Coproduct for symmetric ordering S. MELJANAC, Z. ŠKODA, *preliminary version*

Abstract. Given a finite-dimensional Lie algebra, and a representation by derivations on the completed symmetric algebra of its dual, a number of interesting twisted constructions appear: certain twisted Weyl algebras, deformed Leibniz rules, quantized "star" product. We first illuminate a number of interrelations between these constructions and then proceed to study a special case in certain precise sense corresponding to the symmetric or Weyl ordering. This case has been known earlier to be related to computations with Hausdorff series, for example the expression for the star product is in such terms. For the deformed Leibniz rule, hence a coproduct, we present here a new nonsymmetric expression, which is then expanded into a sum of expressions labelled by a class of planar trees, and for a given tree evaluated by Feynman-like rules. These expressions are filtered by a bidegree and we show recursion formulas for the sums of expressions of a given bidegree. and compare the recursions to recursions for Hausdorff series, including the comparison of initial conditions. This way we show a direct correspondence between the Hausdorff series and the expression for twisted coproduct.

1. Fix a *n*-dimensional Lie algebra \mathfrak{g} over a field k. The main message in our first several pages consists of the correspondences between several kinds of data:

- k-linear maps $\phi : \mathfrak{g} \to \operatorname{Hom}_{k}(\mathfrak{g}, \hat{S}(\mathfrak{g}^{*}))$
- **k**-linear maps $\tilde{\phi} : \mathfrak{g} \to \text{Der}_{k}(\hat{S}(\mathfrak{g}^{*}), \hat{S}(\mathfrak{g}^{*}))$
- Matrices $(\phi_{\beta}^{\alpha})_{\alpha,\beta=1,\dots,n}$ of elements $\phi_{\beta}^{\alpha} \in \hat{S}(\mathfrak{g}^*)$ satisfing the system of formal differential equations (4).
- Hopf actions of $U(\mathfrak{g})$ on $\hat{S}(\mathfrak{g}^*)$.
- Algebra homomorphisms $U(\mathfrak{g}) \to \hat{A}_{n,\mathbf{k}}$ (the codomain is the *n*-th Weyl algebra completed with respect to the powers of ∂^i -s) which is of the form $\hat{x}_{\mu} \mapsto \sum_{\alpha=1}^{n} x_{\alpha} \phi_{\mu}^{\alpha}$ on a basis $\hat{x}_{1}, \ldots, \hat{x}_{n}$ of \mathfrak{g} , with $\phi_{\beta}^{\alpha} \in \hat{S}(\mathfrak{g}^{*})$
- Coalgebra isomorphisms $\xi : S(\mathfrak{g}) \to U(\mathfrak{g})$ which are identity on $\mathfrak{g} \oplus \mathbf{k} = U^1(\mathfrak{g}) \subset U(\mathfrak{g})$.

These correspondences are pretty easy to observe and the list can be meaningfully extended. On the other hand, the many special cases of such data studied in (mainly recent physics) literature, are treated with confusion about the definitions, nature and correspondences between these data and related constructions. The list can be meaningfully extended. For example, there are popular "ordering prescriptions" which are various concrete ways determining the coalgebra isomorphism ξ above (as the isomorphism is trivial on generators in \mathfrak{g} one needs to know what to do with higher polynomials, hence "ordering prescriptions"). Another set of data, more loosely defined: if one extends ξ to the completion (power series) then ξ can be evaluated on some interesting dense set, and exponentials $\exp(ikx)$ are a good candidate and $\xi(\exp(ikx))$ is of the form $\xi(\exp(iK(k)x))$ where $K: \mathbf{k}^n \to \mathbf{k}^n$ is a bijection which is determined by ϕ and determines ϕ (however we do not know general rule which bijections K are admisible, though we do have classification results for some very special \mathfrak{g}). For the correspondences to be bijections we need the assumption that the maps ϕ etc. are close to the "unit" case: for example in the case of ϕ^{α}_{β} the unit case is δ^{α}_{β} and the near by is in the sense of topology for the ring of formal power series in ∂ . Nonformal case of the correspondences is interesting as well, but more difficult and we have no closed sufficiently general results of that form.

2. Morphism ϕ and the equation it satisfies.

Suppose we are given a Lie algebra \mathfrak{g} and a finite-dimensional vector space V over a field \mathbf{k} . By $\widehat{S(V)}$ or $\widehat{S}(V)$ we will denote the completed symmetric algebra on V, which may be viewed as a formal power series in $m = \dim V$ variables. Later we will set $V = \mathfrak{g}^*$, but for the moment we consider the full generality. Suppose we are also given a linear map $\boldsymbol{\phi} : \mathfrak{g} \to \operatorname{Hom}_{\mathbf{k}}(V, \widehat{S(V)})$. We want to extend this map to a \mathbf{k} -linear map into continuous derivations also denoted $\tilde{\boldsymbol{\phi}} : \mathfrak{g} \to \operatorname{Der}_{\mathbf{k}}(\widehat{S(V)}, \widehat{S(V)})$. By the commutativity of $\widehat{S(V)}$, it must hold that

$$\tilde{\boldsymbol{\phi}}(\hat{x})(v_1\cdots v_n) = \sum_{i=1}^n v_1\cdots v_{i-1}v_{i+1}\cdots v_n \boldsymbol{\phi}(\hat{x})(v_i).$$
(1)

