Finite Element Discretisation of 3D Solids and 3D Beams Obtained by Constraining the Continuum

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Summary

In this note the finite element equations of motion for 3D solids and 3D beams are consistently derived from the underlying Cauchy's equations of motion.

1. Introduction

In this note the finite element equations of motion for 3D solids and 3D beams are derived from the underlying Cauchy's equations of motion. The following aspects are addressed in formulating the finite element equations for 3D solids in Section 2 (References 1-8 should be consulted for a deeper understanding):

1. Formulation of the weak form of the problem
2. Integration by parts and manipulation of differential operators
3. Piola transform (change of variables and pull-back)
4. Interpolation of test functions and additional tensorial transformations

In Section 3, the beam kinematics is presented and the finite element equations are derived by introducing it into the continuous weak form of the equations of motion for 3D solids given in Section 2. The following aspects are stressed (References 8-14 should be consulted for a deeper understanding):

5. Configuration space and its kinematically admissible perturbation

6. Geometric properties of the cross section as a consequence of the beam kinematics

7. Angular velocities and accelerations and specific momenta due to the beam kinematics

8. Stress resultants and applied loading due to the beam kinematics

9. Vector form of the strain measures due to the beam kinematics

In Section 4, an attempt at justifying the linear relationship between the stress resultants and the vector strain measures for a linear elastic material is made.

2. 3D elastodynamics

2.1 Kinematics of the deformation

Consider a body, denoted as $\mathcal{B}$, assumed to be subject to a smooth invertible mapping into the ambient space $\phi : \mathcal{B} \rightarrow \mathcal{R}^3$. Let the surface of the body $\partial\mathcal{B}$ be Lipshitz-continuous (smooth and simply connected, loosely speaking) and consist of two parts: part $\mathcal{E}_u$ with prescribed kinematics and parts part $\mathcal{E}_p$ with prescribed surface tractions such that

$$\mathcal{E}_u \cap \mathcal{E}_p = \emptyset \tag{2.1}$$

$$\mathcal{E}_u \cup \mathcal{E}_p = \partial\mathcal{B}. \tag{2.2}$$

Normally, we will distinguish between the mappings that take place at different times by defining a mapping $\phi_t : \mathcal{B} \times \mathcal{R}_+ \rightarrow \mathcal{R}^3$. At time t, $\mathcal{B}$, $\mathcal{E}_u$ and $\mathcal{E}_p$ are mapped into $\phi_t(\mathcal{B}) \subset \mathcal{R}^3$, $\phi_t(\mathcal{E}_u) \subset \mathcal{R}^2$ and $\phi_t(\mathcal{E}_p) \subset \mathcal{R}^2$, where $\phi_t(\bullet)$ is a shorthand notation for $\phi(\bullet, t)$.

For a chosen material particle X in a body define a mapping $\boldsymbol{\kappa} : \mathcal{B} \rightarrow \mathcal{R}^3$ and denote the *reference position vector* of material particle X at a chosen time $t = \bar{t}$ as $\boldsymbol{X} = \boldsymbol{\kappa}(X)$. Also define

$$\boldsymbol{x} = \phi_t(\boldsymbol{X}) \tag{2.3}$$

as the position vector of material particle X (pointed at by the position vector $\boldsymbol{X}$ in the reference state) in the *current state*, at time t.

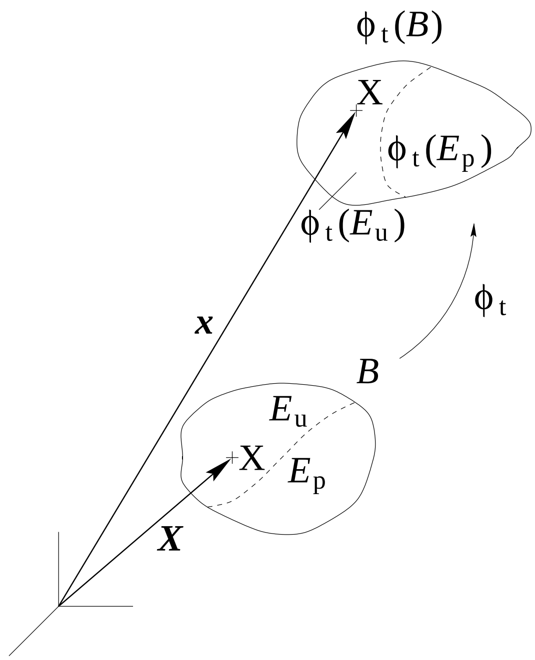

Figure 1. Mappings between the reference state and the current state

2.2 Cauchy's equations of motion and their weak form

The (local) Cauchy's equations of motion are defined in the current state (at time t) as [1]

$$\text{div}\boldsymbol{\sigma} + \boldsymbol{p}_v = \rho\ddot{\boldsymbol{x}} \quad \text{on} \quad \phi_t(\mathcal{B}), \tag{2.4}$$

where $\boldsymbol{\sigma}$ is the Cauchy (true) stress tensor, $\text{div}\boldsymbol{\sigma} = \boldsymbol{\sigma}\nabla\boldsymbol{x}$, $\boldsymbol{p}_v$ is a distributed loading per unit volume, ρ is the current density of the material and a superimposed dot indicates a material time differentiation. For an arbitrary position vector $\boldsymbol{x}$, given componentially as $\boldsymbol{x} = x^i\boldsymbol{E}_i = \begin{Bmatrix} x^1 \\ x^2 \\ x^3 \end{Bmatrix}$, the differential operator $\nabla\boldsymbol{x}$ is defined as $\nabla\boldsymbol{x} = \frac{\partial}{\partial x^i}\boldsymbol{E}_i = \begin{Bmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^3} \end{Bmatrix}$. This equation has to be complemented with the natural boundary conditions

$$\boldsymbol{\sigma n} - \boldsymbol{p}_s = \boldsymbol{0} \quad \text{on} \quad \phi_t(\mathcal{E}_p), \tag{2.5}$$

where $\boldsymbol{n}$ is an outward normal to surface $\phi_t(\mathcal{E}_p)$ and $\boldsymbol{p}_s$ is a distributed loading per unit area of this surface. Furthermore, there are also essential boundary conditions

$$\boldsymbol{x} = \bar{\boldsymbol{x}} \quad \text{on} \quad \phi_t(\mathcal{E}_u). \tag{2.6}$$

The weak form of Cauchy's equation of motion is obtained by taking a dot product of Eqns (2.4) and (2.5) with some test function $\boldsymbol{v}$, which belongs to the same set of functions as the admissible variations of the configuration. The test functions are required to vanish on $\phi_t(\mathcal{E}_u)$, thus identically satisfying Eqn (2.6). By integrating the dot product of Eqns (2.4) and (2.5) with $\boldsymbol{v}$ over the domain of the definition of the problem we obtain the following weak form

$$G(\boldsymbol{x}, \boldsymbol{v}) \equiv \int_{\phi_t(\mathcal{B})} \boldsymbol{v} \cdot (\text{div}\boldsymbol{\sigma} + \boldsymbol{p}_v - \rho\ddot{\boldsymbol{x}})\, dV - \int_{\phi_t(\mathcal{E}_p)} \boldsymbol{v} \cdot (\boldsymbol{\sigma n} - \boldsymbol{p}_s)\, dS = 0. \tag{2.7}$$

