Every quantum minor generates an Ore set

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Abstract. *The subset multiplicatively generated by any given set of quantum minors and the unit element in the quantum matrix bialgebra satisfies the left and right Ore conditions.*

Quantum matrix groups ([9, 11, 12]) have remarkable algebraic properties and connections to several branches of mathematics and mathematical physics. A viewpoint of the noncommutative geometry may elucidate some of their properties. In the formalism in which a quantum group is described by a matrix bialgebra $\mathcal{G}$, e.g. $\mathcal{SL}_q(n)$, the shifts of the main Bruhat cells are expected also to have quantum analogues which are localizations of $\mathcal{G}$. The geometry is richer and more akin to the classical case, if these localizations have good flatness properties. Ore localization is the most well-understood kind of localizations. Ore localizations are biflat (in terminology of [13]), and appear often in "quantum" situation, that is, when the noncommutative algebra is just a deformation of a commutative algebra. Thus one expects to realize the quantum main Bruhat cell and its Weyl group shifts as the (noncommutative spectra of) the localized algebras of the form $\mathcal{G}[S_w^{-1}]$ where S_w are certain Ore subsets in $\mathcal{G}$, depending on the element w in the Weyl group W. Furher support of this conjectural picture is a result of A. Joseph [5], that there is a natural family of Ore sets $\bar{S}_w$, $w \in W$, in the graded algebra $\mathcal{R}$ representing the quantum analogue of the basic affine space G/U, such that in the commutative case the spectra of the localizations $\mathcal{R}[\bar{S}_w^{-1}]$ are exactly the images of the Bruhat cells in the basic affine space.

However, the Ore property for the natural candidates for $\mathcal{S}_w$ has not been proved so far. Trying to answer the question of V. Drinfeld to find the quantum analogue of the Beilinson-Bernstein localization theorem, Y. Soibelman has shown the satisfactory localization picture for $\mathcal{SL}_q(2)$ (unpublished), and came to the conclusion that for $\mathcal{SL}_q(n)$, $n > 2$, already the Ore property of S_w is far from obvious, if not even wrong. Quantum Beilinson-Bernstein theorem has been further studied by Lunts and Rosenberg [7], Tanisaki [20] and others, but in different approaches.

In his thesis, the present author has developed a direct localization approach ([15, 17]) to the construction of the coset spaces of the quantum linear groups and the locally trivial quantum principal fibrations deforming the classical fibrations of the type $G \to G/B$ and having Hopf algebras as the replacements of the structure groups. Apart from the sketch in [15], the main part of that work has been still unpublished (however, a nontrivial application to quantum group coherent states and appropriate measure is exhibited in [16]). The present paper fills a part of this gap in view of the observation that the sets S_w in $\mathcal{SL}_q(n)$ are sets multiplicatively generated by a specific set of quantum minors attached to the permutation matrix w, namely the set of all principal (=lower right corner) quantum minors of the row-permuted matrix of generators $T = (t_j^{w^{-1}(i)})$. Note that this solves the problem weather the quantum Bruhat cells are realizable as Ore localizations only for quantum matrix groups of type A (in the sense of Lie theory).

A subset S in a ring R is **multiplicative** if $1 \in S$ and $s, s' \in S$ implies $ss' \in S$. A subset $S \in R$ satisfies the **left Ore condition** if $\forall r \in S, \forall r \in R$, $\exists r' \in R\, \exists s' \in S$ with $s'r = r's$. If R has no zero divisors, then a multiplicative subset $S \subset R$ satisfying the left Ore condition is called a **left Ore set**. The right Ore condition is the left Ore condition in the ring with opposite multiplication $R^{\rm op}$. A subset $S \in R$ is Ore if it is left Ore in R and $R^{\rm op}$ simultaneously. For more details on Ore sets see [19, 21].