This formula is linear in all arguments and symmetric under their permutations, hence by linearity in all arguments it defines a unique extension of $\phi(\hat{x})$ to a well-defined map $\tilde{\phi}(x) \in \operatorname{Hom}_{k}(S(V), \widehat{S(V)})$. It is straightforward to check that $\tilde{\phi}(\hat{x})$ defined via (1) is indeed a derivation. By abuse of notation, we will henceforth denote the extension $\tilde{\phi}$ also by ϕ . Let $\partial^1, \ldots, \partial^m$ is a vector space basis of V. Then, in terms of (algebraically defined) partial derivatives $\frac{\partial}{\partial(\partial^i)}$, the condition (1) generalizes to the usual chain rule on $\widehat{S(V)}$

$$\boldsymbol{\phi}(\hat{x})(f) = \sum_{i=1}^{m} \frac{\partial}{\partial(\partial^{i})}(f)\boldsymbol{\phi}(\hat{x})(\partial^{i})$$
(2)

Finally, we continuously extend ϕ to a k-linear map $\phi : \mathfrak{g} \to \text{Der}_k(\widehat{S(V)}, \widehat{S(V)})$.

The enveloping algebra $U(\mathfrak{g})$ is a Hopf algebra with elements of $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$ being primitive. If the linear map $\phi : \mathfrak{g} \to \operatorname{Der}_{k}(\widehat{S(V)})$ is a homomorphism of Lie algebras, i.e.

$$\boldsymbol{\phi}(\hat{x})\boldsymbol{\phi}(\hat{y}) - \boldsymbol{\phi}(\hat{y})\boldsymbol{\phi}(\hat{x}) - \boldsymbol{\phi}([\hat{x},\hat{y}]) = 0, \quad \hat{x}, \hat{y} \in \mathfrak{g},$$
(3)

then ϕ extends multiplicatively to a unique Hopf action of $U(\mathfrak{g})$, i.e. to a homomorphism $\phi : U(\mathfrak{g}) \to \operatorname{End}_{k}(\widehat{S(V)})$ satisfying $\phi(u)(fg) = m_{U(\mathfrak{g})}(\phi \otimes \phi)\Delta(u)(f \otimes g) = \sum \phi(u_{(1)})(f)\phi(u_{(2)})(g)$, for all $f, g \in \widehat{S(V)}$ and $u \in U(\mathfrak{g})$, where $m_{U(\mathfrak{g})}$ is the multiplication map on $U(\mathfrak{g})$. From now on, let \mathfrak{g} be finite-dimensional as well and let $\hat{x}_1, \ldots, \hat{x}_n$ be a k-basis of \mathfrak{g} . Denote

$$\phi_{\beta}^{\alpha} = \phi_{\beta}^{\alpha}(\partial^{1}, \dots, \partial^{m}) := \boldsymbol{\phi}(-\hat{x}_{\beta})(\partial^{\alpha}) \in \widehat{S(V)}.$$

The formal power series $\phi_{\beta}^{\alpha} = \phi_{\beta}^{\alpha}(\partial^1, \dots, \partial^m)$ has algebraically defined partial derivatives

$$\frac{\partial}{\partial(\partial^i)}\phi^{\alpha}_{\beta}\in\widehat{S(V)}.$$

Then $\phi(\hat{x}_i)\phi(\hat{x}_j)(\partial^k) = \phi(\hat{x}_i)(-\phi_j^k) = -\frac{\partial}{\partial(\partial^l)}(\phi_j^k)\phi(\hat{x}_i)(\partial^l) = -\frac{\partial}{\partial(\partial^l)}(\phi_j^k)\phi_i^l$. Thus the condition (3) reads for $\hat{x} = \hat{x}_i$ and $\hat{x} = \hat{x}_j$

$$\phi_j^l \frac{\partial}{\partial(\partial^l)} (\phi_i^k) - \phi_i^l \frac{\partial}{\partial(\partial^l)} (\phi_j^k) = C_{ij}^s \phi_s^k \tag{4}$$

Consider the usual Weyl algebra $A_{n,\mathbf{k}}$ with generators $x_1, \ldots, x_n, \partial^1, \ldots, \partial^n$, and its completion $\hat{A}_{n,\mathbf{k}}$ along the filtration by the degree of differential operator. Then the correspondence $\hat{x}_i \mapsto \hat{x}_i^{\phi} := \sum_{j=1}^m x_j \phi_i^j$ extends to an algebra homomorphism $()^{\phi} : U(\mathfrak{g}) \to \hat{A}_{n,\mathbf{k}}$ iff (4) holds.

Many solutions for ϕ satisfying (4) and hence homomorphisms ϕ , injective or not, for particular k and particular k-Lie algebras, with m equal n

or not, exist. For example, in the article of Berceanu [2], such realizations with ϕ faithful have been found for \mathfrak{g} semisimple over $\mathbf{k} = \mathbb{C}$. In that case, dim $V < \dim \mathfrak{g}$ and dim V may be calculated in terms of the combinatorics of root systems.

3. A universal formula for a "symmetric" solution to (4) has been found ([3]), for any ring $\mathbf{k} \supset \mathbb{Q}$, \mathfrak{g} finite rank free module over \mathbf{k} and $V = \mathfrak{g}^*$ (in particular, m = n), where $\boldsymbol{\phi}$ is a monomorphism.

4. Hopf algebras. All bialgebras in the article will be associative, coassociative, with unit map η and counit ϵ , without gradings. Hopf algebras will be bialgebras with an antipode and the standard Sweedler notation for th coproduct $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ is few times used, with or without the summation sign. Recall that the elements $h \in H$ such that $\Delta(h) = 1 \otimes h + h \otimes 1$ are called primitive.