By introducing the identity $\boldsymbol{v} \cdot \text{div}\boldsymbol{\sigma} = \text{div}\,(\boldsymbol{\sigma v}) - \text{grad}\boldsymbol{v} : \boldsymbol{\sigma}$, where $\text{grad}\boldsymbol{v} = \boldsymbol{v} \otimes \nabla_{\boldsymbol{x}} = \frac{\partial v^i}{\partial x^j} \boldsymbol{E}_i \otimes \boldsymbol{E}_j = \begin{bmatrix} \frac{\partial v^1}{\partial x^1} & \frac{\partial v^1}{\partial x^2} & \frac{\partial v^1}{\partial x^3} \\ \frac{\partial v^2}{\partial x^1} & \frac{\partial v^2}{\partial x^2} & \frac{\partial v^2}{\partial x^3} \\ \frac{\partial v^3}{\partial x^1} & \frac{\partial v^3}{\partial x^2} & \frac{\partial v^3}{\partial x^3} \end{bmatrix}$, a colon indicates a double tensor contraction, and applying the divergence theorem via $\int_{\phi_t(\mathcal{B})} \text{div}(\boldsymbol{\sigma v}) dV = \int_{\partial\phi_t(\mathcal{B})} \boldsymbol{v} \cdot \boldsymbol{\sigma n} dS$ and noting that $\int_{\partial\phi_t(\mathcal{E}_p)} \boldsymbol{v} \cdot \boldsymbol{\sigma n} dS = 0$ due to the kinematic admissibility of the test functions, Eqn (2.7) turns into

$$G(\boldsymbol{x}, \boldsymbol{v}) \equiv -\int_{\phi_t(\mathcal{B})} [\text{grad}\boldsymbol{v} : \boldsymbol{\sigma} + \boldsymbol{v} \cdot (\rho\ddot{\boldsymbol{x}} - \boldsymbol{p}_v)]\, dV + \int_{\phi_t(\mathcal{E}_p)} \boldsymbol{v} \cdot \boldsymbol{p}_s dS = 0. \tag{2.8}$$

Let us introduce the Piola transform of $\boldsymbol{\sigma}$ via $\boldsymbol{\sigma} = \frac{1}{J} \boldsymbol{P F}^t$ with $J = \text{det}\boldsymbol{F}$, $\boldsymbol{F} = \text{GRAD}\boldsymbol{x} = \boldsymbol{x} \otimes \nabla_{\boldsymbol{X}}$ and $\boldsymbol{P}$ the first Piola-Kirchhoff stress tensor [1]. The first term in Eqn (2.8) then transforms as $\int_{\phi_t(\mathcal{B})} \text{grad}\boldsymbol{v} : \boldsymbol{\sigma} dV = \int_{\mathcal{B}} \text{grad}\boldsymbol{v} : \boldsymbol{P}\boldsymbol{F}^t dV_0$. By evaluating the tensor contraction via $\boldsymbol{A} : \boldsymbol{B} = \text{tr}\left(\boldsymbol{A}\boldsymbol{B}^t\right) \ \forall \ \boldsymbol{A}, \boldsymbol{B} \in \text{End}(n), n \in \mathcal{N}$ and using the identity $\text{grad}\boldsymbol{v}\boldsymbol{F} = \text{GRAD}\boldsymbol{v}$, we further obtain $\int_{\mathcal{B}} \text{grad}\boldsymbol{v} : \boldsymbol{P}\boldsymbol{F}^t dV_0 = \int_{\mathcal{B}} \text{GRAD}\boldsymbol{v} : \boldsymbol{P} dV_0$. Introducing this result and $\rho dV = \rho_0 dV_0$ as well as Nanson's formula $\boldsymbol{n} dS = J\boldsymbol{F}^{-t}\boldsymbol{n}_0 dS_0$ [2] into Eqn (2.8) gives

$$G(\boldsymbol{x}, \boldsymbol{v}) \equiv -\int_{\mathcal{B}} \left[\text{GRAD}\boldsymbol{v} : \boldsymbol{P} + \boldsymbol{v} \cdot (\rho_0 \ddot{\boldsymbol{x}} - \boldsymbol{p}_{v0})\right] dV_0 + \int_{\mathcal{E}_p} \boldsymbol{v} \cdot \boldsymbol{p}_{s0} dS_0 = 0. \qquad (2.9)$$

2.3 Interpolation of test functions

Let us spatially discretise the problem by choosing N_i nodal points on body $\mathcal{B}_i$ and approximating the test function $\boldsymbol{v}^{(i)}$ as follows

$$\boldsymbol{v}(\boldsymbol{X}) \doteq \boldsymbol{v}^h(\boldsymbol{X}) = I^j(\boldsymbol{X})\boldsymbol{v}_j \ , \quad j = 1, \ldots, N, \qquad (2.10)$$

where $I^j(\boldsymbol{X})$ are Lagrangian polynomials satisfying the standard conditions

$$I^j(\boldsymbol{X}_k) = \delta^j_k \ , \quad \sum_{j=1}^{N_i} I^j(\boldsymbol{X}) = 1 \ , \quad \sum_{j=1}^{N_i} \nabla I^j(\boldsymbol{X}) = 0$$

and the summation convention on the repeated indices is assumed. By noting the following identity

$$\begin{aligned} \text{GRAD}\boldsymbol{v}^h : \boldsymbol{P} &= [(I^j \boldsymbol{v}_j) \otimes \nabla_{\boldsymbol{X}}] : \boldsymbol{P} = [\boldsymbol{v}_j \otimes (\nabla_{\boldsymbol{X}} I^j)] : \boldsymbol{P} = \text{tr}[\boldsymbol{v}_j \otimes (\nabla_{\boldsymbol{X}} I^j)\boldsymbol{P}^t] \\ &= \text{tr}[\boldsymbol{v}_j \otimes (\boldsymbol{P}\nabla_{\boldsymbol{X}} I^j)] = \boldsymbol{v}_j \cdot \boldsymbol{P}\nabla I^j(\boldsymbol{X}), \end{aligned}$$

the approximate weak form follows from Eqns (2.9) and (2.10) as

$$G^h(\boldsymbol{x}, \boldsymbol{v}_j^{(i)}) \equiv -\boldsymbol{v}_j \cdot \underbrace{\left[\left(\boldsymbol{q}_k^j + \boldsymbol{q}_m^j\right) - \boldsymbol{q}_e^j\right]}_{\boldsymbol{g}^j} = 0.$$

$$\boldsymbol{q}_k^j = \int_{\mathcal{B}} \boldsymbol{P} \nabla I^j dV_0$$

$$\boldsymbol{q}_m^j = \int_{\mathcal{B}} I^j \rho_0 \ddot{\boldsymbol{x}} dV_0$$

$$\boldsymbol{q}_e^j = \int_{\mathcal{B}} I^j \boldsymbol{p}_{v0} dV_0 + \int_{\mathcal{E}_p} I^j \boldsymbol{p}_{s0} dS_0$$

are the standard vectors of internal, inertial and external forces, respectively, at node j of the element (body) and $\boldsymbol{g}^j$ is the standard dynamic residual vector at node j of the element (body).

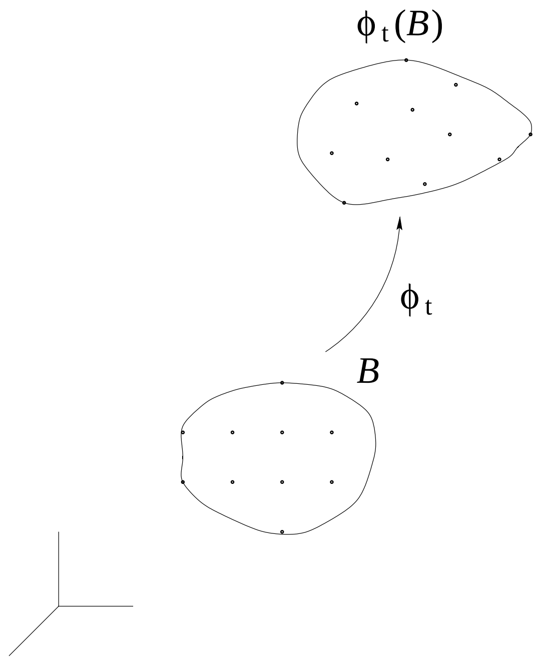

Figure 2. Finite element discretisation

3. 3D beam theory obtained by constraining 3D continuum

3.1 Beam kinematics

A solid body is referred to as a *beam* if two of its dimensions are much smaller than its third dimension. For such a body, it is useful to define the *centroid axis* of the beam as the collection of centroids of the two-dimensional segments (called the *cross sections*) spanned by the base vectors pointing along the two smaller dimensions of the beam. Let us denote the position vector of the centroid axis at time t as $\boldsymbol{r}_t$. For a beam of initial length L, which undergoes an arbitrary deformation in the ambient space $\mathcal{R}^3$, we therefore have

