5th International Congress
of Croatian Society of Mechanics
September, 21-23, 2006
Trogir/Split, Croatia

IMPROVED IMPLICIT NUMERICAL SCHEME FOR ONE-DIMENSIONAL OPEN CHANNEL FLOW EQUATIONS

L. Kranjčević, B. Crnković and N. Črnjarić Žic

Keywords: *open channel flow, implicit numerical scheme, balance laws, source term decomposition, channel with irregular geometry*

1. Introduction

Implicit schemes are well known for the property of allowing numerical stability in the resolution of partial differential equations in presence of time steps not restricted by the Courant-Friedrichs-Lewy (CFL) condition. In this work upwind implicit scheme is used for resolution of the one-dimensional open channel flow equations driven by the interest in accurate and efficient numerical modeling of nonhomogeneous hyperbolic system of conservation laws emphasizing space dependency of the flux and significant geometrical source term variation. Some research has been recently oriented to the development of implicit techniques able to capture transcritical transitions and discontinuities. The implicit first-order upwind scheme proposed by Yee [9] was the first non-oscillatory shock capturing implicit scheme. Some authors used implicit numerical approach in solution of open channel flow equations but not accounting for the flux space dependency [1], and in [2] even without the source splitting. Numerically balanced approximations of flux and source terms including the flux space dependency have been introduced by some authors [8], [7], [5], [4] but employing exclusively explicit numerical approach. In this paper we modified original finite volume Linearised Conservative Implicit (LCI) scheme in order to account for the spatially variable flux dependency, and consequently the source term was appropriately decomposed to balance the upwind flux decomposition. These numerical modifications enabled the use of implicit numerical scheme in modeling of the open channel flow equations involving nonprismatic channels with rectangular cross section geometry.

2. The one-dimensional open channel flow equations

The governing one-dimensional open channel flow equations [6] model the homogeneous, incompressible, viscous water flow in rivers and channels characterized by a hydrostatic pressure distribution. The one-dimensional open channel flow equations are based on conservation of mass and conservation of momentum

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0 \tag{1}$$

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\left(\frac{Q^2}{A}+g\cdot I_1(x,A)\right)=g\cdot I_2(x,A)+gA\cdot\left(S_b-S_f\right) \tag{2}$$

where t is time; x is the horizontal distance along the channel; A is the wetted cross-sectional area; Q represents discharge and g is the gravitational acceleration. The friction slope S_f and the bed slope S_b are defined as

$$S_f = \frac{n^2 Q|Q|}{A^2 R_H^{4/3}}, \qquad S_b = -\frac{dz}{dx}(x)$$

while I_1 represents the hydrostatic pressure force term, and I_2 the pressure force term due to longitudinal channel width variation, *h* is the total water depth; *n* the Manning's roughness coefficient; R_H the hydraulic radius. System of partial differential equations (1,2) describes a set of balance laws that written in vector form reads:

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(x,\mathbf{w}) = \mathbf{s}(x,\mathbf{w})$$

$$\mathbf{w} = \begin{pmatrix} A \\ Q \end{pmatrix},\ \mathbf{f}(x,\mathbf{w}) = \begin{pmatrix} Q \\ \frac{Q^2}{A} + gI_1(x,A) \end{pmatrix},\ \mathbf{s}(x,\mathbf{w}) = \begin{pmatrix} 0 \\ gI_2(x,A) + gA(S_b - S_f) \end{pmatrix} \tag{3}$$

in which ***w*** is a vector of flow variables, ***f*** represents flux vector, ***s*** source term vector. Mentioned term I_1 that represents a hydrostatic pressure force term reads

$$I_1 = \int_0^h (h-\eta)\cdot b(x,\eta)\cdot d\eta\,,$$

and I_2 that stands for the pressure forces due to longitudinal width variations reads

$$I_2 = \int_0^h (h-\eta)\cdot \frac{\partial b}{\partial x}(x,\eta)\cdot d\eta$$

or in the partial derivative form

$$I_2 = \frac{\partial I_1}{\partial x} - A\frac{\partial h}{\partial x}\,. \tag{4}$$

