Theoretical Distributions in Risk Measuring on Stock Market

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Abstract: For any investor on stock market it is very important to predict possible loss, depending on if he holds "long" or "short" position. By forecasting stock risk investor can be ensured "a priori" from estimated market risk, using financial derivatives, i.e. options, forwards, futures and other instruments. In that sense we find financial econometrics as the most useful tool for modeling conditional mean and conditional variance of nonstationary financial time series. Besides the assumption of normal distributed returns does not represent asymmetry of information influence, normal distribution also is not the most appropriate approximation of the real data on the stock market. Using assumption of heavy tailed distribution, such as Student's t-distribution in GARCH(p,q) model, it becomes possible to forecast market risk much more precisely. Even more, using Student's distribution with non-integer degrees of freedom leads approximation to minimal differences between theoretical and real values. Such modeling enables time-varying risk forecasting, because the assumption of constant risk measures between stocks is unrealistic. The complete procedure of analysis has been established using real observed data at Zagreb Stock Exchange. For this purpose daily returns of the most frequently traded stocks from CROBEX index is used.

Key-Words: theoretical distribution comparison, non-integer degrees of freedom, heavy-tails, scale and shape parameters, risk measuring, conditional variance, risk forecasting of stock returns

1 Introduction

Value at Risk (VaR) has become the most common measure that financial analysts use to quantify market risk. Even so VaR is proposed, by Basel Acords, as the basis for calculation of capital requirements for risk hedging. In category of parametric models the most are used GARCH(p,q) models in forecasting conditional mean and conditional variance within VaR framework. During optimization procedure it is important to take into account, not only first two moments, but also skewness and kurtosis of empirical distribution.

Unfortunately the assumption that the returns are independently and identically normally distributed is unrealistic.

Furthermore, empirical research about financial markets reveals following facts:

- financial return distributions are leptokurtic, i.e. they have heavy tails and a higher peak than a normal distribution,
- equity returns are typically negatively skewed and
- squared return series shows significant autocorrelation, i.e. volatilities tends to cluster

Returns from financial instruments such as exchange rates, equity prices and interest rates measured over short time intervals, i.e. daily or weekly, are characterized by high kurtosis.

The complete procedure of analysis is established using daily observations of Pliva stocks as the most frequently traded stock from CROBEX index at Zagreb Stock Exchange. If the distribution of returns is heavy tailed, the VaR and conditional Value at Risk (CVaR) calculated using normal assumption differs significantly from Student's t-distribution. As it's known Student's distribution belongs to family of extreme value distributions. In case of volatility modeling and CVaR estimating of Pliva stock returns it's found that kurtosis and degrees of freedom from Student's t-distribution are closely related.

Statistical significance of existing heavy tailed distribution has been shown by Q-Q plot and tested using Jarque-Bera test.

2 Extreme value diagnostics

To identify outliers and another extreme values Box and Whisker plot has been used.

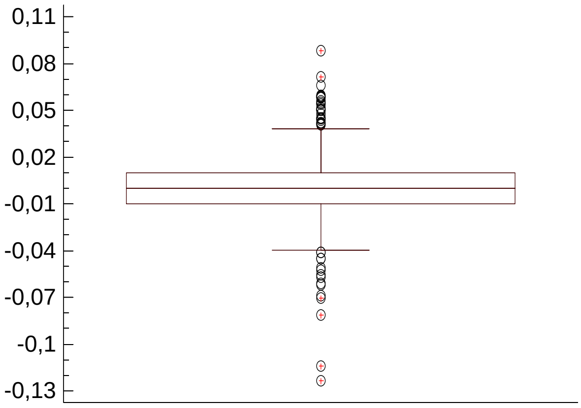
From Figure 1. outliers can be identify as Pliva returns which deviates from quartiles more than $\frac{3}{2}(Q_3 - Q_1)$. The extreme values are Pliva returns which deviates from quartiles more than $3(Q_3 - Q_1)$.

Extreme values from Box and Whisker plot are perceived as circles with plus sign on both side of distribution.

These extreme values and outliers are cause of existence fat tailed distribution.

Figure 1.