$$\boldsymbol{r}_t(p) = \boldsymbol{r}(p,t) : [0,L] \times \mathcal{R}_+ \rightarrow \mathcal{R}^3, \tag{3.1}$$

where $p \in [0,L]$ is an arc-length parameter of the beam, to be defined in more detail later. Also, it is reasonable to introduce certain kinematic hypotheses (the beam hypotheses), which serve to reduce the total number of degrees of freedom of a discretised system to be arrived at at a later stage. Furthermore, they will prove indispensable in the process of applying the theory from Section 2. The most important beam hypothesis is that of the undeformability of cross sections. By denoting the cross section at $p \in [0,L]$ as $\mathcal{A}(p)$ and by introducing a set of *material* coordinates $\{X^1 = p, X^2, X^3\}$ which parametrise the initial configuration of the beam with respect to an orthonormal basis of base vectors $\boldsymbol{G}_i$ $(i = 1,2,3)$ as depicted in Fig. 3, we have the spatial position vector of an arbitrary material particle within the beam body as the following mapping

$$\boldsymbol{x}_t(X^i) = \boldsymbol{x}(X^i,t) : [0,L] \times \mathcal{A}(X^1) \times \mathcal{R}_+ \rightarrow \mathcal{R}^3. \tag{3.2}$$

Here and throughout the text, we *choose* to parametrise the kinematic quantities with respect to the initial state (other choices are possible). We will focus our attention only to beams which are initially straight and have uniform cross section $\mathcal{A}(X^1) = \text{const}$. We also define a vector basis of a spatially fixed *inertial* frame and denote it as $\boldsymbol{E}_i$ $(i = 1,2,3)$ with co-ordinates $\{X, Y, Z\}$, so that the material frame is defined with respect to it via

$\boldsymbol{G}_i = \boldsymbol{E}_i$. At time t, we define an orthonormal "moving" frame rigidly attached to a cross section at X^1 via

$$\boldsymbol{g}_{i,t}(X^1) = \boldsymbol{g}_i(X^1, t) = \boldsymbol{\Lambda}(X^1, t)\boldsymbol{E}_i, \tag{3.3}$$

where $\boldsymbol{\Lambda} \in SO(3)$ is a proper orthogonal transformation satisfying $\det\boldsymbol{\Lambda} = 1$ and $\boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}^t$.

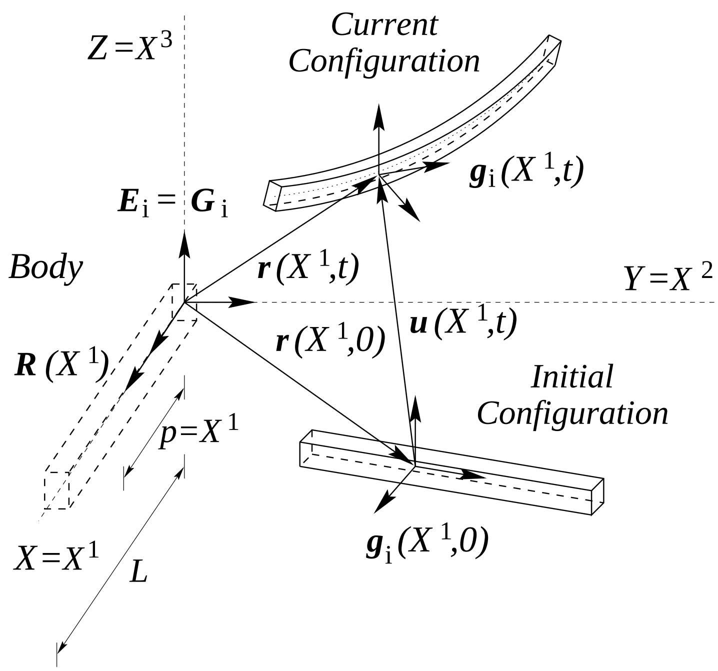

Figure 3. Beam kinematics

Note that in general $\frac{\partial}{\partial X^1}\boldsymbol{r}(X^1, t) \neq \boldsymbol{g}_1(X^1, t)$ unless $t = 0$. The position vector $\boldsymbol{x}_t(X^i) = \boldsymbol{x}(X^i, t)$ can now be expressed as

$$\boldsymbol{x}_t(X^i) = \boldsymbol{r}_t(X^1) + X^\alpha \boldsymbol{g}_\alpha(X^1) = \boldsymbol{r}_t(X^1) + \boldsymbol{\Lambda}(X^1, t)X^\alpha \boldsymbol{E}_\alpha, \tag{3.4}$$

where $\alpha = 2, 3$ and $(X^2, X^3) \in \mathcal{A}$. Greek indices will be used whenever summation runs over values 2 and 3. The deformed configuration of the beam is therefore completely defined by its deformed position vector and the collection of the moving frames along the deformed centroid axis. We say that a *configuration space* $\mathcal{C}$ is defined as $\mathcal{C} = \{(\boldsymbol{r}, \boldsymbol{\Lambda}) : [0, L] \rightarrow \mathcal{R}^3 \times SO(3)\}$. By superposing a perturbation $\epsilon\boldsymbol{\nu} = (\epsilon\boldsymbol{\eta}, \epsilon\boldsymbol{\mu})$ onto configuration $\phi = (\boldsymbol{r}, \boldsymbol{\Lambda})$ we arrive at an adjacent configuration

$$\phi_\epsilon = (\boldsymbol{r}_\epsilon, \boldsymbol{\Lambda}_\epsilon) = (\boldsymbol{r} + \epsilon\boldsymbol{\eta}, \exp \widehat{\epsilon\boldsymbol{\mu}} \boldsymbol{\Lambda}) \,. \tag{3.5}$$

The *linearised kinematics* follows from the definition $D\phi \cdot \boldsymbol{\nu} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \phi_\epsilon$ as

$$D\phi \cdot \boldsymbol{\nu} = (\boldsymbol{\eta}, \widehat{\boldsymbol{\mu}}\boldsymbol{\Lambda}) \,. \tag{3.6}$$

Note that $\widehat{\boldsymbol{\mu}}\boldsymbol{\Lambda} \notin SO(3)$ hence $D\phi \cdot \boldsymbol{\nu} \notin \mathcal{C}$. The definition of a *tangent* space to $\mathcal{C}$ (at identity) then follows as $\mathcal{T}_{(\boldsymbol{r},\boldsymbol{I})}\mathcal{C} = \{(\boldsymbol{\eta}, \widehat{\boldsymbol{\mu}}) : [0, L] \rightarrow \mathcal{R}^3 \times so(3)\}$. Due to a co-ordinate transformation character of $\boldsymbol{\Lambda}$ and due to a topological equivalence of $\mathcal{R}^3$ and $so(3)$ (to which $\boldsymbol{v}$ and $\widehat{\boldsymbol{v}}$ respectively belong) we are justified in employing a less stringent definition of the tangent space via

$$\mathcal{T}_{(\boldsymbol{r},\boldsymbol{\Lambda})}\mathcal{C} = \{\boldsymbol{\nu} = (\boldsymbol{\eta}, \boldsymbol{\mu}) : [0, L] \rightarrow \mathcal{R}^3 \times \mathcal{R}^3\} \,. \tag{3.7}$$

As a result, the relationship between an admissible variation $\boldsymbol{v}$ of a position vector $\boldsymbol{x}$, which follows by taking a directional derivative of $\boldsymbol{x}$ in the direction of $\boldsymbol{v}$ as $\boldsymbol{v} = D\boldsymbol{x} \cdot \boldsymbol{v} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \boldsymbol{x}_\epsilon = \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\boldsymbol{x} + \epsilon\boldsymbol{v})$, and the admissible translation and rotation follows from Eqn (3.4) as

$$\boldsymbol{v} = \boldsymbol{\eta} + \boldsymbol{\mu} \times (\boldsymbol{x} - \boldsymbol{r}). \tag{3.8}$$