The bed slope S_b is the partial derivative of the bottom elevation z :

$$S_b = -\frac{dz}{dx}\,. \tag{5}$$

Equation (3) can be further expressed in quasilinear form as

$$\frac{\partial \mathbf{w}}{\partial t} + J\frac{\partial \mathbf{w}}{\partial x} = \mathbf{s}(x,\mathbf{w})$$

where ***J*** is the Jacobian matrix of ***f*** with respect to ***w***

$$J = \frac{\partial \mathbf{f}}{\partial \mathbf{w}}(\mathbf{w}) = \begin{pmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{pmatrix},\ \text{with velocity } u = \frac{Q}{A},\ \text{and celerity } c = \sqrt{g\frac{\partial I_1}{\partial A}(x,A)}$$

For nonprismatic channels with rectangular cross sections, terms for celerity *c*, I_1, and I_2 simplify to:

$$c = \sqrt{g\frac{A}{b}}\,, \qquad I_1 = \frac{1}{2}h^2 b\,, \qquad I_2 = \frac{1}{2}h^2\frac{\partial b}{\partial x}\,.$$

Since the Jacobian ***J*** has a set real and different eigenvalues,

$$\lambda^{(1)} = u - c\,,\quad \lambda^{(2)} = u + c\,,$$

the nature of partial differential equations (1), (2) is hyperbolic. These eigenvalues correspond to the two characteristic wavespeeds with their signs providing information about the flow direction. The corresponding right eigenvectors of the Jacobian are

$$\mathbf{r}^{(1)} = \begin{pmatrix} 1 \\ \lambda^{(1)} \end{pmatrix} \quad \mathbf{r}^{(2)} = \begin{pmatrix} 1 \\ \lambda^{(2)} \end{pmatrix}. \tag{6}$$

3. Improved upwind implicit scheme with source term decomposed

In this section implicit upwind TVD numerical scheme used is briefly presented, since primary goal was to construct implicit numerical scheme able to deal with channels with variable width and strong variation in bed geometry with friction term included. Implicit scheme used in this paper is based upon Roe's scheme solving Riemann problems approximately on the cell boundaries. Finite volume method is employed, with uniform space step Δx, and adjustable time step Δt. Spatial domain is discretised into a set of cells $[x_{i-1/2}, x_{i+1/2}]$ with cell centers x_i, i=1,…,N. The general form of finite volume numerical scheme used can be written as

$$\mathbf{w}_i^{n+1} + \beta\theta\left(\tilde{\mathbf{f}}_{i+1/2}^{n+1} - \tilde{\mathbf{f}}_{i-1/2}^{n+1}\right) - \theta \cdot \Delta t\, \mathbf{s}_i^{n+1} = \mathbf{w}_i^n - \beta(1-\theta)\left(\tilde{\mathbf{f}}_{i+1/2}^{n} - \tilde{\mathbf{f}}_{i-1/2}^{n}\right) + (1-\theta)\Delta t\, \mathbf{s}_i^n$$

where $\beta = \Delta t / \Delta x$, $0 \le \theta \le 1$ and $\tilde{\mathbf{f}}_{i\pm1/2}^{n,\,n+1}$ numerical flux vector. For $\theta = 0$, the scheme is explicit and for $\theta = 1$ Euler implicit time integration is achieved, i.e., it is the fully implicit case. For implicit case numerical flux vector is evaluated