Box and Whisker plot of Pliva returns

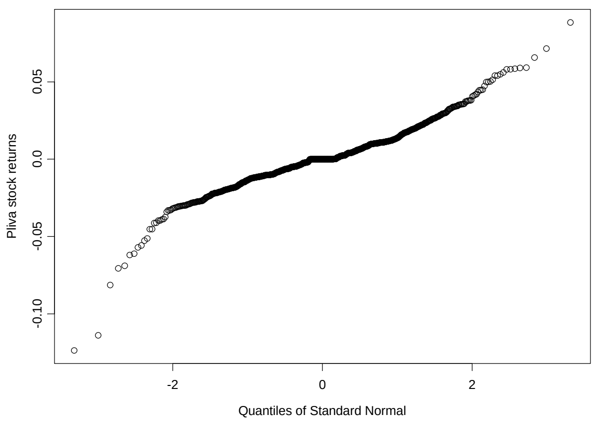


Source: According to ZSE

There are various analytical and graphical methods to detect heavy tails from observed distribution. The most common used are Jarque-Bera test, while Q-Q plot graphically determines fat tails.

Figure 2.

Q-Q plot of Pliva stock returns distribution



Source: According to ZSE

In table 1. essential statistics are presented.

Table 1.

Essential statistics of Pliva stock returns distribution

stats	plivaret
mean	.0002185
min	-.123698
max	.088411
skewness	-.0651446
kurtosis	7.615428
sd	.0181509

Source: According to ZSE

Each of shape measures, i.e. skewness and kurtosis are tested separately, indicating that skewness isn't statistically significant whereas excess kurtosis of 4.6154 is significantly greater than 3. In general, joint test shows that null hypothesis of normality distribution assumption can't be accepted. This joint test is presented as Jarque-Bera test in table 2.

Table 2.

Normaly test of Pliva return distribution

Skewness/Kurtosis tests for Normality

			------- joint ------	
Variable	Pr(Skewness)	Pr(Kurtosis)	chi2(2)	Prob>chi2
plivaret	0.781	0.000	103.70	0.0000

Source: According to ZSE

From results presented in table 2, it's obvious that skewness, which is very close to zero, is not statisticaly significant at empirical p-value of 78.1%. From the other side high kurtosis (7.6154) is statistically significant. According to joint chi-square test null hypotheses of normality can not be accepted.

3 Degrees of freedom estimation using method of moments

In practice, the kurtosis is often larger than six (which is confirmed in this empirical example), leading to estimation of non-integer degrees of freedom between four and six. Thus, degrees of freedom can easily be estimated using the method of moments.

Generally, there are three parameters that define a probability density function (pdf):

- location parameter,
- scale parameter and
- shape parameter.

The most common measure of location parameter is the mean. The scale parameter measure variability of pdf, and the most commonly used is variance or standard deviation. The shape parameter (skewness and/or kurtosis) determines how the variations are distributed about the location parameter.

The density of a non-central Student t-distribution has the following form:

$$f(x)=\frac{\Gamma\left(\frac{df+1}{2}\right)}{\Gamma\left(\frac{df}{2}\right)\sqrt{\pi\cdot\beta\cdot df}}\left(1+\frac{(x-\mu)^2}{\beta\cdot df}\right)^{-\frac{1+df}{2}}, \qquad (1)$$

where μ is location parameter, β is scale parameter and *df* is a shape parameter, or degrees of freedom and $\Gamma(\cdot)$ is gamma function. Standard t-distribution assumes $\mu = 0$, $\beta = 1$, with integer *df*. However, there are no mathematical reasons why the degrees of freedom should be an integer. Even so, the degrees of freedom can be estimated using method of moments, which means that kurtosis and degrees of freedom are closely related:

$$k = \frac{6}{df-4} + 3 \quad \forall \quad df > 4 . \tag{2}$$

So, when empirical distribution is leptokurtic, then Student's t-distribution with parameter $4 < df \le 30$ should be used to allow heavy tails of high kurtosis distribution.