Let I, J be some linearly ordered sets of equal cardinality $|I| = |J| = n > 0$, where the elements of K and L are called labels. Given a field $\mathbf{k}$ of characteristic zero, and number $0 \neq q \in \mathbf{k}$, the underlying $\mathbf{k}$-algebra of the matrix bialgebra $\mathcal{M}_q(I, J)$ is the free $\mathbf{k}$-algebra on generators t^i_j where $i \in I$, $j \in J$, modulo the relations

$$\begin{array}{ll} t^\alpha_\gamma t^\alpha_\delta = q t^\alpha_\delta t^\alpha_\gamma, & \alpha = \beta,\ \gamma < \delta\ \text{(same row)} \\ t^\alpha_\gamma t^\beta_\gamma = q t^\beta_\gamma t^\alpha_\gamma, & \alpha < \beta,\ \gamma = \delta\ \text{(same column)} \\ t^\alpha_\gamma t^\beta_\delta - t^\beta_\delta t^\alpha_\gamma = \left\{ \begin{array}{ll} (q - q^{-1}) t^\beta_\gamma t^\alpha_\delta, & \alpha < \beta,\ \gamma < \delta \\ 0, & \alpha < \beta,\ \gamma > \delta \end{array} \right. & \end{array}$$

The isomorphism class of algebra $\mathcal{M}_q(K, L)$ depends only on the cardinality n of K and L, and we denote a representative of that class $\mathcal{M}_q(n)$, e.g. where $K = L = \mathbf{n} = \{1, \ldots, n\}$.

Let $K = (k_1, \ldots, k_m) \subset I$, $L = (l_1, \ldots, l_m) \subset J$ be subsets of equal cardinality $m < n$ where the labels in the round brackets are ordered according to the subset order from I and J. Then T^K_L will be the submatrix of $T = (t^i_j)^{i \in I}_{j \in J}$ whose rows have labels in K and whose columns have labels in L. Denote $D^K_L = \det_q T^K_L = \sum_{\sigma \in \Sigma(m)} (-q)^{l(\sigma)} t^{k_1}_{l_{\sigma(1)}} \cdots t^{k_m}_{l_{\sigma(m)}}$ where $l(\sigma)$ denotes the number of inversions in the permutation σ on the set of m labels. Any element of this form is called a **quantum minor**. D^K_L is central in $\mathcal{M}_q(K, L)$, but not in the whole $\mathcal{M}_q(I, J)$, unless $K = I$ and $L = J$.

Lemma 1. *Let $S \subset R$ be a multiplicative subset and $E = E(S)$ be the set of all $e \in R$ which satisfy the following 'partial' left Ore condition*

$$\forall s \in S, \exists s' \in S, \exists r' \in R \quad r's = s'e \tag{1}$$

Suppose also that $S \subset E$. Then (i) E is a subalgebra of R.

(ii) If $r \in R$ satisfies the following E-relative partial left Ore condition

$$\forall s \in S, \exists s' \in S, \exists r' \in R \quad r's - s'r \in E \tag{2}$$

then $r \in E$.

(iii) Let $S_0 \subset S$ multiplicatively generates S. Then for left Ore condition to hold, that is $E(S) = R$, it is enough to check (1) *for all $e \in R$ but only $s \in S_0$ (still with $s' \in S$).*

Proof. The reasoning method here is pretty standard (cf. [19], Ch. 6).

(i) Given $e_1, e_2 \in E$, we prove that $e_1 e_2 \in E$ as follows. By the assumption, given any $s \in S$, $\exists r' \in R, \exists s' \in S$, with $r's = s'e_1$; and $\exists r'' \in R, \exists s'' \in S$ such that $r''s' = s''e_2$. Then $s''(e_1 e_2) = r''s'e_2 = (r''r')s$.

To prove that $e_1 + e_2 \in E$ we reason as follows. Given any $s \in S$, we first find $s_1, s_2 \in S, r_1, r_2 \in R$ such that $s_1 e_1 = r_1 s$ and $s_2 e_2 = r_2 s$. By $S \subset E$, also $\exists s_* \in S$, $r_* \in R$ such that $s_* s_1 = r_* s_2$. Then

$$\begin{aligned}(s_* s_1)(e_1 + e_2) &= s_* r_1 s + s_* s_1 e_2 \\ &= s_* r_1 s + r_* s_2 e_2 \\ &= s_* r_1 s + r_* r_2 s \\ &= (s_* r_1 + r_* r_2)s, \qquad \text{with } s_* s_1 \in S, \text{ as required.}\end{aligned}$$

(ii) Let $r's - s'r = e \in E$. By $e \in E$ we can find r'', s'' such that $r''s = s''e$. Then $(s''r' - r'')s = s''(r's - e) = s''s'r$ with $s''s' \in S$, as required.