$$\tilde{\mathbf{f}}_{i+1/2}^{n+1} = \frac{1}{2}\left(\mathbf{f}_{i+1}^{n+1} + \mathbf{f}_i^{n+1}\right) - \frac{1}{2}\left|\tilde{\mathbf{Q}}_{i+1/2}\right|\left(\mathbf{w}_{i+1}^{n+1} - \mathbf{w}_i^{n+1}\right) - \frac{1}{2}\tilde{\mathbf{Q}}_{i+1/2}^{-1}\left|\tilde{\mathbf{Q}}_{i+1/2}\right|\mathbf{V}_{i+1/2}^{n+1}\Delta x \tag{7}$$

with matrix $\tilde{\mathbf{Q}}_{i+1/2}$ being approximate Jacobian of the flux at the cell interface

$$\tilde{\mathbf{Q}}_{i+1/2} = \begin{bmatrix} 0 & 1 \\ c_{i+1/2}^2 - u_{i+1/2}^2 & 2u_{i+1/2} \end{bmatrix}$$

and matrix

$$\left|\tilde{\mathbf{Q}}_{i+1/2}\right| = \mathbf{R}_{i+1/2}\, diag\left[\Phi_{i+1/2}^k\right]\mathbf{R}_{i+1/2}^{-1}.$$

where $\mathbf{R}$ is the matrix containing right eigenvectors (6). Choice of matrix $\Phi_{i+1/2}$ defines the type of scheme from family of schemes. For *Symmetric LCI Form*

$$\Phi_{i+1/2}^k = -\psi\left(\tilde{\lambda}_{i+1/2}^k\right)\left[1 - \left(L_{i+1/2}^k\right)/\alpha_{i+1/2}^k\right] \;, k = 1,2$$

where α is wave strength, L some limiter function and Ψ entropy correction function to eigenvalues λ of Jacobian $\mathbf{J}$ that gurantees physically valid discontinuities in the solution [3]. Accuracy of this scheme is second order in time and space. Limiter function was set to zero and scheme became first order in space, but it made it more robust admitting bigger CFL values. Vector $\mathbf{V}$ in (7) accounts for space dependency of the flux

$$\mathbf{V} = \frac{\partial \mathbf{f}}{\partial x} = \begin{bmatrix} 0 \\ g\dfrac{\partial I_1}{\partial x} \end{bmatrix}.$$

Source term is appropriately decomposed to balance the flux upwind decomposition in (7) [8]:

$$\mathbf{s}_i^{n+1} = \mathbf{s}_{i+1/2}^{n+1,L} + \mathbf{s}_{i-1/2}^{n+1,R}$$

where

$$\mathbf{s}_{i+1/2}^{n+1,L} = \frac{1}{2}\left(I - \tilde{\mathbf{Q}}_{i+1/2}^{-1}\left|\tilde{\mathbf{Q}}_{i+1/2}\right|\right)\mathbf{S}_{i+1/2}^{n+1}$$

$$\mathbf{s}_{i-1/2}^{n+1,R} = \frac{1}{2}\left(I + \tilde{\mathbf{Q}}_{i-1/2}^{-1}\left|\tilde{\mathbf{Q}}_{i-1/2}\right|\right)\mathbf{S}_{i-1/2}^{n+1}$$

In above expressions vector $\mathbf{S}$ represents the part of the source term (3) that is decomposed

$$\begin{bmatrix} 0 \\ gI_2 + gAS_b \end{bmatrix}. \tag{8}$$

Balance between flux and source terms decomposition can only be obtained locally by balancing non-zero fluxes through the edges of a finite volume, because the numerical source integral cannot, in general, be written as a difference. It is possible to overcome this problem when the source term takes the form of a derivative (4), (5) and that is the case with part of the source term (8) that can simply augment the conservative flux. The remaining part of the source term

$$\begin{bmatrix} 0 \\ -gAS_f \end{bmatrix}$$

is applied in pointwise manner.
Different terms need to be linearised using Taylor expansion around w^n:

$$\mathbf{f}_i^{n+1} \approx \mathbf{f}_i^n + \mathbf{J}_i^n \left(\mathbf{w}_i^{n+1} - \mathbf{w}_i^n \right)$$