Second and fourth central moments are given as:

$$\mu_2 = E\left[(x-\mu)^2\right] = \frac{\beta \cdot df}{df-2}$$
$$\mu_4 = E\left[(x-\mu)^4\right] = \frac{3\beta^2 df^2}{(df-2)(df-4)} , \tag{3}$$

with excess kurtosis (greater then 3):

$$k^* = \frac{\mu_4}{\mu_2^2} - 3 = \frac{6}{df-4} . \tag{4}$$

Hence, we may apply method of moments to get consistent estimators:

$$\hat{\mu}_2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$
$$\hat{\mu}_4 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^4}{n}$$
$$\hat{k}^* = \frac{\hat{\mu}_4}{\hat{\mu}_2^2} - 3 , \tag{5}$$
$$\hat{df} = 4 + \frac{6}{\hat{k}^*}$$
$$\hat{\beta} = \left(\frac{3+\hat{k}^*}{3+2\cdot\hat{k}^*}\right)\cdot\hat{\sigma}^2$$

where variance from sample σ^2 is biased estimator of scale parameter β.

According to equations (2) to (5) estimated degrees of freedom equal 5.3. Thus, non-integer degrees of freedom are used:

- during optimization of likelihood function to estimate GARCH parameters, within quasi-Newton algorithm, and
- to precisely calculate left percentile of heavy tailed distribution for VaR and CVaR estimation.

4 Identifying ARCH and leverage effects

Before we continue to create the model to capture volatility of Pliva returns it is necessary to investigate if there is asymmetry in volatility clustering, i.e. if there is leverage effect. The tendency for volatility to decline when returns rise and to rise when returns fall is called the leverage effect, i.e. "bad" news seems to have a more effect on volatility than does "good" news.

A simple test to investigate the leverage effect is to calculate first-order autocorrelation coefficient between lagged returns and current squared returns:

$$\frac{\sum_{i=2}^{n} r_t^2 r_{t-1}}{\sqrt{\sum_{i=2}^{n} r_t^4 \sum_{i=1}^{n} r_{t-1}^2}} . \tag{6}$$

Table 3.

Testing for leverage effects

Correlations

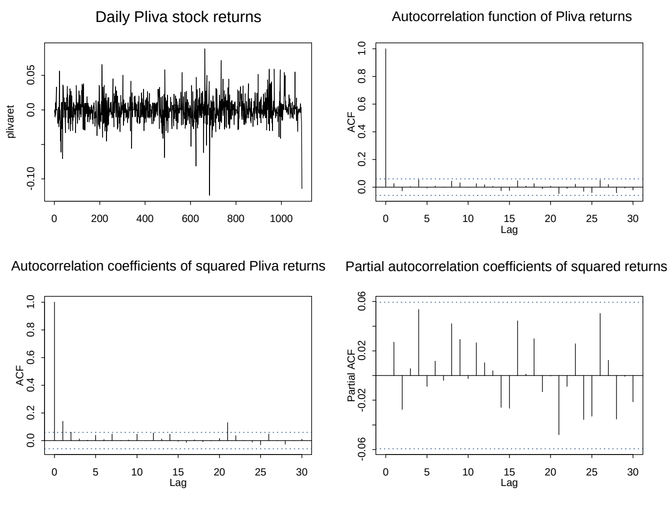
		Laged Pliva returns	Squared Pliva returns
Laged Pliva returns	Pearson Correlation	1	,030
	Sig. (2-tailed)		,328
	N	1089	1089
Squared Pliva returns	Pearson Correlation	,030	1
	Sig. (2-tailed)	,328	
	N	1089	1090

Source: Tested according to data on ZSE

It can be concluded that there is no asymmetric volatility clustering of Pliva returns at p-value of 33%, because the above autocorrelation coefficient is positivee and it is not significantly different from zero.

Figure 3.

Pliva's stock returns from June 2002 to October 2006 (1090 trading days), ACF of returns, ACF and PACF of squared returns



Source: According to data on ZSE

From figure 3. it's obvious that there is significant autocorrelation in squared return series of Pliva stocks for almost each time lag. It means that return series contain ARCH effects. These ARCH effects are also tested using Lagrange multiplier test, which results are given in table 4.

Table 4.

Lagrange multiplier test

```
--------------------------------------------------
Test for ARCH Effects: LM Test
Null Hypothesis: no ARCH effects

Test Stat 68.8194
  p.value  0.0001

Dist: chi-square with 30 degrees of freedom
Total Observ.: 1090
--------------------------------------------------
```

Source: Tested according to data on ZSE

From table 4. it can be seen that variance is heteroscedastic because the square unexpected returns follows autoregressive process. Even more LM test value, for large samples, is significant at 0.01%. It means that variance is time-varying.