5. Smash product algebras. Given any Hopf algebra H and a, say left, Hopf action of H on algebra $\mathcal{S}, h \otimes s \mapsto h \triangleright s$, one forms a crossed product algebra (in Hopf literature "smash product") $\mathcal{S} \ddagger H$. As a vector space, it is simply the tensor product vector space $S \otimes \mathcal{H}$ and the associative product is given by

$$(s \otimes h)(s' \otimes h') = \sum s(h_{(1)} \triangleright s') \otimes h_{(2)}h'.$$

The canonical embeddings $S \hookrightarrow S \sharp H$ and $H \hookrightarrow S \sharp H$ will be considered identifications, and one usually omits the tensor sign because $s \otimes h = sh$ with respect to these embeddings and the product in $S \sharp H$. Then $h \triangleright s =$ $\sum h_{(1)} s S_H(h_{(2)})$ where $S_H : H \to H$ is the antipode. Furthermore, the rule

$$(s\sharp h) \triangleright s' := s(h \triangleright s') \tag{5}$$

defines an action of $\mathcal{S} \sharp H$ on \mathcal{S} .

Analogously, for any *right* action of H on S one defines the crossed product denoted by $H \sharp S$, whose underlying vector space is $H \otimes S$. If the antipode $S_H : H \to H^{\text{op}}$ is bijective, there is a bijective correspondence between the left and right actions (namely, composing with S_H) and the crossed products for the two corresponding (left and right) actions are canonically isomorphic and we often identify them throughout the article.

6. (\mathfrak{g}, ϕ) -deformed Weyl algebras. Regarding that for any \mathfrak{g} , V and ϕ such that (4) holds, the action of $U(\mathfrak{g})$ on $\widehat{S(V)}$ is a Hopf action, we may

define the smash product algebra

$$A_{\mathfrak{g},\phi} := \widehat{S(V)} \sharp U(\mathfrak{g}) = \widehat{S(V)} \sharp_{\phi} U(\mathfrak{g})$$

where the action $u \triangleright v := \phi(u)(v)$ is uniquely determined by the values $\phi(-\hat{x}_i)(\partial^j) = \phi_j^i$ as explained above. The rule (5) specializes to a (dual) "natural" action of $A_{\mathfrak{g},\phi}$ on $\widehat{S(V)}$. In particular, if $\mathfrak{g} = \mathfrak{a}$ is an abelian Lie algebra, $V = \mathfrak{g}^*$ and ϕ is given simply by the bilinear pairing $\phi(\hat{x}_i)(\partial^j) = \delta_i^j$, then $A_{\mathfrak{g},\phi}$ is isomorphic to the usual (semi)completed Weyl algebra $\widehat{A}_{n,\mathbf{k}}$ and the action is the usual action of $S(\mathfrak{a})$ on $\widehat{S(\mathfrak{a}^*)}$.

7. From now on we suppose

(i) $\phi : \mathfrak{g} \to \operatorname{Der}(\tilde{S}(\mathfrak{g}^*))$ is a homomorphism of Lie algebras

(ii) the matrix ϕ (not bold) with entries $\phi_j^i := \phi(-\hat{x}_i)(\partial^j)$ has the unit matrix as its constant term, i.e. $\phi_j^i = \delta_j^i + O(\partial)$.

8. Under the assumptions from 7, ϕ is invertible as a matrix over the formal power series ring $\mathbf{k}[[\partial^1, \ldots, \partial^n]]$ and the homomorphism $U(\mathfrak{g})\sharp \widehat{S(\mathfrak{g}^*)} \cong S(\mathfrak{g})\sharp \widehat{S(\mathfrak{g}^*)}$ given on generators by

$$\hat{x}_{\alpha} \mapsto x_{\beta} \phi_{\alpha}^{\beta}, \quad \partial^{\mu} \mapsto \partial^{\mu}$$

is an isomorphism. Hence the (one-sidedly) completed deformed and undeformed Weyl algebras are isomorphic via a nontrivial map and we often identify them when doing calculations.

9. This isomorphism enables us to consider the homomorphism

$$()^{\phi}: U(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \not \cong \widehat{S(\mathfrak{g}^*)} \cong S(\mathfrak{g}) \not \cong \widehat{S(\mathfrak{g}^*)} = \widehat{A}_{n,k}$$

which agrees with the unique homomorphism $U(\mathfrak{g}) \to \hat{A}_{n,\mathbf{k}}$ extending the rule

$$\hat{x}_{\alpha} \mapsto \hat{x}_{\alpha}^{\phi} := x_{\beta} \phi_{\alpha}^{\beta} \in \hat{A}_{n,\boldsymbol{k}}$$

Furthermore, we may identify $S(\mathfrak{g})\sharp\widehat{S(\mathfrak{g}^*)} \cong \operatorname{Hom}_{k}(S(\mathfrak{g}), S(\mathfrak{g}))$. Here $\phi_{\alpha}^{\beta} = \phi_{\alpha}^{\beta}(\partial^{1}, \ldots, \partial^{n})$ is understood as an element of the completed Weyl algebra $\widehat{A}_{n,k} \cong S(\mathfrak{g}) \sharp \widehat{S(\mathfrak{g}^*)}$ acting in the usual way (as differential operator, this one with constant coefficients) on the polynomial algebra. Therefore we obtained an action, depending on ϕ , of $U(\mathfrak{g})$ on $S(\mathfrak{g})$.