With Eqn (3.8) at hand and by evaluating the tensor contraction via $\boldsymbol{A} : \boldsymbol{B} = \operatorname{tr}\left(\boldsymbol{A}\boldsymbol{B}^t\right)$ $\forall\ \boldsymbol{A}, \boldsymbol{B} \in \operatorname{End}(n), n \in \mathcal{N}$ and utilising the first Piola-Kirchhoff stress vectors $\boldsymbol{T}^i$ which measure internal forces with respect to a unit of initial area defined by unit normals $\boldsymbol{G}_i$, defined as $\boldsymbol{T}^i = \boldsymbol{P}\boldsymbol{G}_i \Longleftrightarrow \boldsymbol{P} = \boldsymbol{T}^i \otimes \boldsymbol{G}_i$, the beam equations of motion for the analysed problem can be derived from the weak form of a 3D solid body

$$G(\boldsymbol{x}, \boldsymbol{v}) \equiv -\int_{\mathcal{B}} \left[\operatorname{GRAD}\boldsymbol{v} : \boldsymbol{P} + \boldsymbol{v} \cdot (\rho_0 \ddot{\boldsymbol{x}} - \boldsymbol{p}_{v0})\right] dV_0 + \int_{\mathcal{E}_p} \boldsymbol{v} \cdot \boldsymbol{p}_{s0} dS_0 = 0.$$

which is taken from Eqn (2.9). In this way, the following result is obtained

$$\begin{aligned} G(\boldsymbol{r}, \boldsymbol{\Lambda}, \boldsymbol{\eta}, \boldsymbol{\mu}) \equiv -\int_{\mathcal{B}} \operatorname{tr}\left\{\frac{\partial}{\partial X^i}\left[\boldsymbol{\eta} + \boldsymbol{\mu} \times (\boldsymbol{x} - \boldsymbol{r})\right] \otimes \boldsymbol{G}_i \boldsymbol{G}_j \otimes \boldsymbol{T}^j\right\} dV_0 \\ -\int_{\mathcal{B}} \left[\boldsymbol{\eta} + \boldsymbol{\mu} \times (\boldsymbol{x} - \boldsymbol{r})\right] \cdot (\rho_0 \ddot{\boldsymbol{x}} dV_0 - \boldsymbol{p}_{v0}) dV_0 + \int_{\mathcal{E}_p} \left[\boldsymbol{\eta} + \boldsymbol{\mu} \times (\boldsymbol{x} - \boldsymbol{r})\right] \cdot \boldsymbol{p}_{s0} dS_0 + . \qquad (3.9) \end{aligned}$$

After noting $\boldsymbol{G}_i \boldsymbol{G}_j = \boldsymbol{G}_i \cdot \boldsymbol{G}_j = \delta_{ij}$ and making use of Eqn (3.4), Eqn (3.9) can be further transformed into

$$\begin{aligned} G(\boldsymbol{r}, \boldsymbol{\Lambda}, \boldsymbol{\eta}, \boldsymbol{\mu}) \equiv \int_{\mathcal{B}} \rho_0 \left[\boldsymbol{\eta} \cdot (\ddot{\boldsymbol{r}} + X^\alpha \ddot{\boldsymbol{g}_\alpha}) + \boldsymbol{\mu} \times (\boldsymbol{x} - \boldsymbol{r}) \cdot (\ddot{\boldsymbol{r}} + X^\alpha \ddot{\boldsymbol{g}_\alpha})\right] dV_0 \\ -\int_{\mathcal{B}} \left(\operatorname{tr}\left\{\left[\frac{\partial \boldsymbol{\eta}}{\partial X^i} + \frac{\partial \boldsymbol{\mu}}{\partial X^i} \times (\boldsymbol{x} - \boldsymbol{r}) - \boldsymbol{\mu} \times \frac{\partial \boldsymbol{r}}{\partial X^i}\right] \otimes \boldsymbol{T}^i\right\} + \operatorname{tr}\left[\boldsymbol{\mu} \times \frac{\partial \boldsymbol{x}}{\partial X^i} \otimes \boldsymbol{G}_i \boldsymbol{P}^t\right]\right) dV_0 \\ +\int_{\mathcal{B}} \left[\boldsymbol{\eta} \cdot \boldsymbol{p}_{v0} + \boldsymbol{\mu} \times (\boldsymbol{x} - \boldsymbol{r}) \cdot \boldsymbol{p}_{v0}\right] dV_0 + \int_{\mathcal{E}_p} \left[\boldsymbol{\eta} \cdot \boldsymbol{p}_{s0} + \boldsymbol{\mu} \times (\boldsymbol{x} - \boldsymbol{r}) \cdot \boldsymbol{p}_{s0}\right] dS_0. \qquad (3.10) \end{aligned}$$

The second term in the second integral vanishes due to $\frac{\partial \boldsymbol{x}}{\partial X^i} \otimes \boldsymbol{G}_i = \operatorname{GRAD}\boldsymbol{x} = \boldsymbol{F}$ and the Piola identity $\boldsymbol{F}\boldsymbol{P}^t = J\boldsymbol{\sigma}$ [1]. Because the Cauchy stress tensor $\boldsymbol{\sigma}$ is *symmetric* and $\widehat{\boldsymbol{\mu}}$ is skew-symmetric, $J\operatorname{tr}(\widehat{\boldsymbol{\mu}}\boldsymbol{\sigma}) = 0$. By following Eqn (3.3) we obtain

$$X^\alpha \ddot{\boldsymbol{g}}_\alpha = \left(\widehat{\dot{\boldsymbol{w}}} + \widehat{\boldsymbol{w}}^2\right)(\boldsymbol{x} - \boldsymbol{r}) = \left(\widehat{\dot{\boldsymbol{w}}} + \widehat{\boldsymbol{w}}^2\right) X^\alpha \boldsymbol{g}_\alpha, \tag{3.11}$$

where $\boldsymbol{w}$ is obviously the angular velocity of any vector $X^\alpha \boldsymbol{g}_\alpha$ within the cross section and thus also the *(spatial) angular velocity* of the orthonormal basis $\boldsymbol{g}_i$. Likewise, $\dot{\boldsymbol{w}}$ is the *(spatial) angular acceleration* of the orthonormal basis $\boldsymbol{g}_i$. Making use of the identity $\widehat{\boldsymbol{a}}\widehat{\boldsymbol{b}}\widehat{\boldsymbol{a}}\boldsymbol{b} = \widehat{\boldsymbol{b}}\widehat{\boldsymbol{a}}\widehat{\boldsymbol{a}}\boldsymbol{b}\ \forall\ \boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}^3$ and Eqns (3.1), (3.2) and (3.7), assuming a homogeneous distribution of density ρ_0 over a cross section, denoting $\mathcal{E}_p = \mathcal{A}_0 \cup \mathcal{A}_L \cup (L \times \partial\mathcal{A}) \backslash \partial\mathcal{A}_u$, where $\partial\mathcal{A}$ is the closed curve surrounding a cross section $\mathcal{A}$ and index to $\mathcal{A}$ indicates the position of the cross section in terms of the arc-length co-ordinate X^1, we obtain