5 Specification of GARCH(p,q) model and parameter estimation by quasi-Newton algorithm

In mean equation constant is entered as regressor, because ACF of Pliva stock returns didn't show statistical significance for any time lag. If there is significant autocorrelation in returns, best fitted ARMA models are usually used, following Box-Jenkins procedure. It has been shown that ARCH(p) process with infinite number of parameters is equivalent to much generalized ARCH process called GARCH(1,1). As the time lag increases in an ARCH(p) model it becomes more difficult to estimate parameters. Besides it is recommended to use parsimonious model as GARCH(1,1) that is much easier to identify and estimate.

In table 5. estimated parameters of GARCH(1,1) model are presented as well as appropriate diagnostics test.

Table 5.

Estimated GARCH(1,1) model

```
------------------------------------------------------
Estimated Coefficients:
               Value  Std.Error t value   Pr(>|t|)
       C -0.00018119 0.00042886 -0.4225 0.33637565
       A  0.00005835 0.00001597  3.6533 0.00013564
 ARCH(1)  0.22614940 0.05345972  4.2303 0.00001265
GARCH(1)  0.63441794 0.07000522  9.0624 0.00000000
------------------------------------------------------
Information criteria:
AIC(4) = -5876.871
BIC(4) = -5856.895
------------------------------------------------------
Normality Test:
 Jarque-Bera P-value Shapiro-Wilk P-value
        1037       0       0.9587       0
------------------------------------------------------
Ljung-Box test for standardized residuals:
 Statistic P-value Chi^2-d.f.
     9.625  0.6489         12
------------------------------------------------------
Ljung-Box test for squared standardized residuals:
 Statistic P-value Chi^2-d.f.
     6.637  0.8807         12
------------------------------------------------------
Lagrange multiplier test:
   Lag 1  Lag 2  Lag 3 Lag 4   Lag 5  Lag 6   Lag 7
 -0.9236 -1.078 -1.012 0.962 -0.5799 0.3408 -0.3561
  TR^2 P-value F-stat P-value
 9.191   0.6865 0.8427  0.7085
------------------------------------------------------
```

Source: Estimated according to data on ZSE

Results from diagnostics test indicates that there are no ARCH effects and no autocorrelation of standardized residuals left. Parameters in table 5. are estimate using BHHH (*Berndt, Hall, Hall, Hausman*) algorithm within quasi-Newton optimization. Maximization of likelihood function procedure is defined by the iteration formula:

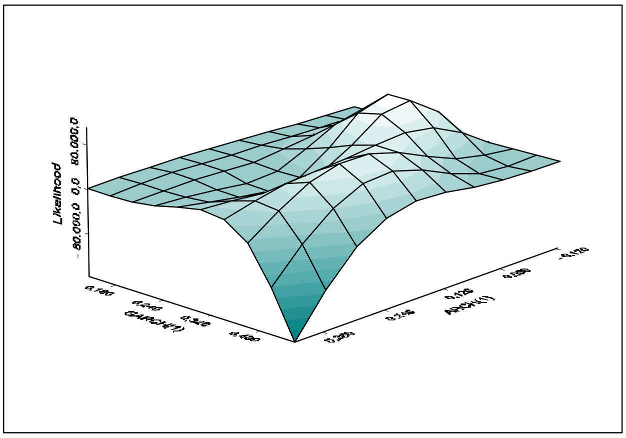
$$\theta_{i+1} = \theta_i + \lambda_i \cdot H_i^{-1} \cdot g_i$$
$$H_i = \frac{1}{T}\sum_{t=1}^{T}\left(g_t \cdot g_t^T\right) \quad . \qquad (7)$$
$$g_i = \sum_{t=1}^{T} g_t$$

In equations (7) $H_i^{-1} g_i$ is the gradient vector premultiplied by the inverse of the Hessian approximation, which determines the direction in i-th iteration. Scalar λ_i is step size which in each iteration provides an increase in log-likelihood function. By assumption of Student's distribution log-likelihood function has following form:

$$\ln L = -\sum_{t=1}^{T}\left[\frac{1+df}{2}\cdot \ln\left(1+\frac{\varepsilon_t^2}{(df-2)\cdot\sigma_t^2}\right)+\frac{1}{2}\ln\left(\sigma_t^2\right)\right]. \qquad (8)$$

Figure 4.