10. Lemma. Let $\chi \in \mathbf{k}[[\partial^1, \ldots, \partial^n]]$, then $x_{\sigma}\chi$ is a coderivation of the polynomial algebra $P = \mathbf{k}[x_1, \ldots, x_n]$. In other words,

$$(x_{\sigma}\chi \otimes \mathrm{id} + \mathrm{id} \otimes x_{\sigma}\chi)(\Delta_P(f)) = \Delta_P(x_{\sigma}\chi(f)), \quad \forall f \in P.$$
(6)

Proof. By linearity it is enough to prove it for f which are monomial. We prove this by induction on the sum of the polynomial degree of f and the order of differential operator Ξ . The identity is clearly true if either the degree of f or order of ξ is 0. Regarding that f is monomial it is of the form $x_{\gamma}g$ where g is some monomial of a lower order. We identify id with 1 in Weyl algebra and x_{μ} with multiplication with x_{μ} . For step of induction we want to prove that

$$(x_{\sigma}\xi \otimes 1 + 1 \otimes x_{\sigma}\xi)\Delta(x_{\gamma}g) = \Delta(x_{\sigma}\xi(x_{\gamma}g))$$

provided this is true for ξ of lower order or $x_{\gamma}g$ replaced by g what is of lower degree. Using the fact that Δ is a homomorphism of algebras and that x_{γ} is primitive, we rewrite this equality using commutators:

$$\begin{aligned} (x_{\sigma}[\xi, x_{\gamma}] \otimes 1 &+ 1 \otimes x_{\sigma}[\xi, x_{\gamma}])\Delta(g) + \\ &+ (x_{\gamma} \otimes 1 + 1 \otimes x_{\gamma})(x_{\sigma}\xi \otimes 1 + 1 \otimes x_{\sigma}\xi)\Delta(g) \\ &= \Delta(x_{\sigma}[\xi, x_{\gamma}](g)) + (x_{\gamma} \otimes 1 + 1 \otimes x_{\gamma})\Delta(x_{\sigma}\xi(g)) \end{aligned}$$

and recall that $[\xi, x_{\gamma}]$ is of lower order. This equality holds because it is a sum of two equations which hold by the assumption of the induction. Q.E.D.

Corollary. The action from **9** restricted on \mathfrak{g} is an action by coderivations with respect to the standard coalgebra structure on $S(\mathfrak{g})$:

$$(x_{\beta}\phi_{\alpha}^{\beta} \otimes \mathrm{id} + \mathrm{id} \otimes x_{\beta}\phi_{\alpha}^{\beta})(\Delta_{S(\mathfrak{g})}(f)) = \Delta_{S(\mathfrak{g})}(x_{\beta}\phi_{\alpha}^{\beta}(f)), \quad \forall f \in S(\mathfrak{g}).$$
(7)

11. For us it is important to consider the special case of the action of $U(\mathfrak{g})$ on $S(\mathfrak{g})$ from 9, when f is 1 (action on "vacuum").

Proposition. The rule $\xi^{-1} : \hat{u} \mapsto \hat{u}(1)$ for $u \in U(\mathfrak{g})$ is an isomorphism of coalgebras, which restricts to the identity on $\mathbf{k} \oplus \mathfrak{g}$.

Of course, the inverse of ξ^{-1} will be some isomorphism of coalgebras $\xi : S(\mathfrak{g}) \to U(\mathfrak{g})$. Conversely, every isomorphism of coalgebras $\xi : S(\mathfrak{g}) \to U(\mathfrak{g})$ which is identity on $\mathbf{k} \oplus \mathfrak{g}$, defines a map $D^T : \mathfrak{g} \to \operatorname{Coder}(S(\mathfrak{g}))$ into coderivations by $D_x^T(f) = D^T(x)(f) = \xi^{-1}(\xi(x) \cdot U(\mathfrak{g})\xi(f))$. The dual map $D_x : \widehat{S(\mathfrak{g}^*)} \to \widehat{S(\mathfrak{g}^*)}$ is a continuous derivation, and one has $D_x^T(f) =$

 $-\sum_{\alpha} x_{\alpha} D_x(\partial^{\alpha})(f)$ where the action on the left is the usual action as differential operator. Here $\sum_{\alpha} x_{\alpha} \otimes \partial^{\alpha} \in \mathfrak{g} \otimes \mathfrak{g}^*$ is the "canonical element" (the image of $\mathrm{id}_{\mathfrak{g}}$ under the isomorphism $\mathrm{Hom}_{k}(\mathfrak{g},\mathfrak{g}) \to \mathfrak{g} \otimes \mathfrak{g}^*$). Thus one defines a Lie homomorphism $\phi: \mathfrak{g} \to \mathrm{Der}(\widehat{S(\mathfrak{g}^*)}, \widehat{S(\mathfrak{g}^*)})$ by $x \mapsto D_x$ such that $\phi_i^i = D_{x_i}(\partial^i)$ and $\phi_i^i = \delta_i^i + O(\partial)$.

12. If some linear isomorphism $S(\mathfrak{g}) \to U(\mathfrak{g})$ preserves the degree filtration, then it clearly extends by continuity to a linear map among the corresponding completions $S(\mathfrak{g}) \to U(\mathfrak{g})$. If the isomorphism is a coalgebra map, then the extension respects the completed coproducts $\Delta: \hat{H} \to \hat{H} \otimes \hat{H}$ $(H = S(\mathfrak{g}) \text{ or } U(\mathfrak{g}))$. Thus, it makes sense to consider the behaviour of exponential series (as a formal series) under coalgebra isomorphism ξ as above. It is also useful to extend the field by $\sqrt{-1}$ if it is not present and consider formal series of the type $\exp(ik^{\alpha}x_{\alpha})$. If the field is \mathbb{C} then such series are specially important because of Fourier integral methods. However, Fourier integral is defined only for some formal series, so the formulas, though useful for other spaces of functions (one can extend our coproducts etc. to various functional spaces, but we will avoid this here) the formulas involving Fourier integrals in this paper will be understood just in the following sense: every abstract series involved is a linear combination of series of the form $\exp(ia^{\alpha}x_{\alpha})$. Such sums are dense in the space of formal power series, so if some identity is proved for finite sums of exponentials (which we heuristically write as integrals, with some kernels). The imaginary unit is just for easiness of applications in physics, one can correct the $\sqrt{-1}$ factors and prove the formulas just for the sums of functions of the form $\exp(ia^{\alpha}x_{\alpha})$ but we will not spend time on these nicetess.