(iii) Suppose $s_1, s_2 \in S_0$. By assumption, $r_2 s_2 = s_2' e$ and $s_* r_2 = r_* s_1$ for some $s_2', s_* \in S$ and some $r_2, r_* \in R$. Then $r_*(s_1 s_2) = s_* r_2 s_2 = (s_* s_2')e$ with $s^* s_2' \in S$. Hence, the equation (1) holds for all $s \in S_0'$ where S_0' consists of all products of pairs of elements in S. Continuing by induction on the length n of a product $s = s_1 \ldots s_n \in S$ we conclude that (1) holds for all $s \in S$.

Lemma 2. *Let $|K| = |L| < n$. Consider the subalgebra $E_0 = E_0(K, L)$ in $\mathcal{M}(n)$ generated by all t^k_l where either $k \in K$ or $l \in L$ (or both). Let $t^{k'}_{l'} \notin E_0$. Then for every $e \in E_0$, $t^{k'}_{l'} e - e t^{k'}_{l'} \in E_0$.*

Proof. This is a linear statement, hence it suffices to prove it for words $e = t^{k_1}_{l_1} \cdots t^{k_r}_{l_r}$ of degree r in generators of E_0. We show this by induction on r.

For $r = 0, 1$, this is obvious. Let $e = e_{r-1} t^{k_r}_{l_r}$ where e_{r-1} is of the degree $r-1$. The commutator $[e_{r-1}, t^{k'}_{l'}] \in E_0$ by the induction hypothesis, and $[t^{k_r}_{l_r}, t^{k'}_{l'}]$ is either zero or proportional to $t^{k'}_{l_r} t^{k_r}_{l'}$. Since E_0 is a subalgebra, these two facts and the identity $[e_{r-1} t^{k_r}_{l_r}, t^{k'}_{l'}] = [e_{r-1}, t^{k'}_{l'}] t^{k_r}_{l_r} + e_{r-1}[t^{k_r}_{l_r}, t^{k'}_{l'}]$ imply that $[e, t^{k'}_{l'}] \in E_0$. Q.E.D.

Notation. If $L = (l_1, \ldots, l_r)$, then $L(l_k \to l) := (l_1, \ldots, l_{k-1}, l, l_{k+1}, \ldots, l_r)$.

Lemma 3. (special cases of commutation relations for quantum minors)

(i) [11] If $k \in K$ and $l \in L$ then $t^k_l D^K_L = D^K_L t^k_l$.

(ii) [11] If either $k > \max K$ and $l \in L$, or $l > \max L$ and $k \in K$, then $t^k_l D^K_L = q^{-1} D^K_L t^k_l$. If either $k < \max K$ and $l \in L$, or $l < \max L$ and $k \in K$, then $t^k_l D^K_L = q D^K_L t^k_l$. Also (say, by Muir's law [6]), if L and L' differ by an interchange of a single label, then $D^K_L D^K_{L'} = q^{\pm 1} D^K_{L'} D^K_L$.

(iii) Let $L = (l_1, \ldots, l_s)$. Suppose $l_1 < l < l_i$, for $i > 1$. Then

$$D^K_L t^k_l - q^{-1} t^k_l D^K_L = (1 - q^{-2}) D^K_{L(l \mapsto l_1)} t^k_{l_1} \tag{3}$$

(iv) Let $L = (l_1, \ldots, l_s)$ be ascendingly ordered and $l_r < l < l_{r+1}$ for some r. Then (iii) generalizes to

$$D^K_L t^k_l - q^{-1} t^k_l D^K_L = q^{-1}(q - q^{-1}) \sum_{u=1}^{r} (-q)^{u-r} t^{k_m}_{l_u} D^K_{L(l_u \to l)} \tag{4}$$

Parts (iii) and (iv) are now widely known among people working on "identities between quantum minors", and were many times independently rediscovered by many people (e.g. by the author around 1998). Both identities are much easier to prove when $k = \max K$ (using for example Laplace identities [4, 6, 11], and simple calculational arguments, and, for (iv), induction). The general case, follows from $k = \max K$ case applying the included-row exchange principle ([14]).