Maximization of likelihood function



Source: According to data on ZSE

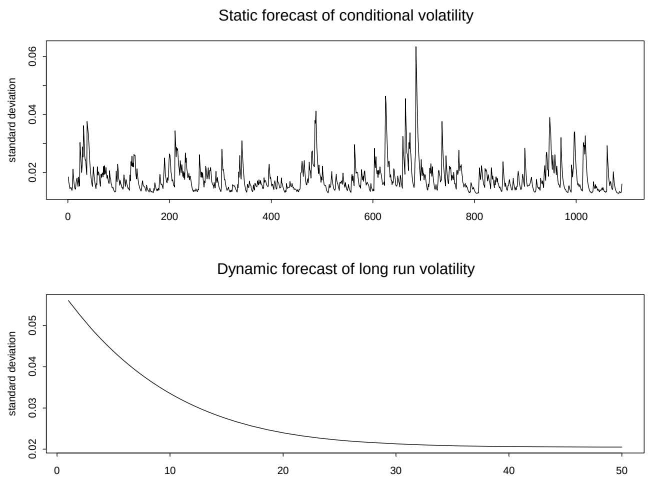
According to results presented in table 5. estimated model has following form:

$$r_t = \varepsilon_t , \qquad \varepsilon_t = u_t \cdot \sqrt{\sigma_t^2} \;, \quad u_t = i.i.d.(0,1)$$
$$\sigma_t^2 = 0.000058 + 0.22615 \cdot \varepsilon_t^2 + 0.634418 \cdot \sigma_{t-1}^2 \quad . \quad (9)$$

In system (9) residuals ε_t, i.e. innovations, are defined with u_t by assumption of symmetric Student's distribution.

Sum of parameters ARCH(1)+GARCH(1), according to equations (9), indicates that there is persistence volatility of 86%, i.e. conditional variance decays slowly, not far from long-memory model. Hence the sum of parameters is less then one the condition of covariance stationary is confirmed.

Figure 5.
Static and dynamic volatility forecasting



Source: According to data on ZSE

Figure 5. shows static forecast of conditional standard deviation and dynamic forecast of unconditional long run variance, using Student's distribution assumption with 5.3 degrees of freedom in GARCH(1,1) model.

6 Tail function as the instrument of VaR and CVaR forecasting

VaR is defined as the maximum potential loss of financial instrument with a given probability (usually 1% or 5%) over a certain time period. Based on the Student's distribution, Value at Risk can be calculated as:

$$VaR_t(\alpha) = \hat{\mu}_t + t_\alpha^{df} \cdot \left[\hat{\sigma}_t \cdot \sqrt{\frac{3+\hat{k}^*}{3+2\cdot\hat{k}^*}} \right], \qquad (10)$$

where $\hat{\mu}_t$ is expected mean and $\hat{\sigma}_t$ expected standard deviation, predicted from estimated GARCH(1,1) model. VaR expressed in equation (10) can be interpreted as expected minimal percentage loss within probability of α, when t_α^{df} is left percentile from standard Student's distribution. This is the case when investor holds "long" position, i.e. if he has bought an asset, in which case he incurs the risk of a loss of value of the asset. When investor holds "short" position (he has sold an asset, in which case he incurs a positive opportunity cost if the asset value increases), variable t_α^{df} presents the right percentile from standard Student's distribution.

In formula (10) expected standard deviation is corrected to get unbiased estimator of standard Student's scale parameter, according to equation (5).

However, there is no rule for selection appropriate confidence level in VaR estimation. Hence, for achieving compromise solution in confidence level selection, it is better to estimate conditional Value at Risk, which includes more information about expected loss. Therefore, CVaR is defined as expected loss under tail area bounded by VaR:

$$CVaR_t(\alpha) = E\left[r_t / r_t \le VaR_t(\alpha)\right]. \qquad (11)$$

According to definition of conditional expectation CVaR is:

$$CVaR(q)_t = \frac{\int_{-\infty}^{-q} x f(x) dx}{F(q)} \;, \qquad (12)$$

where $f(x)$ is density function of Student's distribution according to equation (1). Value q presents left percentile of standard Student's distribution, i.e. standardized VaR.