13. Coalgebra isomorphisms $\xi : S(\mathfrak{g}) \to U(\mathfrak{g})$ which are identity on $\mathbf{k} \oplus \mathfrak{g}$, and which are extended to the completions have the property

$$\xi(\exp(ik^{\alpha}x_{\alpha})) = \exp(iK(\vec{k})^{\beta}\hat{x}_{\beta})$$
(8)

for some bijection $K : \mathbf{k}^n \to \mathbf{k}^n$. (Proof: All group like elements both in $\hat{S}(\mathbf{g})$ and in $\hat{U}(\mathbf{g})$ are of such exponential form. ξ is a bijection and preserves the group like elements because it is a coalgebra map.) For example, if K is the identity map, this is the case of symmetric ordering: ξ is the coexponential map (when considered defined on $S(\mathbf{g})$ only). Furthermore, one can get a very large class of other solutions which satisfy 8 using certain inner automorphisms of Weyl algebra.

Namely, let $S = \exp(x^{\alpha}R_{\alpha} + B)$ where $R_{\alpha} = R_{\alpha}(\partial), B = B(\partial)$ are some formal series in variables $\partial^{1}, \ldots, \partial^{n} \in \hat{A}_{n,k}$. Then define the formal power series

$$y_{\alpha} := S x_{\alpha} S^{-1}, \qquad \partial_{y}^{\alpha} := S \partial^{\alpha} S^{-1}.$$

They again satify canonical commutation relations: $\partial_y^{\alpha}, y_{\beta}] = \delta_{\beta}^{\alpha}$ (this does not depend on the special form of S) and $y_{\alpha} = x_{\rho}\psi_{\alpha}^{\rho}$ for some formal power series in ∂ -s $\psi_{\alpha}^{\rho} = \psi_{\alpha}^{\rho}(\partial)$ (this follows by the special form of S). Moreover, $\partial_y^{\alpha} = d(\partial)$ is also a power series in ∂ -s only.

Now $x_{\alpha}\phi_{\beta}^{\alpha} = y_{\rho}\psi_{\tau}^{\rho}\phi_{\beta}^{\tau}$. This way ϕ_{β}^{α} is in the new basis replaced by $\psi_{\tau}^{\rho}(\partial)\phi_{\beta}^{\tau}(\partial)$ what should be expressed in terms of ∂_y -s (what is done computing the inverse transformation S^{-1}). This way we get some $\Xi_{\beta}^{\alpha} = \Xi_{\beta}^{\alpha}(\partial_y)$ in place of ϕ_{β}^{α} . This procedure can be accomplished in some special cases (for κ -Minkowski space see [7, 8]), but also some general statements may be proved for Ξ_{β}^{α} ontained by this procedure, where the starting ϕ_{β}^{α} corresponds to the symmetric ordering.

14. Given a Hopf algebra H acting by a right Hopf action on an algebra S and a homomorphism of unital algebras $\epsilon^{S} : S \to k$, one defines a k-linear map $(H \ddagger S) \otimes H \to H$ as the following composition:

$$(H\sharp\mathcal{S})\otimes H \hookrightarrow (H\sharp\mathcal{S})\otimes (H\sharp\mathcal{S}) \xrightarrow{m_{H\sharp\mathcal{S}}} H\sharp\mathcal{S} \xrightarrow{H\sharp\epsilon^{S}} H\otimes \mathbf{k} \cong H.$$

This map is a left *action* of the smash product algebra $H \sharp S$ on H. Algebra embedding $S \hookrightarrow H \sharp S$, $s \mapsto 1 \otimes s$, gives rise to the restriction of the above action to a left action $S \otimes H \to H$. If the antipode $S_H : H \to H^{\text{op}}$ is an isomorphism, the corresponding representation $\rho : S \to \text{End}_k(H)$ is faithful.

15. In the case when $\mathcal{S} = \widehat{S(\mathfrak{g}^*)}, H = U(\mathfrak{g})$ and the Hopf action is induced by $\phi : U(\mathfrak{g}) \to \operatorname{Der}(\widehat{S(\mathfrak{g}^*)}, \widehat{S(\mathfrak{g}^*)})$, the action $\rho_{\phi} : \mathcal{S} \to \operatorname{End}_{\mathbf{k}}(H)$ from 14 may be alternatively described in terms of values on the standard generators $\widehat{\partial}^{\mu} = \rho_{\phi}(\partial^{\mu}) \in \operatorname{End}_{\mathbf{k}}(U(\mathfrak{g})), \mu = 1, \ldots, n$. We describe the action of $\widehat{\partial}^{\mu}$ on $U(\mathfrak{g})$ inductively on the order of monomials in $U(\mathfrak{g})$. First of all, $\widehat{\partial}^{\mu}(1) = 0$ and $\widehat{\partial}^{\mu}(\widehat{x}_{\nu}) = \delta^{\mu}_{\nu}$. Then suppose $\widehat{\partial}^{\mu}$ is defined on monomials of order up to n. Then any monomial of order n + 1 is of the form $\widehat{x}_{\nu}\widehat{f}$ where $\widehat{\partial}(\widehat{f})$ is already defined. We set