Fact. (e.g. [11]) *$\mathcal{M}_q(n)$ has no zero divisors.*

Theorem. *Each quantum minor $D^K_L = \det_{\mathrm{q}} T^K_L$ (where K and L are the subsets of $\{1, \ldots, n\}$ of the same cardinality) and 1 multiplicatively generate an Ore set $S = S^K_L$ in $R = \mathcal{M}_q(n)$.*

Proof. We prove the right Ore condition. The left Ore condition then follows by using the algebra automorphism $t^i_j \to t^j_i$.

According to the Lemma 1 (i), it is sufficient to prove that all the generators t^k_l are in $E = E(S^K_L)$, and $S \subset E$. The latter is clear as S is multiplicatively generated by only one element. Furthermore, by Lemma 1

(iii) we will be checking the condition (1) for $s = D^K_L$ only, and not for the higher powers of D^K_L. We first prove for $t^k_l \in E_0(K, L) \subset R$, (in notation of Lemma 2).

If $k \in K$ and $l \in L$ this is obvious because then t^k_l and D^K_L commute. If $k \in K$ and l is smaller than the minimum of L or larger then the maximum of L then $t^k_l D^K_L = q^{\pm 1} D^K_L t^k_l$ what again proves the condition. The same way conclude that in the case that $l \in L$ and k is smaller than the minimum of K or larger than the maximum of K.

Now consider the case when $k \in K$ and $l_r < l < l_{r+1}$ for some r. If $r = 1$ then the formula (3) applies. Notice that the RHS is in E, because $t^k_{l_1}$ is already proven to be in E, and D^K_L and $D^K_{L'}$ differ only in one column, hence q-commute according to Lemma 3 (ii) and E is a subalgebra. Hence the formula (3) is easily recognized to be of type (2) and therefore $t^k_l \in E$.

Now we use induction on r to prove the same for all r. So suppose we have proved for $r-1$. Then, the formula (4) applies. Using the induction hypothesis, one easily sees that the factors involved in each summand on the right-hand side of (4) are in E: Namely, $D^K_{L(l_u \to l)}$ has at most $(r-1)$ columns with indices less than l, and $t^{k_m}_{l_u}$ is one of the entries in T^K_L, hence it commutes with D^K_L.

Thus we have proved the claim for all $t^k_l \in E_0 = E_0(K, L)$, and therefore $E_0 \subset E$. We are left with the case of $t^k_l \notin E_0$, i.e. where both $k \notin K$ and $l \notin L$. Surely $D^K_L \in E_0$, hence by Lemma 2, $t^k_l D^K_L - D^K_L t^k_l \in E_0 \subset E$. By Lemma 1 (ii) $t^k_l \in E$. Hence by Lemma 1 (i) we are done.

Corollary. *(0) The theorem holds for $\mathcal{S}L_q(n)$ and $\mathcal{G}L_q(n)$.*

(i) Every set of quantum minors multiplicatively generates an Ore set in $R = \mathcal{M}_q(n), \mathcal{S}L_q(n)$ or $\mathcal{G}L_q(n)$.

(ii) The product of finitely many q^r-commuting (various r-s) quantum minors is a multiplicative generator of an Ore set in R.

(iii) The product of all principal quantum minors (lower right corner) of sizes 1×1 to $(n-1) \times (n-1)$ of any row and column-permuted matrix $G = T^\sigma_\tau$ of T is a multiplicative generator of an Ore set in R.

(iv) The theorem holds for the multiparametric quantum groups obtained by twisting ([1, 3, 10, 18]).

To observe (0), notice that $\mathcal{S}L_q(n))$ has no zero divisors and it is a quotient of $\mathcal{M}_q(n)$. The projection map sends all quantum minors to nonzero elements, so the Ore conditions are fullfilled by using the projection map directly. $\mathcal{G}L_q(n))$ is a *central* localization of $\mathcal{M}_q(n)$, hence compatible with any Ore set in $\mathcal{M}_q(n)$: any central localization sends Ore sets to Ore sets.

(i-iii) easily follow from the theorem by general principles on recognizing Ore sets ([19], Ch. 6).