$$\begin{aligned}
G(\boldsymbol{r}, \boldsymbol{\Lambda}, \boldsymbol{\eta}, \boldsymbol{\mu}) \equiv & -\int_{\mathcal{B}} \operatorname{tr}\left\{[\boldsymbol{\eta}' + \boldsymbol{\mu}' \times (\boldsymbol{x} - \boldsymbol{r}) - \boldsymbol{\mu} \times \boldsymbol{r}'] \otimes \boldsymbol{T}^1\right\} dV_0 - \int_0^L \rho_0 \Big\{ \boldsymbol{\eta} \cdot \int_{\mathcal{A}} (\ddot{\boldsymbol{r}} - \boldsymbol{p}_{v0}) dA \\
& + \boldsymbol{\eta} \cdot \left(\widehat{\dot{\boldsymbol{w}}} + \widehat{\boldsymbol{w}}^2\right) \int_{\mathcal{A}} (\boldsymbol{x} - \boldsymbol{r})\, dA + \boldsymbol{\mu} \cdot \int_{\mathcal{A}} (\boldsymbol{x} - \boldsymbol{r}) \times (\ddot{\boldsymbol{r}} - \boldsymbol{p}_{v0}) dA - \boldsymbol{\mu} \cdot \int_{\mathcal{A}} \widehat{(\boldsymbol{x} - \boldsymbol{r})}^2 dA \dot{\boldsymbol{w}} \\
& - \boldsymbol{\mu} \cdot \widehat{\boldsymbol{w}} \int_{\mathcal{A}} \widehat{(\boldsymbol{x} - \boldsymbol{r})}^2 dA \boldsymbol{w} \Big\} dX^1 + \int_0^L \left[\boldsymbol{\eta} \cdot \oint_{\partial\mathcal{A}} \boldsymbol{p}_{s0} d(\partial A) + \boldsymbol{\mu} \cdot \oint_{\partial\mathcal{A}} (\boldsymbol{x} - \boldsymbol{r}) \times \boldsymbol{p}_{s0} d(\partial A) \right] dX^1 \\
& + \boldsymbol{\eta}_0 \cdot \int_{\mathcal{A}_0} \boldsymbol{p}_{v0} dA + \boldsymbol{\mu}_0 \cdot \int_{\mathcal{A}_0} (\boldsymbol{x} - \boldsymbol{r}) \times \boldsymbol{p}_{v0} dA + \boldsymbol{\eta}_L \cdot \int_{\mathcal{A}_L} \boldsymbol{p}_{v0} dA + \boldsymbol{\mu}_L \cdot \int_{\mathcal{A}_L} (\boldsymbol{x} - \boldsymbol{r}) \times \boldsymbol{p}_{v0} dA = 0,
\end{aligned}$$

where a dash denotes a differentiation with respect to X^1. By introducing the material angular velocity $\boldsymbol{W} = \boldsymbol{\Lambda}^t \boldsymbol{w}$ and acceleration $\dot{\boldsymbol{W}} = \boldsymbol{\Lambda}^t \dot{\boldsymbol{w}}$ and the following notation for the area and the mass moment of inertia tensor of a cross section

$$A = \int_{\mathcal{A}} dA\ , \qquad \boldsymbol{J}_\rho = \rho_0 \boldsymbol{J} = -\int_{\mathcal{A}} \rho_0 X^\alpha X^\beta dA \widehat{\boldsymbol{G}}_\alpha \widehat{\boldsymbol{G}}_\beta$$

and the following notation for the applied loading

$$\boldsymbol{f} = \int_{\mathcal{A}} \boldsymbol{p}_{v0} dA + \oint_{\partial\mathcal{A}} \boldsymbol{p}_{s0} d(\partial A)$$

$$\boldsymbol{t} = \int_{\mathcal{A}} (\boldsymbol{x}-\boldsymbol{r}) \times \boldsymbol{p}_{v0} dA + \oint_{\partial\mathcal{A}} (\boldsymbol{x}-\boldsymbol{r}) \times \boldsymbol{p}_{s0} d(\partial A)$$

$$\boldsymbol{F}^0 = \int_{\mathcal{A}_0} \boldsymbol{p}_{v0} dA, \quad \boldsymbol{T}^0 = \int_{\mathcal{A}_0} (\boldsymbol{x}-\boldsymbol{r}) \times \boldsymbol{p}_{v0} dA$$

$$\boldsymbol{F}^L = \int_{\mathcal{A}_L} \boldsymbol{p}_{v0} dA, \quad \boldsymbol{T}^L = \int_{\mathcal{A}_L} (\boldsymbol{x}-\boldsymbol{r}) \times \boldsymbol{p}_{v0} dA$$

and by noting that $\int_{\mathcal{A}} (\boldsymbol{x}-\boldsymbol{r})\, dA = \boldsymbol{0}$ so long as $\boldsymbol{r}$ is the centroidal axis of the beam, we further obtain

$$G(\boldsymbol{r},\boldsymbol{\Lambda},\boldsymbol{\eta},\boldsymbol{\mu}) \equiv -\int_0^L \left[\boldsymbol{\eta}'\cdot\int_{\mathcal{A}} \boldsymbol{T}^1 dA + \boldsymbol{\mu}'\cdot\int_{\mathcal{A}} (\boldsymbol{x}-\boldsymbol{r}) \times \boldsymbol{T}^1 dA - (\boldsymbol{\mu}\times\boldsymbol{r}')\cdot\int_{\mathcal{A}} \boldsymbol{T}^1 dA\right] dX^1$$
$$-\int_0^L \left[\boldsymbol{\eta}\cdot A\rho_0\ddot{\boldsymbol{r}} + \boldsymbol{\mu}\cdot\boldsymbol{\Lambda}\left(\boldsymbol{J}_\rho\dot{\boldsymbol{W}} + \boldsymbol{W}\times\boldsymbol{J}_\rho\boldsymbol{W}\right)\right] dX^1 + \int_0^L (\boldsymbol{\eta}\cdot\boldsymbol{f} + \boldsymbol{\mu}\cdot\boldsymbol{t})\, dX^1$$
$$+\boldsymbol{\eta}_0\cdot\boldsymbol{F}^0 + \boldsymbol{\mu}_0\cdot\boldsymbol{T}^0 + \boldsymbol{\eta}_L\cdot\boldsymbol{F}^L + \boldsymbol{\mu}_L\cdot\boldsymbol{T}^L = 0.$$

By introducing the specific translational momentum $\boldsymbol{k} = A\rho_0\dot{\boldsymbol{u}}$, the specific angular momentum $\boldsymbol{\pi} = \boldsymbol{\Lambda}\boldsymbol{J}_\rho\boldsymbol{W}$, the stress resultant $\boldsymbol{N} = \boldsymbol{\Lambda}^t \int_{\mathcal{A}} \boldsymbol{T}^1 dA = \boldsymbol{\Lambda}^t \int_{\mathcal{A}} \boldsymbol{P}\boldsymbol{G}_1 dA$ and the stress-couple resultant $\boldsymbol{M} = \boldsymbol{\Lambda}^t \int_{\mathcal{A}} (\boldsymbol{x}-\boldsymbol{r}) \times \boldsymbol{T}^1 dA = \boldsymbol{\Lambda}^t \int_{\mathcal{A}} (\boldsymbol{x}-\boldsymbol{r}) \times \boldsymbol{P}\boldsymbol{G}_1 dA$, we get

$$G(\boldsymbol{r},\boldsymbol{\Lambda},\boldsymbol{\eta},\boldsymbol{\mu}) \equiv -\int_0^L \left[(\boldsymbol{\eta}' - \boldsymbol{\mu}\times\boldsymbol{r}')\cdot\boldsymbol{\Lambda}\boldsymbol{N} + \boldsymbol{\mu}'\cdot\boldsymbol{\Lambda}\boldsymbol{M} + \boldsymbol{\eta}\cdot(\dot{\boldsymbol{k}} - \boldsymbol{f}) + \boldsymbol{\mu}\cdot(\dot{\boldsymbol{\pi}} - \boldsymbol{t})\right] dX^1$$
$$+\boldsymbol{\eta}_0\cdot\boldsymbol{F}^0 + \boldsymbol{\mu}_0\cdot\boldsymbol{T}^0 + \boldsymbol{\eta}_L\cdot\boldsymbol{F}^L + \boldsymbol{\mu}_L\cdot\boldsymbol{T}^L = 0. \tag{3.12}$$

For a linear elastic material, the relationship between the stress resultants and some energy conjugate strain resultants $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ is normally defined as

$$\boldsymbol{N} = \boldsymbol{C}_N\boldsymbol{\gamma} \quad \text{and} \quad \boldsymbol{M} = \boldsymbol{C}_M\boldsymbol{\kappa}, \tag{3.13}$$

where $\boldsymbol{C}_N$ and $\boldsymbol{C}_M$ are given as

$$\boldsymbol{C}_N = (E\boldsymbol{G}_1 \otimes \boldsymbol{G}_1 + G\boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\alpha)A \tag{3.14}$$

$$\boldsymbol{C}_M = (G\boldsymbol{G}_1 \otimes \boldsymbol{G}_1 + E\boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\alpha)\boldsymbol{J}, \tag{3.15}$$

with E and G being Young's and shear moduli and A and $\boldsymbol{J}$ being given in Eqns (3.11). These results may be rigorously derived from a continuum form of a linear material in terms of a particular choice of strain and stress tensors upon introduction of beam kinematic hypotheses (see Section 4). In engineering beam theory, however, the shear modulus in Eqn (3.14) is usually corrected due to the non-linearity of the distribution of the shear stresses over cross sections. In a similar vein, the shear modulus in Eqn (3.15) is corrected to take into account the effect of Saint-Venant torsion for non-circular cross sections.