$$\hat{\partial}^{\mu}(\hat{x}_{\nu}\hat{f}) := [\hat{\partial}^{\mu}, \hat{x}_{\nu}](\hat{f}) + \hat{x}_{\nu}\hat{\partial}^{\mu}(\hat{f}) := \phi^{\mu}_{\nu}(\hat{f}) + \hat{x}_{\nu}\hat{\partial}^{\mu}(\hat{f}),$$

where $\phi^{\mu}_{\nu} = \phi^{\mu}_{\nu}(\hat{\partial})$ (we can substitute $\hat{\partial}$ because $S(\mathfrak{g}^*)$ is a free commutative algebra and $\hat{\partial}^{\mu}$ mutually commute as it may be shown a posteriori). $\hat{\partial}$ is

well defined on $S(\mathfrak{g}^*)$ (hence by continuity on $\widehat{S(\mathfrak{g}^*)}$), namely it is obviously well defined linear operator from the free algebra on abstract variables \hat{x}_{α} to $U(\mathfrak{g})$, and if one takes the generators of the defining ideal of the enveloping algebra $i_{\nu_1\nu_2} = \hat{x}_{\nu_1}\hat{x}_{\nu_2} - \hat{x}_{\nu_2}\hat{x}_{\nu_1} - C^{\alpha}_{\nu_1\nu_2}\hat{x}_{\alpha}$ then, applying our inductive rules for every of the three monomials on RHS, then for every $\hat{f} \in U(\mathfrak{g})$,

$$\hat{\partial}^{\gamma}(i_{\nu_{1}\nu_{2}}\hat{f}) = \phi_{\nu_{1}}^{\gamma}(\hat{x}_{\nu_{2}}\hat{f}) + \hat{x}_{\nu_{1}}\hat{\partial}^{\gamma}\hat{x}_{\nu_{2}}(\hat{f}) - \phi_{\nu_{2}}^{\gamma}(\hat{x}_{\nu_{1}}\hat{f}) - \\ -\hat{x}_{\nu_{2}}\hat{\partial}^{\gamma}\hat{x}_{\nu_{1}}(\hat{f}) - C_{\nu_{1}\nu_{2}}^{\alpha}\phi_{\alpha}^{\gamma}(\hat{f}) - C_{\nu_{1}\nu_{2}}^{\alpha}\hat{x}_{\alpha}\hat{\partial}^{\gamma}(f) \\ = \frac{\partial}{\partial(\partial^{\nu_{2}})}(\phi_{\nu_{1}}^{\gamma})(\hat{f}) + \hat{x}_{\nu_{2}}\phi_{\nu_{1}}^{\gamma}(\hat{f}) + \hat{x}_{\nu_{1}}\phi_{\nu_{2}}^{\gamma}(\hat{f}) + \hat{x}_{\nu_{1}}\hat{x}_{\nu_{2}}\hat{\partial}^{\gamma}(\hat{f}) \\ - \left(\frac{\partial}{\partial(\partial^{\nu_{1}})}(\phi_{\nu_{2}}^{\gamma})(\hat{f}) + \hat{x}_{\nu_{1}}\phi_{\nu_{2}}^{\gamma}(\hat{f}) + \hat{x}_{\nu_{2}}\phi_{\nu_{1}}^{\gamma}(\hat{f}) + \hat{x}_{\nu_{1}}\hat{x}_{\nu_{2}}\hat{\partial}^{\mu}(\hat{f})\right) \\ - C_{\nu_{1}\nu_{2}}^{\alpha}\phi_{\alpha}^{\gamma}(\hat{f}) - C_{\nu_{1}\nu_{2}}^{\alpha}\hat{x}_{\alpha}\hat{\partial}^{\gamma}(\hat{f}) \\ = \left(\frac{\partial}{\partial(\partial^{\nu_{2}})}(\phi_{\nu_{1}}^{\gamma}) - \frac{\partial}{\partial(\partial^{\nu_{1}})}(\phi_{\nu_{2}}^{\gamma}) - C_{\nu_{1}\nu_{2}}^{\alpha}\phi_{\alpha}^{\gamma})(\hat{f}) \right)$$

The injectivity of ρ implies that $\hat{\partial}^{\gamma}(i_{\mu\nu}\hat{f}) = 0$ for every \hat{f} iff the operator in the brackets on RHS vanishes, what amount to our main assumption (4). It is trivial that $\hat{\partial}(\hat{f}i_{\mu\nu}) = 0$ as well, namely this is sufficient to check for monomial \hat{f} , but this is $\hat{f}\hat{\partial}(i_{\mu\nu}) + [\hat{\partial}, \hat{f}](i_{\mu\nu})$. We already know that the first summand is zero. The commutator in the second summand is some polynomial in $\hat{\partial}$ -s, hence it is clearly zero modulo $i_{\mu\nu}$ by induction on the degree of monomials and linearity.

Notice for the classical case of the abelian Lie algebra, that $[\partial, \hat{f}] = \hat{\partial}(\hat{f})$, while this is not true in general (the equality always makes sense: LHS is the bracket $\partial \hat{f} - \hat{f} \partial$ in the smash product $\widehat{S(\mathfrak{g}^*)} \sharp U(\mathfrak{g})$, while RHS is in $U(\mathfrak{g}) \hookrightarrow \widehat{S(\mathfrak{g}^*)} \sharp U(\mathfrak{g})$).