$$\mathbf{V}_{i+1/2}^{n+1} \approx \mathbf{V}_{i+1/2}^n + \mathbf{J}_{V\,i+1/2}^{\;n} \left(\mathbf{w}_{i+1/2}^{n+1} - \mathbf{w}_{i+1/2}^n \right)$$

$$\mathbf{S}_{i+1/2}^{n+1} \approx \mathbf{S}_{i+1/2}^n + \mathbf{J}_{S\,i+1/2}^{\;n} \left(\mathbf{w}_{i+1/2}^{n+1} - \mathbf{w}_{i+1/2}^n \right)$$

The numerical procedure described produces block tridiagonal linear system of equations with block matrices
[2 x 2], that is to be solved in each time step:

$$\mathbf{A}_i \mathbf{w}_{i-1}^{n+1} + \mathbf{B}_i \mathbf{w}_i^{n+1} + \mathbf{C}_i \mathbf{w}_{i+1}^{n+1} = \mathbf{b}_i, \qquad i = 1, \ldots, N.$$

where $\mathbf{A_i}$, $\mathbf{B_i}$, $\mathbf{C_i}$ are block matrices that are elements of the global matrix of the system and $\mathbf{d_i}$ represents RHS vector containing source term and some data from previous time level t^n . For brevity, treatment of boundary conditions in implicit case will not be discussed here.

4. Numerical tests

In this section, numerical simulations of two test cases are presented. Chosen test problems validate the correct balancing of the flux gradient and the source term. In order to illustrate the quality and the achieved improvements, these balanced schemes are compared with their versions with pointwise evaluated source terms. Since the first order and flux limited versions of the scheme give similar results, from the point of view of obtained improvements, tests are computed using non-limited schemes.

4.1 Dam break test problem with variable bed with friction

Dam break test problem was used to investigate and compare the ability of balanced and pointwise scheme to deal with propagating discontinuities. Since implicit method for solving open channel flow problems was originally derived for solving Euler equations dealing with steady state problems in gas dynamics, their behavior in unsteady calculation of dam break problem is shown.

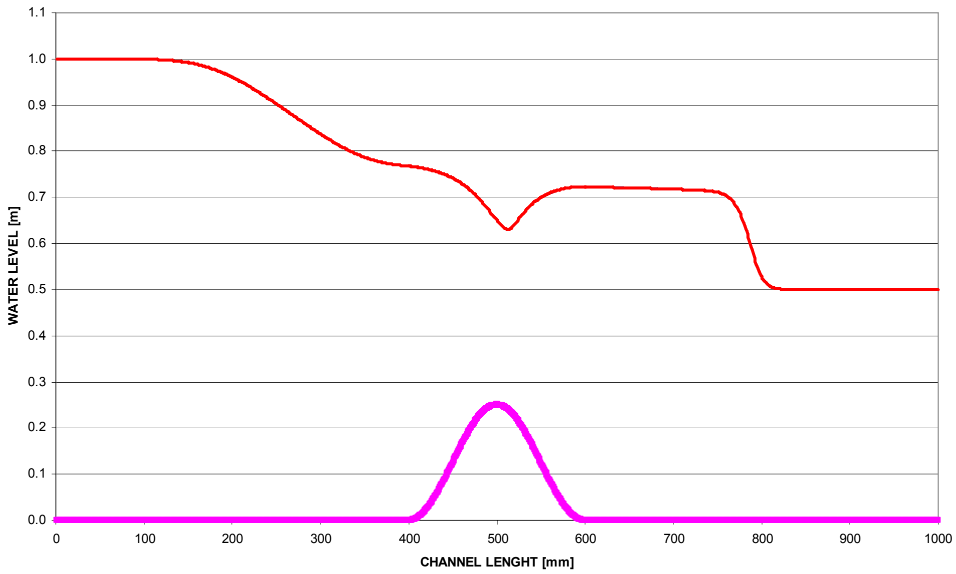

Figure 1. TEST 4.1 - Dam break –water levels with pointwise and balanced scheme, t=0.1s