It follows from the equivalence between Hamilton's principle and the weak form of Cauchy's equations of motion, that the linearised strain resultants $\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{\gamma}_\epsilon$ and $\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{\kappa}_\epsilon$ must belong to the same space as $\boldsymbol{\Lambda}^t\left(\boldsymbol{\eta}' - \boldsymbol{\mu}\times\boldsymbol{r}'\right)$ and $\boldsymbol{\Lambda}^t\boldsymbol{\mu}'$, shown to be duals to $\boldsymbol{N}$ and $\boldsymbol{M}$,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{\gamma}_\epsilon = \boldsymbol{\Lambda}^t\left(\boldsymbol{\eta}' - \boldsymbol{\mu}\times\boldsymbol{r}'\right) \quad \text{and} \quad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{\kappa}_\epsilon = \boldsymbol{\Lambda}^t\boldsymbol{\mu}'. \tag{3.16}$$

Using Eqns (3.5), (3.6) and (3.16) the following relationships between the configuration and the adopted strain resultants are obtained

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{\gamma}_\epsilon = \boldsymbol{\Lambda}^t\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{r}'_\epsilon + \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{\Lambda}^t_\epsilon\boldsymbol{r}' = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}(\boldsymbol{\Lambda}^t_\epsilon\boldsymbol{r}'_\epsilon) \Longleftrightarrow \boldsymbol{\gamma} = \boldsymbol{\Lambda}^t\boldsymbol{r}' - \boldsymbol{G}_1 \tag{3.17}$$

$$\begin{aligned}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\widehat{\boldsymbol{\kappa}}_\epsilon &= \boldsymbol{\Lambda}^t\widehat{\boldsymbol{\mu}}'\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^t(\widehat{\boldsymbol{\mu}}\boldsymbol{\Lambda})' - \boldsymbol{\Lambda}^t\widehat{\boldsymbol{\mu}}\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}^t(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{\Lambda}_\epsilon)' + \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\boldsymbol{\Lambda}^t_\epsilon\boldsymbol{\Lambda}' \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}(\boldsymbol{\Lambda}^t_\epsilon\boldsymbol{\Lambda}'_\epsilon) \Longleftrightarrow \widehat{\boldsymbol{\kappa}} = \boldsymbol{\Lambda}^t\boldsymbol{\Lambda}'.\end{aligned} \tag{3.18}$$

In order to apply the finite element approach to the present problem it is necessary to *assume* a shape of the kinematically admissible test functions $\boldsymbol{\nu}^t = \langle \boldsymbol{\eta}^t \, \boldsymbol{\mu}^t \rangle$. On the segment $[0, L]$ we choose to interpolate $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ as

$$\boldsymbol{\eta}(X^1) \doteq \boldsymbol{\eta}^h(X^1) = \sum_{I=1}^{N} I^I(X^1)\boldsymbol{\eta}_I$$

$$\boldsymbol{\mu}(X^1) \doteq \boldsymbol{\mu}^h(X^1) = \sum_{I=1}^{N} I^I(X^1)\boldsymbol{\mu}_I,$$

where N is the number of nodes on beam $\mathcal{B}$ and $I^I(X^1) : [0, L] \rightarrow [-1, 1] \subset \mathcal{R}$ are Lagrangian polynomials of the degree $N-1$. We will often use the following notation

$$\boldsymbol{\nu}(X^1) \doteq \boldsymbol{\nu}^h(X^1) = \sum_{I=1}^{N} I^I(X^1)\boldsymbol{\nu}_I,$$

with $\boldsymbol{\nu}^t(X^1) = \langle \boldsymbol{\eta}^t(X^1) \, \boldsymbol{\mu}^t(X^1) \rangle$ and $\boldsymbol{\nu}_I^t = \langle \boldsymbol{\eta}_I^t \, \boldsymbol{\mu}_I^t \rangle$. By applying this interpolation to the weak form (3.12), the approximated weak form is obtained as

$$G^h(\boldsymbol{r}, \boldsymbol{\Lambda}, \boldsymbol{\nu}_I) \equiv -\boldsymbol{\nu}_I \cdot \Bigg[\overbrace{\int_0^L \left\{ \begin{array}{c} I^{I'} \boldsymbol{\Lambda N} \\ -I^I \boldsymbol{r}' \times \boldsymbol{\Lambda N} + I^{I'} \boldsymbol{\Lambda M} \end{array} \right\} dX^1}^{\boldsymbol{q}_k^I} + \overbrace{\int_0^L \left\{ \begin{array}{c} I^I \dot{\boldsymbol{k}} \\ I^I \dot{\boldsymbol{\pi}} \end{array} \right\} dX^1}^{\boldsymbol{q}_m^I}$$
$$\underbrace{- \left(\int_0^L \left\{ \begin{array}{c} I^I \boldsymbol{f} \\ I^I \boldsymbol{t} \end{array} \right\} dX1 + \delta_1^I \left\{ \begin{array}{c} \boldsymbol{F}^0 \\ \boldsymbol{T}^0 \end{array} \right\} + \delta_N^I \left\{ \begin{array}{c} \boldsymbol{F}^L \\ \boldsymbol{T}^L \end{array} \right\} \right)}_{\boldsymbol{q}_e^I} \Bigg] = 0, \qquad (3.19)$$

or, in a compact notation, $G^h(\boldsymbol{r}, \boldsymbol{\Lambda}, \boldsymbol{\nu}_I) \equiv -\boldsymbol{\nu}_I \cdot \boldsymbol{g}^I = 0$, where $\boldsymbol{g}^I = \boldsymbol{q}_k^I + \boldsymbol{q}_m^I - \boldsymbol{q}_e^I$ is the dynamic residual vector at node I of the element and the definition of the corresponding nodal vectors of internal, inertial and external loads, $\boldsymbol{q}_k^I$, $\boldsymbol{q}_m^I$ and $\boldsymbol{q}_e^I$ is obvious from Eqn (3.19).