So, if on 12 October 2006 (the last day of observed period) investor has bought Pliva stocks at price of 700,50 kunas, it can be predicted, for twenty days ahead, that his loss wouldn't exceed 26.39 kunas per stock with probability of 95%:

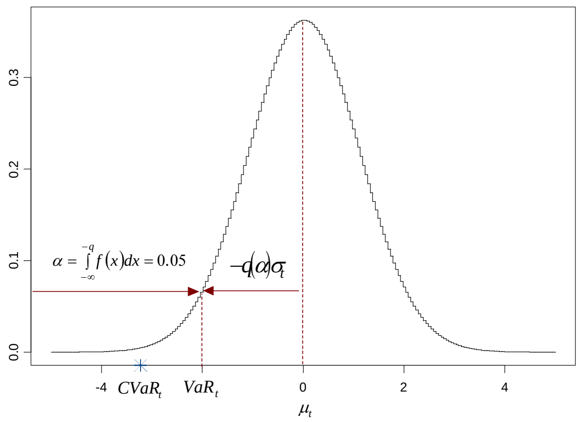
$$V\hat{a}R_{t+20}(0.05) =$$
$$= 700.50 \left[0 - 1.990124 \cdot 0.0239933 \sqrt{\frac{3+4.61543}{3+2\cdot 4.61543}} \right] =$$
$$= -26.39$$

According to estimated non-integer degrees of freedom that in this case amount 5.3, function $f(x)$ has the following expression:

$$f(x) = \frac{\Gamma(3.15)}{\Gamma(2.65)\sqrt{\pi \cdot 5.3}} \left(1 + \frac{x^2}{5.3} \right)^{\frac{-6.3}{2}} =$$
$$= 0.3806635 \left(1 + \frac{x^2}{5.3} \right)^{-3.15}$$

The above defined function $f(x)$ in interval $\langle -\infty , -26.39]$ is in fact tail function. Tail function of $f(x)$ for estimated $V\hat{a}R_{t+20}(0.05)$ is shown on figure 7.

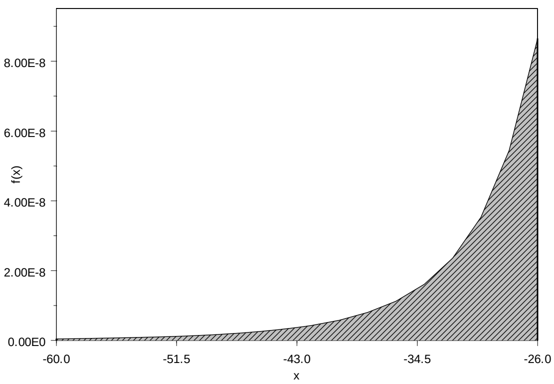
Figure 6.
VaR and CVaR according to Student's distribution



Source: Authors construction

From the figure 6. theoretical difference among VaR and CVaR is obvious.

Figure 7.
Tail function



Source: According to data on ZSE

In the treated case by substitution of empirical data conditional expectation of continuously random variable can be calculated:

$$C V\hat{a}R_{t+20}(q)=\frac{\int_{-\infty}^{-1.990124} x\cdot 0.3806635\left(1+\frac{x^2}{5.3}\right)^{-3.15}}{F(-1.990124)}=$$
$$=-2.82682$$

Therefore, conditional expectation is:
$E(x/x\leq -26.39)=-37.49$,
which means that investor can expect average loss of -37.49 kunas per Pliva stock with confidence level of 95%.

7 Conclusion

This paper deals with modeling volatility of returns of Pliva stocks on Zagreb Stock Exchange, measuring volatility reaction on market movements and the persistence of volatility. These financial time series contain volatility which is time-varying. As the most appropriate model for those analyses is GARCH(p,q) model.

According to the market efficiency hypothesis (*random walk hypothesis*), the returns are serially uncorrelated with a zero mean and hence unpredictable random variables, but autocorrelation of the squared returns suggests high dependency between them. This means that volatility changes over time and it is conditioned on its past values. In estimation procedure the assumption of Student's distribution is used to capture fat tails.

Likelihood function is maximized with non-integer degrees of freedom, which are related with appropriate kurtosis of empirical distribution. Moreover, estimated degrees of freedom are used for precisely forecasting VaR and CVaR under non-normality assumption. Namely, in this paper the conditional expectation of continuously random variable is calculated under tail area. Therefore, depending on if investor on capital market holds "long" or "short" position it's essentially important to predict possible and expected loss with appropriate probability density function, which was the basic aim to be shown by this paper.

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