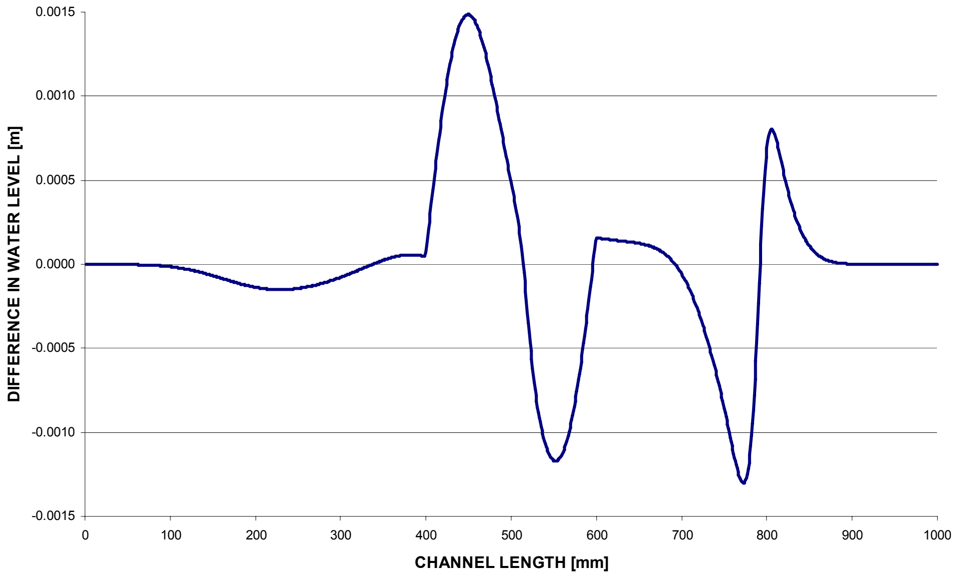

Figure 2. TEST 4.1 - Dam break – Difference in calculated water levels with pointwise and balanced scheme, t=0.1s

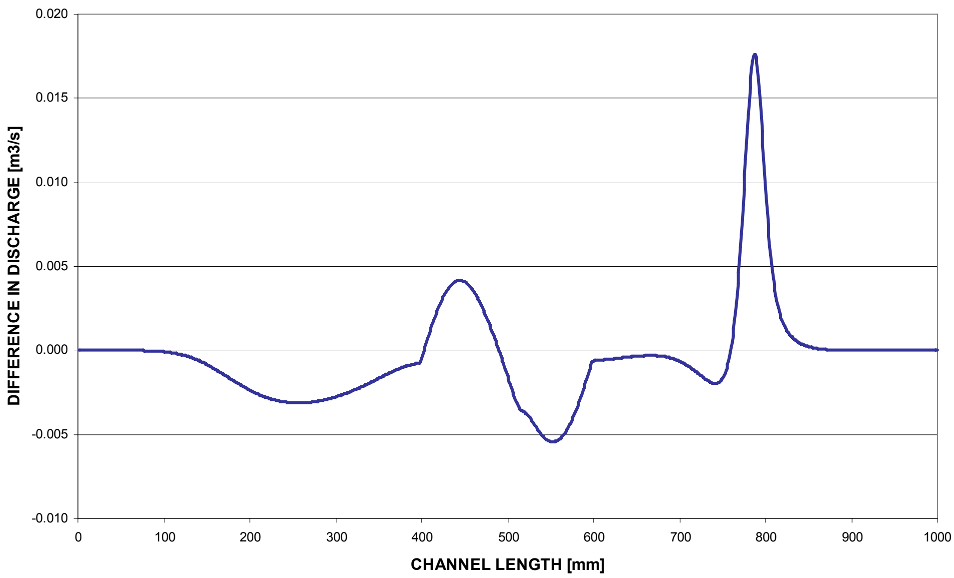

Figure 3. TEST 4.1 - Dam break – Difference in discharge *Q* with pointwise and balanced scheme, t=0.1s