4. Derivation of constitutive relations (3.13)-(3.15)

From Eqn (3.4), the deformation gradient is obtained as

$$\boldsymbol{F} = \boldsymbol{x} \otimes \nabla_{\boldsymbol{X}} = \frac{\partial}{\partial X^i}(\boldsymbol{r} + \boldsymbol{\Lambda} X^\alpha \boldsymbol{G}_\alpha) \otimes \boldsymbol{G}_i = (\boldsymbol{r}' + \boldsymbol{\Lambda}' X^\alpha \boldsymbol{G}_\alpha) \otimes \boldsymbol{G}_1 + \boldsymbol{\Lambda} \boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\alpha$$

$$= \boldsymbol{\Lambda}[\boldsymbol{G}_i \otimes \boldsymbol{G}_i + \boldsymbol{\Lambda}^t(\boldsymbol{r}' - \boldsymbol{g}_1 + \boldsymbol{\Lambda}' X^\alpha \boldsymbol{G}_\alpha) \otimes \boldsymbol{G}_1],$$

which, after introducing the strain measure resultants defined by Eqns (3.17) and (3.18), becomes

$$\boldsymbol{F} = \boldsymbol{\Lambda}[\boldsymbol{I} + (\boldsymbol{\gamma} + \boldsymbol{\kappa} \times X^\alpha \boldsymbol{G}_\alpha) \otimes \boldsymbol{G}_1] = \boldsymbol{\Lambda}[\boldsymbol{I} + \boldsymbol{\epsilon} \otimes \boldsymbol{G}_1], \tag{4.1}$$

where we have introduced the shorthand notation

$$\boldsymbol{\epsilon} = \boldsymbol{\gamma} + \boldsymbol{\kappa} \times X^\alpha \boldsymbol{G}_\alpha. \tag{4.2}$$

In order to show that the linear relationships between the stress and strain resultants defined by Eqns (3.13)-(3.15) follow from a linear constitutive law applied to a suitably chosen virtual-work conjugate stress and strain tensors, it is useful to consider the specific virtual work (over the unit of initial volume) due to the action of internal (elastic) forces, $\bar{W}$, which in terms of the deformation gradient and first Piola-Kirchhoff stress tensor reads

$$\bar{W} = \boldsymbol{P} : \bar{\boldsymbol{F}}, \tag{4.3}$$

where $\bar{\boldsymbol{F}}$ is the kinematically admissible variation of the deformation gradient, which follows from Eqn (4.1) as $\bar{\boldsymbol{F}} = \widehat{\boldsymbol{\mu}}\boldsymbol{F} + \boldsymbol{\Lambda}\bar{\boldsymbol{\epsilon}} \otimes \boldsymbol{G}_1$, with $\bar{\boldsymbol{\epsilon}} = \boldsymbol{\Lambda}^t[\boldsymbol{\eta}' - \boldsymbol{\mu} \times \boldsymbol{r}' + \boldsymbol{\mu}' \times (\boldsymbol{x} - \boldsymbol{r})]$.

By noting that $\boldsymbol{P} : (\widehat{\boldsymbol{\mu}}\boldsymbol{F})$ vanishes as explained in the paragraph between Eqns (3.10) and (3.11), Eqn (4.3) turns into $\boldsymbol{P} : \bar{\boldsymbol{F}} = (\boldsymbol{\Lambda}^t\boldsymbol{P}) : (\bar{\boldsymbol{\epsilon}} \otimes \boldsymbol{G}_1)$, i.e. $\boldsymbol{\Lambda}^t\boldsymbol{P}$ and $\boldsymbol{\epsilon} \otimes \boldsymbol{G}_1$ make another pair of virtual-work conjugate stress and strain tensors.

The stress tensor $\boldsymbol{\Lambda}^t\boldsymbol{P}$, which is obtained by rotating the first leg of the first Piola-Kirchhoff stress tensor by the rotation between the spatial and the material basis, bears some resemblance to the Biot stress tensor $\boldsymbol{B} = \boldsymbol{R}^t\boldsymbol{P}$, which is obtained by rotating the first leg of the first Piola-Kirchhoff stress tensor by the rotation $\boldsymbol{R}$ from the polar decomposition of the deformation gradient $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$, where $\boldsymbol{U}$ is a symmetric right-stretch tensor. An attempt to impose a linear constitutive law between $\boldsymbol{\epsilon} \otimes \boldsymbol{G}_1$ and $\boldsymbol{\Lambda}^t\boldsymbol{P}$ would not result in the linear relationship between the beam strain and stress resultants (3.13)-(3.15). Instead, it is necessary to define a new stress tensor $\boldsymbol{\Sigma}$, which is (i) virtual-work conjugate to $\mathrm{sym}(\boldsymbol{\epsilon} \otimes \boldsymbol{G}_1)$, i.e.

$$(\boldsymbol{\Lambda}^t\boldsymbol{P}) : (\bar{\boldsymbol{\epsilon}} \otimes \boldsymbol{G}_1) = \boldsymbol{\Sigma} : \mathrm{sym}(\bar{\boldsymbol{\epsilon}} \otimes \boldsymbol{G}_1) \tag{4.4}$$

and which (ii) preserves the stress vector acting on the cross section of the beam $\boldsymbol{\Lambda}^t\boldsymbol{P}\boldsymbol{G}_1$,

$$\boldsymbol{\Lambda}^t\boldsymbol{P}\boldsymbol{G}_1 = \boldsymbol{\Sigma}\boldsymbol{G}_1. \tag{4.5}$$

Eqn (4.4) results in $\bar{\boldsymbol{\epsilon}} \cdot \frac{1}{2}(\boldsymbol{\Sigma} + \boldsymbol{\Sigma}^t)\boldsymbol{G}_1 = \bar{\boldsymbol{\epsilon}} \cdot \boldsymbol{\Lambda}^t\boldsymbol{P}\boldsymbol{G}_1$, which along with Eqn (4.5) leads to

$$\boldsymbol{\Sigma}^t\boldsymbol{G}_1 = \boldsymbol{\Lambda}^t\boldsymbol{P}\boldsymbol{G}_1. \tag{4.6}$$

With as yet undefined components $\boldsymbol{G}_\alpha \cdot \boldsymbol{\Sigma}\boldsymbol{G}_\beta$ of the stress tensor $\boldsymbol{\Sigma}$, we now impose the linear constitutive law between stress tensor $\boldsymbol{\Sigma}$ and its energy-conjugate strain tensor $\boldsymbol{E}_\Sigma$

$$\boldsymbol{\Sigma} = (2\mu \mathbb{I} + \lambda \boldsymbol{I} \otimes \boldsymbol{I})\boldsymbol{E}_\Sigma = \frac{E}{1+\nu}\boldsymbol{E}_\Sigma + \frac{E\nu}{(1-2\nu)(1+\nu)}\mathrm{tr}\boldsymbol{E}_\Sigma \boldsymbol{I}, \tag{4.7}$$

where $\mathbb{I} = \boldsymbol{G}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{G}_i$ is the fourth order identity tensor, $\mu = \frac{E}{2(1+\nu)}$ and $\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}$ are Lamé's constants, E is Young's modulus and ν is Poisson's ratio. At this stage it is necessary to introduce the beam stress condition

$$\boldsymbol{G}_\alpha \cdot \boldsymbol{\Sigma} \boldsymbol{G}_\beta = o, \tag{4.8}$$

which along with Eqns (4.5) and (4.6) makes it clear that the stress tensor $\boldsymbol{\Sigma}$ is *symmetric*. Results (4.5), (4.6) and (4.8) all follow from the following relationship between $\boldsymbol{\Sigma}$, $\boldsymbol{\Lambda}$ and $\boldsymbol{P}$:

$$\boldsymbol{\Sigma} = \mathrm{sym}[(\boldsymbol{I} + \boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\alpha)\boldsymbol{\Lambda}^t \boldsymbol{P}(\boldsymbol{G}_1 \otimes \boldsymbol{G}_1)]$$

Condition (4.8) leads to $\frac{E}{1+\nu}\boldsymbol{G}_\alpha \cdot (\boldsymbol{E}_\Sigma + \frac{\nu}{1-2\nu}\mathrm{tr}\boldsymbol{E}_\Sigma \boldsymbol{I})\boldsymbol{G}_\beta = 0 \Longleftrightarrow E_{\Sigma,\alpha\beta} = -\delta_{\alpha\beta}\frac{\nu}{1-2\nu}E_{\Sigma,ii}$, which results in

$$E_{\Sigma,23} = E_{\Sigma,32} = 0\ , E_{\Sigma,22} = E_{\Sigma,33} = -\nu E_{\Sigma,11} \Longrightarrow \mathrm{tr}\boldsymbol{E}_\Sigma = (1-2\nu)E_{\Sigma,11}. \tag{4.9}$$

