
\quad \text{and}\quad
2^\text{nd} \sigma_{r,s}^{e,f}(\mathcal{S} / \mathcal{S}_1) = \begin{bmatrix} \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \\ \frac{15}{60} & \frac{5}{60} & \frac{6}{60} & \frac{4}{60} \end{bmatrix}.
\]

The uncertainties of subsystems of sequences can now be evaluated according to (8) as
\[
H\left[\sigma_{r,s}^{e,f}(\mathcal{S} / \mathcal{S}_1)\right] = 0.92 \text{ bits} \quad \text{and}\quad
H\left[\sigma_{r,s}^{e,f}(\mathcal{S} / \mathcal{S}_2)\right] = 1.78 \text{ bits}.
\]

It can be shown that the relation (9) holds.

\section*{5. Example}

The system under consideration is built up of three independent devices denoted L-left, C-central, R-right, arranged as a combination of two series in parallel [8], Fig. 1.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Event} & \textbf{Mode} & \textbf{p(Ei)} \\
\hline
E_1 & L C R & 0.9000 \\
E_2 & L C R & 0.0391 \\
E_3 & L C R & 0.0391 \\
E_4 & L C R & 0.0017 \\
E_5 & L C R & 0.0183 \\
E_6 & L C R & 0.0008 \\
E_7 & L C R & 0.0008 \\
E_8 & L C R & 0.0008 \\
\hline
\end{tabular}
\caption{Example of three component system}
\end{table}

For assumed device reliabilities \(p(L) = p(R) = 0.9583\) the probabilities of mutually exclusive modes are given in Table 2.

Note that the underlined symbols \(L C R\) denote failed left, central and right device, respectively. The whole system of all the possible events is:
\[
\mathcal{S} = (E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8)
\]
with appropriate entropy \(H(\mathcal{S}) = 0.6414 \text{ bits}\).

The event oriented system analysis [6] identifies subsystems of operational and failure modes based on the system functional diagram, Fig. 1 as
\[
\mathcal{S}^{o} = (E_1, E_2, E_3, E_4, E_5) \quad \text{and} \quad \mathcal{S}^{f} = (E_6, E_7, E_8)
\]
with probabilities (7)
\[
p(\mathcal{S}^{o}) = 0.9984 \quad \text{and} \quad p(\mathcal{S}^{f}) = 0.0016.
\]

The conditional entropy of operational and failure modes in bits (8), sometimes denoted as system redundancy and robustness [7], are:
\[
H(\mathcal{S}^{o} / \mathcal{S}) = 0.6231 \quad \text{and} \quad H(\mathcal{S}^{f} / \mathcal{S}) = 1.1273.
\]

The relation of uncertainty conservation (9) holds.

The sequences of events leading to system collapse are investigated next. The first sequence corresponds to the failure path in four steps starting with the intact mode, followed first by the right device failure, after the left device failure and at the end system collapses when the central device fails. The second sequence corresponds to the failure path in three steps starting with the intact mode, followed first by the central device failure and the system collapses when the right device fails.
There are six sequences of interest in this example appropriate to six possible failure paths:

\( S_1 = (L \; C \; R \; L \; C \; R \; L \; C \; R) \),
\( S_2 = (L \; C \; R \; L \; C \; R \; L \; C \; R) \),
\( S_3 = (L \; C \; R \; L \; C \; R \; L \; C \; R) \),
\( S_4 = (L \; C \; R \; L \; C \; R \; L \; C \; R) \),
\( S_5 = (L \; C \; R \; L \; C \; R \; L \; C \; R) \),
\( S_6 = (L \; C \; R \; L \; C \; R \; L \; C \; R) \).

The calculation of all possible failure paths sequence probabilities according to (5) gives the following system (set) of sequences:

\[
\sigma(\mathcal{J}) = \begin{pmatrix}
0.0000005 & 0.0016 & 0.0046 & 0.0000005 & 0.0016 & 0.0046 \\
\end{pmatrix}
\]

Note that the least probable sequences are \( S_1 \) and \( S_4 \) when first fail the right (left) device, after the left (right) device and at the end system collapse when the central device fails. The most probable sequences are \( S_3 \) and \( S_6 \).

The probability is \( p[\sigma(\mathcal{J})] = 0.0125 \) and the entropy (9) of the set of sequences is:

\[
H[\sigma(\mathcal{J})] = -\sum_{i=1}^{6} \frac{p(S_i)}{p(S)} \log \frac{p(S_i)}{p(S)} = 1.8353 \text{ bits}
\]

The helpfulness of the uncertainty assessment of sequences of occurrences is investigated on a device selection procedure. Under the assumption of constant probability of the system intactness the effect of variation of device reliabilities on the entropy of the sequences of events are given on Fig. 2.

**6. Conclusions**

The article reveals that the probabilistic approach to uncertainty assessment of occurrence of events can be extended to uncertainty assessment of sequences of occurrences of events employing the entropy concept in probability theory.

In case of jointly occurring repeatable events or in case of equi-probable non-repeatable successive events, the probability of sequences is identical and therefore no ordering is available and the entropy attains its maximal value.

However, in case of different probabilities of successive non-repeatable events there are different probabilities for particular sequences of events and the most and least probable sequences of events can be identified by ascending and descending probabilities, respectively. The entropy value in this case is always less of the maximal attainable value.

The intention in this paper was to provide opportunity for evaluation of system properties in engineering problems when progression of events, for example sequences of operational and failure modes leading to collapse are of interest.

**7. References**