Test channel is prismatic and rectangular with bottom width b=1m, channel length L=1 m and bed friction given by Manning factor n=0.03. Initially, water is at rest with water level $h_{0,L}$=1 m in the left half of the channel and $h_{0,R}$=0.5 m in the right part. At the time level t=0 s imaginary dam located in the middle of the channel is removed instantly.

Propagating shock is tracked satisfactory with both versions of the scheme, but it is obvious that shock front "smears" as time steps and CFL numbers increase and numerical dissipation effects become significant. Test is calculated with Courant number CFL=3 with still significant gains in computational time and moderately “smeared” shock front. Balanced and pointwise scheme showing matching result in Figure 1 in raster given in the picture in fact showed difference in water level and discharge results in order of 10^{-3} m Figure 2, Figure 3.

4.2 The quiescent flow test in rectangular prismatic channel with variable bed with friction

Good shock tracking capabilities of the both versions of the scheme being proved in the previous test case, this test tries to emphasize the quality of the balanced scheme with the quiescent flow test case. Channel specifications are identical to the previous test case (4.1) and only difference is in water level being initially set to constant level h_0= 1m throughout the whole channel length.

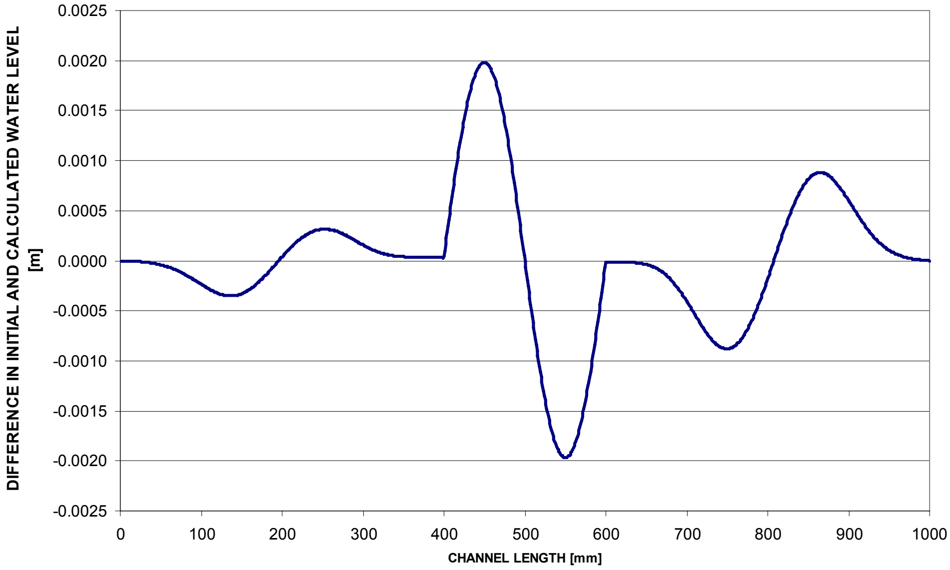

Figure 4. TEST 4.2 – Quiescent flow – Difference in water level between initial and calculated water levels calculated with pointwise scheme