Introducing this result into Eqn (4.7) gives $\boldsymbol{\Sigma} = \frac{E}{1+\nu}(\boldsymbol{E}_\Sigma + \nu E_{\Sigma,11}\boldsymbol{G}_1 \otimes \boldsymbol{G}_1)$. After adopting $\frac{1}{2}(\boldsymbol{G}_1 \otimes \boldsymbol{\epsilon} + \boldsymbol{\epsilon} \otimes \boldsymbol{G}_1)$ for $\boldsymbol{E}_\Sigma$, which makes a perfect sense due to the virtual work conjugacy between $\frac{1}{2}(\boldsymbol{G}_1 \otimes \bar{\boldsymbol{\epsilon}} + \bar{\boldsymbol{\epsilon}} \otimes \boldsymbol{G}_1)$ and $\boldsymbol{\Sigma}$, we finally obtain

$$\boldsymbol{\Sigma} = [2G\mathbb{I} + (E-2G)\boldsymbol{I} \otimes \boldsymbol{I}]\boldsymbol{E}_\Sigma\ ,\ \boldsymbol{E}_\Sigma = \frac{1}{2}(\boldsymbol{G}_1 \otimes \boldsymbol{\epsilon} + \boldsymbol{\epsilon} \otimes \boldsymbol{G}_1). \tag{4.10}$$

It should be noted that the last result in Eqn (4.9) in conjunction with the adopted definition of the strain tensor $\boldsymbol{E}_\Sigma$ leads to a paradoxical result $\text{tr}\boldsymbol{E}_\Sigma = (1-2\nu)\text{tr}\boldsymbol{E}_\Sigma$, which is a consequence of introducing a plane-stress type condition (4.8) into a problem where the kinematical constraints (the rigidity of cross sections) dictate a plane-strain type condition. Obviously, the two conflicting requirements are reconciled for $\nu = 0$. It may be said that only in this case can the beam theories be consistently derived from the equations of 3D continuum mechanics.

Using $\boldsymbol{N} = \boldsymbol{\Lambda}^t \int_{\mathcal{A}} \boldsymbol{T}^1 dA = \boldsymbol{\Lambda}^t \int_{\mathcal{A}} \boldsymbol{P}\boldsymbol{G}_1 dA$ and Eqns (4.5) and (4.10) we obtain

$$\boldsymbol{N} = \int_{\mathcal{A}} \boldsymbol{\Sigma}\boldsymbol{G}_1 dA = \int_{\mathcal{A}} [G(\boldsymbol{G}_1 \otimes \boldsymbol{\epsilon} + \boldsymbol{\epsilon} \otimes \boldsymbol{G}_1) + \frac{1}{2}(E-2G)\text{tr}(\boldsymbol{G}_1 \otimes \boldsymbol{\epsilon} + \boldsymbol{\epsilon} \otimes \boldsymbol{G}_1)\boldsymbol{I}]\boldsymbol{G}_1 dA =$$

$$[G(\boldsymbol{I} + \boldsymbol{G}_1 \otimes \boldsymbol{G}_1) + (E-2G)\boldsymbol{G}_1 \otimes \boldsymbol{G}_1] \int_{\mathcal{A}} \boldsymbol{\epsilon} dA,$$

or after introducing $A = \int_{\mathcal{A}} dA$ and Eqn (4.2) and noting $\int_{\mathcal{A}} X^\alpha dA = 0$,

$$\boldsymbol{N} = (E\boldsymbol{G}_1 \otimes \boldsymbol{G}_1 + G\boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\alpha)A\boldsymbol{\gamma} = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & GA \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{Bmatrix}.$$

Using $\boldsymbol{M} = \boldsymbol{\Lambda}^t \int_{\mathcal{A}} (\boldsymbol{x} - \boldsymbol{r}) \times \boldsymbol{T}^1 dA = \boldsymbol{\Lambda}^t \int_{\mathcal{A}} (\boldsymbol{x} - \boldsymbol{r}) \times \boldsymbol{P}\boldsymbol{G}_1 dA$ and Eqns (4.2), (4.5) and (4.10) and noting $\int_{\mathcal{A}} X^\alpha dA = 0$ we obtain

$$\boldsymbol{M} = \boldsymbol{G}_\alpha \times \int_{\mathcal{A}} X^\alpha \boldsymbol{\Sigma}\boldsymbol{G}_1 dA = \boldsymbol{G}_\alpha \times (E\boldsymbol{G}_1 \otimes \boldsymbol{G}_1 + G\boldsymbol{G}_\gamma \otimes \boldsymbol{G}_\gamma) \int_{\mathcal{A}} X^\alpha X^\beta dA \widehat{\boldsymbol{\kappa}} \boldsymbol{G}_\beta$$

$$= -\widehat{\boldsymbol{G}}_\alpha (E\boldsymbol{G}_1 \otimes \boldsymbol{G}_1 + G\boldsymbol{G}_\gamma \otimes \boldsymbol{G}_\gamma) \widehat{\boldsymbol{G}}_\beta \int_{\mathcal{A}} X^\alpha X^\beta dA \boldsymbol{\kappa}. \tag{4.11}$$

By observing the following results

$$\widehat{\boldsymbol{G}}_\alpha \boldsymbol{G}_1 \otimes \boldsymbol{G}_1 = e_{\alpha 1 \gamma} \boldsymbol{G}_\gamma \otimes \boldsymbol{G}_1 = -e_{\alpha \gamma 1} \boldsymbol{G}_\gamma \otimes \boldsymbol{G}_1 = -\boldsymbol{G}_\gamma \otimes (\boldsymbol{G}_\alpha \times \boldsymbol{G}_\gamma) = \boldsymbol{G}_\gamma \otimes \boldsymbol{G}_\gamma \widehat{\boldsymbol{G}}_\alpha,$$

where e_{ijk} is the permutation symbol defined as $e_{123} = e_{231} = e_{312} = -e_{132} = -e_{321} = -e_{213} = 1$ and $e_{ijk} = 0$ in all other cases, and

$$\widehat{\boldsymbol{G}}_\alpha \boldsymbol{G}_\gamma \otimes \boldsymbol{G}_\gamma = \widehat{\boldsymbol{G}}_\alpha (\boldsymbol{I} - \boldsymbol{G}_1 \otimes \boldsymbol{G}_1) = (\boldsymbol{I} - \boldsymbol{G}_\gamma \otimes \boldsymbol{G}_\gamma) \widehat{\boldsymbol{G}}_\alpha = \boldsymbol{G}_1 \otimes \boldsymbol{G}_1 \widehat{\boldsymbol{G}}_\alpha,$$

and making use of $\boldsymbol{J} = -\int_{\mathcal{A}} X^\alpha X^\beta dA \widehat{\boldsymbol{G}}_\alpha \widehat{\boldsymbol{G}}_\beta = -J_3 \widehat{\boldsymbol{G}}_2^2 - J_2 \widehat{\boldsymbol{G}}_3^2$, with $J_3 = \int_{\mathcal{A}} X^2 X^2 dA$ and $J_2 = \int_{\mathcal{A}} X^3 X^3 dA$, Eqn (4.11) results in

$$\boldsymbol{M} = (G\boldsymbol{G}_1 \otimes \boldsymbol{G}_1 + E\boldsymbol{G}_\gamma \otimes \boldsymbol{G}_\gamma)\boldsymbol{J}\boldsymbol{\kappa} = \begin{bmatrix} G(J_2 + J_3) & 0 & 0 \\ 0 & EJ_2 & 0 \\ 0 & 0 & EJ_3 \end{bmatrix} \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}.$$

It should be noted that these results are derived without ever introducing restrictions to small strains. The linear constitutive relationship between the stress and strain resultants in beams follows rigorously from the linear constitutive law defined between the stress tensor $\boldsymbol{\Sigma} = \mathrm{sym}[(\boldsymbol{I} + \boldsymbol{G}_\alpha \otimes \boldsymbol{G}_\alpha)\boldsymbol{\Lambda}^t \boldsymbol{P} (\boldsymbol{G}_1 \otimes \boldsymbol{G}_1)]$ and the strain tensor $\boldsymbol{E}_\Sigma = \mathrm{sym}(\boldsymbol{\Lambda}^t \boldsymbol{F} - \boldsymbol{I})$, where $\boldsymbol{F}$ is the deformation gradient, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor and $\boldsymbol{\Lambda}$ is the rotation of the cross section.

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