16. We saw in 11 that giving the Lie homomorphism ϕ for which the matrix $\phi(-\hat{x}_i)(\partial^j) = \delta^i_j + O(\partial)$ is equivalent to giving a coalgebra isomorphism $\xi : S(\mathfrak{g}) \to U(\mathfrak{g})$ which is identity when restricted to $\mathbf{k} \oplus \mathfrak{g}$. This homomorphism helps us define the star product

$$\star : S(\mathfrak{g}) \otimes S(\mathfrak{g}) \to S(\mathfrak{g}), \quad f \star g = \xi^{-1}(\xi(f) \cdot_{U(\mathfrak{g})} \xi(g)). \tag{9}$$

17. One can alternatively describe operators $\hat{\partial}^{\mu} = \rho_{\phi}(\partial^{\mu})$ from 15 by the formula

$$\hat{\partial}^{\mu}(\xi(f)) = \xi(\partial^{\mu}(f)), \quad f \in S(\mathfrak{g}),$$

where $\xi = \xi_{\phi}$ is described in **11**. Therefore also $\xi^{-1}\hat{\partial}^{\mu} = \partial^{\mu}\xi^{-1}$. It is straightforward to check that this description agrees with the inductive description of $\hat{\partial}^{\mu}$ in **11**. It is of course convenient to have such an invariant

description. Moreover, for any $P \in \hat{A}_{nk}$, understood in a usual way as an operator $S(\mathfrak{g}) \to S(\mathfrak{g})$, we form $\hat{P} : U(\mathfrak{g}) \to U(\mathfrak{g})$ by the same transport rule $\hat{P}(\xi(f)) = \xi(P(f))$.

The deformed coproduct $\Delta(\hat{\partial}^{\mu}) = \sum \hat{\partial}^{\mu}_{(1)} \otimes \hat{\partial}^{\mu}_{(2)}$ is defined by $\hat{\partial}(u \cdot U(\mathfrak{g}) v) = \sum \hat{\partial}^{\mu}_{(1)}(u) \cdot U(\mathfrak{g}) \hat{\partial}^{\mu}_{(2)}(v)$ for $u, v \in U(\mathfrak{g})$. This is equivalent to the "deformed Leibniz rule", popular in some physics works:

$$\partial^{\mu}(f \star g) = \sum_{i} \partial^{\mu}_{(1)} f \star \partial^{\mu}_{(2)} g, \qquad f, g \in S(\mathfrak{g}),$$

as the following calculation shows: $\partial^{\mu}(f \star g) = \partial^{\mu}(\xi^{-1}(\xi(f) \cdot_{U(\mathfrak{g})} \xi(g)) = \xi^{-1}(\hat{\partial}^{\mu}_{(1)}(\xi(f) \cdot_{U(\mathfrak{g})} \xi(g))) = \xi^{-1}(\hat{\partial}^{\mu}_{(1)}(f) \cdot_{U(\mathfrak{g})} \hat{\partial}^{\mu}_{(2)}(g)) = \xi^{-1}(\xi \hat{\partial}^{\mu}_{(1)}(f) \cdot_{U(\mathfrak{g})} \xi \hat{\partial}^{\mu}_{(2)}(g)) = \partial^{\mu}_{(1)}(f) \star \partial^{\mu}_{(2)}(g).$

From now on, when commuting with elements in $U(\mathfrak{g}) \hookrightarrow A_{\mathfrak{g},\phi}$ we will by $\partial \in S(\mathfrak{g}^*)$ mean $\hat{\partial} \in S(\mathfrak{g}^*) \hookrightarrow A_{\mathfrak{g},\phi}$ and so on – this amounts to the identification of the deformed and undeformed Weyl algebras, cf. 8. The main purpose of this article is describing more concretely this deformed coproduct.

18. This coproduct is related to but different from the dual coproduct $(S(\mathfrak{g}))^* \cong \widehat{S(\mathfrak{g}^*)} \xrightarrow{\Delta'} \widehat{S(\mathfrak{g}^*)} \widehat{\otimes} \widehat{S(\mathfrak{g}^*)}$ to the star product (9). The defining property of Δ' is $\langle u_{1'}, f \rangle \langle u_{2'}, g \rangle \equiv \langle \Delta'(u), f \otimes g \rangle = \langle u, f \star g \rangle$ for $u \in S(\mathfrak{g})^* \cong \widehat{S(\mathfrak{g}^*)}$, $f, g \in S(\mathfrak{g})$.

The coorespondence $P \mapsto (f \mapsto P(f)(0))$ is the linear isomorphism from the space of derivations of $S(\mathfrak{g})$ to the space of linear functionals $S(\mathfrak{g})^*$. Evaluating at zero the *n*-th partial derivative is the same as evaluating the product of first partial derivatives except that one has to adjust the factor of *n*! what amounts to a different pairing between the graded components $S^n(\mathfrak{g})$ and $S^n(\mathfrak{g}^*)$ (i.e. a different identification $S^n(\mathfrak{g}^*) \cong S^n(\mathfrak{g})^*$).