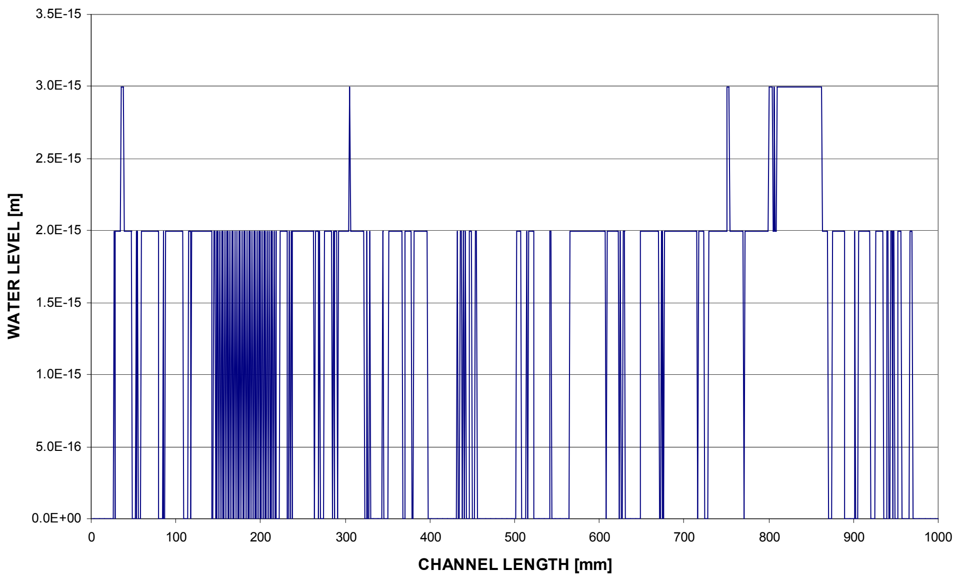

Figure 5. TEST 4.2 – Quiescent flow – Difference in water level between initial and calculated water levels calculated with balanced scheme

In this test case numerical gains with balanced scheme are fully revealed. Presence of the channel source term (the channel bed "bump") in this test case, shows the necessity of source term decomposition and furthermore, decomposition has to be done in balance with flux upwind decomposition. Pointwise scheme showed errors of the order 10^{-3} m Figure 4, while balanced scheme gave excellent results with error in water level of the order 10^{-15} m (Figure 5) that is in fact product of the computer double precision round off error.

4.3 The quiescent flow in a channel proposed by the working group on dam break modeling

Finally, numerical capabilities of improved balanced scheme proposed are fully revealed through the demanding test case involving channel with strong variations in bed geometry and strong channel width variations with friction. The channel geometry is proposed by the working group on dam break modeling [7] .

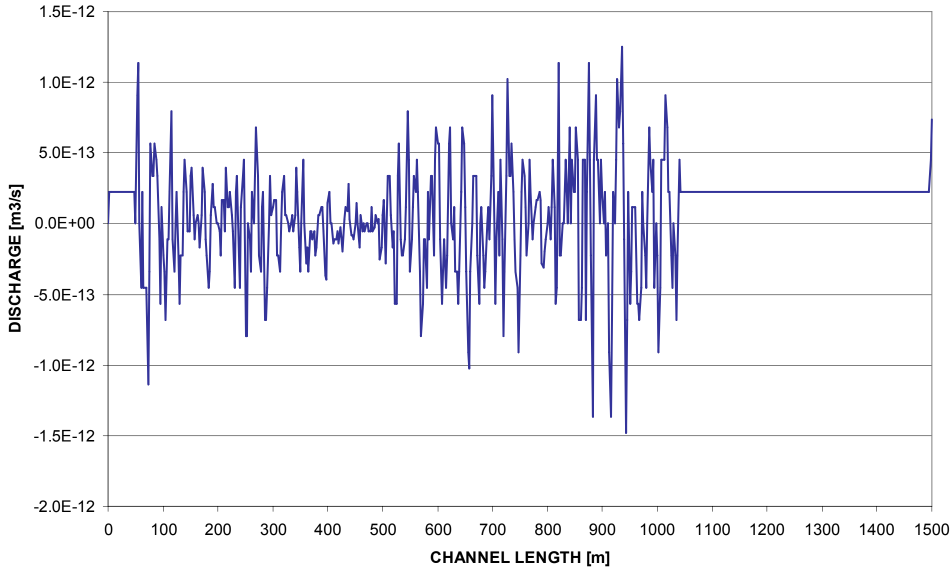

Figure 6. TEST 4.3 – Quiescent flow – Discharge calculated with balanced scheme

Computation is performed using constant space step $\Delta x = 2.5m$. Figure 6 shows excellent results with discharge oscillations along the channel reach of the order 10^{-12} . Error for the water level is of the order smaller than 10^{-20} while pointwise version of the numerical scheme produces unacceptably large errors as expected.