(iv) It is not difficult to observe, and it was shown in detail in [18], the twisting of [1] changes the relations (in Lemma 3) among the quantum minors by nonzero factors in each of the summands, and the rest of the reasoning follows the same way, up to numerical factors, which can always be absorbed in r' in Ore condition.

In fact, one can directly use the theorem in its 1-parameter version, and an isomorphism from the multiparametric quantum bialgebra into the $(0,0)$-bidegree component of certain tensor product $S_l \otimes \mathcal{M}_q(n) \otimes S_r$. This isomorphism appropriately transfers the quantum minors and Ore conditions involving certain class of homogeneous elements in general ([18]).

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References

[1] M. Artin, W. Schelter, J. Tate, *Quantum deformations of* GL_n, Comm. Pure Appl. Math. 44 (1991), no. 8-9, 879–895.

[2] A. Beilinson, J. Bernstein, *Localization de $\mathcal{G}$-modules*, Comp. Ren. Acad. Sci. Paris 292 (1981), pp. 15-18.

[3] E. E. Demidov, *Multiparameter quantum deformations of the group* $\mathrm{GL}(n)$*. (Russian) Uspehi Mat. Nauk 46 (1991), no. 4(280), pp. 147–148; translation in Russian Math. Surveys 46 (1991), no. 4, pp. 169–171.*

[4] M. Hashimoto, T. Hayashi, *Quantum multilinear algebra*, Tôhoku Math J. 44 (1992), pp. 471-521.

[5] A. Joseph, *Faithfully flat embeddings for minimal primitive quotients of quantized enveloping algebras*, Quantum deformations of alg-s and their rep-s (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), 79–106, Israel Math. Conf. Proc., 7, Bar-Ilan Univ., Ramat Gan, 1993.

[6] D. Krob, B. Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Comm.Math.Phys. 169 (1995), pp. 1–23.

[7] V. A. Lunts, A. L. Rosenberg, *Differential operators on noncommutative rings*, Selecta Math. (N. S.) 3 (1997), pp. 335–359.

[8] V. A. Lunts, A. L. Rosenberg, *Localization for quantum groups*, Selecta Math. (N. S.) 5 (1999), no. 1, pp. 123–159.

[9] Yu. I. Manin, *Quantum groups and non-commutative geometry*, CRM, Montreal 1988.

[10] Yu. I. Manin, *Multiparametric quantum deformation of the general linear supergroup*, Comm. Math. Phys. 123 (1989), pp. 163–175.

[11] B. Parshall, J. Wang, *Quantum linear groups*, Memoirs AMS 439, 1991.

[12] N.Yu.Reshetikhin, L.A. Takhtajan, L.D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra i analiz **1**, 178 (1989) (Russian), English translation in Leningrad Math. J. 1.

[13] A. L. Rosenberg, *Noncommutative schemes*, Comp. Math. 112 (1998), pp. 93–125.

[14] Z. Škoda, *Included-row exchange principle for quantum minors*, `math.QA/0510512`

[15] Z. Škoda, *Localizations for construction of quantum coset spaces*, "Noncommutative geometry and quantum groups" (conf. Warszawa 2001), P. M. Hajac, W. Pusz eds. Banach Center Publications vol.61, pp. 265–298, Polish Acad. Sci. Warszawa 2003; `arXiv:math.QA/0301090`.

[16] Z. Škoda, *Coherent states for Hopf algebras*, `math.QA/0303357`.

[17] Z. Škoda, *The geometric meaning of quantum Gauss decomposition*, preprint.

[18] Z. Škoda, *A simple algorithm for extending the identities for quantum minors to multiparametric case*, preprint.

[19] Z. Škoda, *Noncommutative localization in noncommutative geometry*, in "Noncommutative localization in topology and algebra", ed. A. Ranicki; London Math. Society Lecture Note Series 330, pp. 220–313, `math.QA/0403276`.

[20] T. Tanisaki, *The Beilinson-Bernstein correspondence for quantized enveloping algebras*, Math. Z. 250 (2005), no. 2, 299–361.

[21] F. van Oystaeyen, *Algebraic geometry for associative algebras*, Marcel Dekker 2000.