5. Conclusion

Two versions of conservative implicit methods belonging to the family of linearized upwind schemes have been presented. Improved balanced implicit scheme was tested and compared with simple non-balanced pointwise version of the scheme on different open channel test problems which include friction, nonuniform bed slopes, strong channel width variations and dam break problem with analysis of transient solution. In unsteady case both methods proved to be nondispersive, showing good shock tracking capability, and with moderate CFL numbers produced acceptably dissipative results. The one-dimensional open channel flow equations numerical modeling approach proposed showed excellent results because successful balancing of the implicit upwind scheme was performed reaching exact conservation property and still keeping all the benefits that implicit numerical approach brings.

References

[1] Bermudez, A., Dervieux, A., Desideri, J., Vazquez, M.E., "Upwind Schemes for the Two-Dimensional Shallow Water Equations with Variable Depth Using Unstructured Meshes", INRIA Rapport de recherche, No 2738, 1995.

[2] Delis, A.I., Skeels, C.P., Ryrie, S.C., "Implicit high resolution methods for nodelling one-dimensional open channel flow", J. of Hydraulic Research. vol. 38, 2000. No. 5
[3] Delis, A.I., Skeels, C.P., *TVD* schemes for open channel flow Int. J. Numer. Meth. Fluids, 26(7):791, 1998.
[4] Garcia-Navarro, P., Vazquez-Cendon, M.E., "On numerical treatment of the source terms in the shallow water equations", Computers & Fluids 29 (2000) 951-979
[5] Hubbard, M.E., Garcia-Navarro, P., "Flux Difference Splitting and the Balancing of SOurce Terms and Flux Gradients", Journal of Comput. Physics 2000..; 165:89-125.
[6] MacDonald, I., "Analysis and Computation of Steady Open Channel Flow", Ph.D. thesis, University of Reading, Department of Mathematics, 1996.
[7] Vazquez-Cendon, M.E., "Improved treatment of source terms in upwind schemes for shallow water equations in channel with irregular geometry", J. Comput. Phys. 148, 497-526, 1999.
[8] Vukovic, S., Sopta, L., "Upwind schemes with exact conservation property for one-dimensional open channel flow equations", SIAM J. SCI. COMPUT., Vol. 24, No. 5, pp. 1630-1649, 2003.
[9] Yee HC, "Construction of explicit and implicit symmetric TVD schemes and their applications", Journal of Comput. Physics 1987.; 68:151-179.

Lado Kranjčević, Ph.D., Assist.
Faculty of Engineering/University of Rijeka, Department of Fluid Mechanics and Computational Engineering, Vukovarska 58, Rijeka, Croatia, Phone 00385 (0)51 651444, Fax 00385 (0)51 651490, lado.kranjcevic@riteh.hr

Bojan Crnković, Assist.
Faculty of Engineering/University of Rijeka, Department of Fluid Mechanics and Computational Engineering, Vukovarska 58, Rijeka, Croatia, Phone 00385 (0)51 651444, Fax 00385 (0)51 651490, bojan.crnkovic@riteh.hr

Nelida Črnjarić Žic, Ph.D., Assist. Prof.
Faculty of Engineering/University of Rijeka, Department of Fluid Mechanics and Computational Engineering, Vukovarska 58, Rijeka, Croatia, Phone 00385 (0)51 651444, Fax 00385 (0)51 651490, nelida@riteh.hr