19. Lemma. If $\hat{a} = a^{\alpha} \hat{x}_{\alpha}$ and $\hat{f} \in U(\mathfrak{g})$ then

$$\hat{\partial}^{\mu}(\hat{a}^{p}\hat{f}) = \sum_{k=0}^{p-1} \binom{n}{k} a^{\alpha_{1}} a^{\alpha_{2}} \cdots a^{\alpha_{k}} \hat{a}^{p-k} [[[\hat{\partial}^{\mu}, \hat{x}_{\alpha_{1}}], \dots, \hat{x}_{\alpha_{k}}](\hat{f})$$
(10)

Proof. This is a tautology for p = 0. Suppose it holds for all p up to some p_0 , and for all \hat{f} . Then set $\hat{g} = \hat{a}\hat{f} = a^{\alpha}\hat{x}_{\alpha}$. Then $\hat{\partial}^{\mu}(\hat{a}^{p_0+1}\hat{f}) =$ and we can

apply (10) to $\hat{\partial}^{\mu}(\hat{a}^{p_0}\hat{g})$. Now

$$\begin{aligned} [[[\hat{\partial}^{\mu}, \hat{x}_{\alpha_{1}}], \ldots], \hat{x}_{\alpha_{k}}](\hat{g}) &= a^{\alpha_{k}} [[[\hat{\partial}^{\mu}, \hat{x}_{\alpha_{1}}], \ldots], \hat{x}_{\alpha_{k}}](\hat{x}_{\alpha_{k}}\hat{g}) \\ &= \hat{a} [[[\hat{\partial}^{\mu}, \hat{x}_{\alpha_{1}}], \ldots], \hat{x}_{\alpha_{k}}](\hat{f}) + \\ &+ a^{\alpha_{k+1}} [[[[\hat{\partial}^{\mu}, \hat{x}_{\alpha_{1}}], \ldots], \hat{x}_{\alpha_{k}}], \hat{x}_{\alpha_{k+1}}](\hat{f}) \end{aligned}$$

Collecting the terms and the Pascal triangle identity complete the induction step.

20. Given a basis $\hat{x}_1, \ldots, \hat{x}_n$ in a Lie algebra \mathfrak{g} , and structure constants defined by $[\hat{x}_i, \hat{x}_j] = C_{ij}^k \hat{x}_k$, denote by \mathbb{C} the matrix with entries in $A_{n,k}$ whose (i, j)-th entry is

$$\mathbb{C}^i_j = C^i_{jk} \partial^k$$

In [3] we have shown that if $\xi : S(\mathfrak{g}) \to U(\mathfrak{g})$ is the coexponential map then the corresponding ϕ is determined by

$$\boldsymbol{\phi}(-\hat{x}_{\beta})(\partial^{\alpha}) = \phi_{\beta}^{\alpha} = \sum_{N=s}^{\infty} (-1)^{N} \frac{B_{N}}{N!} (\mathbb{C}^{N})_{\beta}^{\alpha}$$

where B_N are the Bernoulli numbers. For the reason of structure of the coexponential map, from now on we will say that this is the case of symmetric ordering. It has the property that $\xi^{-1}(\exp(a^{\alpha}\hat{x}_{\alpha})) = \exp(a^{\alpha}x_{\alpha})$. In fact there is a bit more general fact, which we will show in [10]:

21. (Symmetric case; tensorial form only) Given C_{ij}^k , \mathbb{C} as above let U be any subalgebra of $A_{n,k}[[t]]$ (a priori not necessarily isomorphic to $U(\mathfrak{g})$) generated by n generators X_1, \ldots, X_n which satisfies the following two conditions

(i) the mapping $x_{\alpha_1} \cdots x_{\alpha_k} \mapsto \frac{1}{|\Sigma(k)|!} \sum_{\sigma \in \Sigma(k)} X_{\alpha_{\sigma_1}} \cdots X_{\alpha_{\sigma_k}}$ extends to a onto map $\xi : \mathbf{k}[x_1, \dots, x_n] \to U$ (ii) $X_i = \sum_{N=0}^{\infty} A_N x_{\alpha}(\mathbb{C}^N)_i^{\alpha}$, where $A_N \in \mathbf{k}$ for all N > 0 are arbitrary,

(ii) $X_i = \sum_{N=0}^{\infty} A_N x_{\alpha} (\mathbb{C}^N)_i^{\alpha}$, where $A_N \in \mathbf{k}$ for all N > 0 are arbitrary, $A_0 = 1$ and where the summation over α is understood. We will denote $\phi = \sum_{N=0}^{\infty} A_N \mathbb{C}^N$, hence $X_i = x_{\alpha} \phi_i^{\alpha}$.

Then the following theorem holds

22. Theorem. Let $\theta: U \to \mathbf{k}[x_1, \ldots, x_n]$ be defined as

$$\theta(P) = P(1).$$

where P(1) is evaluated in the sense of the natural action of $A_{n,k}[[t]]$ on $k[x_1, \ldots, x_n][[t]]$. Then $\theta \circ \xi = \text{id.}$ In particular, ξ is then injective, hence by (i) an isomorphism of vector spaces.

23.
$$[\hat{\partial}^{\mu}, \hat{x}_{\alpha}] = \phi^{\mu}_{\alpha}, \ [\hat{\partial}^{\mu}, \hat{x}_{\alpha}](1) = \delta^{\mu}_{\alpha},$$

 $[[\hat{\partial}^{\mu}, \hat{x}_{\alpha}], \hat{x}_{\beta}] = \frac{\partial}{\partial(\partial^{\rho})}(\phi^{\mu}_{\alpha})\phi^{\rho}_{\beta} = \phi^{\mu}_{\alpha,\rho}\phi^{\rho}_{\beta}$

In the case of the symmetric ordering (cf. 20),

$$\phi^{\mu}_{\alpha,\rho}\phi^{\rho}_{\beta}(1) = \frac{1}{2}C^{\mu}_{\alpha\beta}$$

when ξ is the coexponential map (the case from the paper with Durov [3]) and the higher order terms are not so easy to evaluate at 1 in a clean form (involves identities between different tensors in C-s, what should be probably handled with tree calculus and so on).

Given $\phi^{\alpha}_{\beta} \in \widehat{S}(\mathfrak{g}^*)$ as above, denote

$$\phi^{\alpha}_{\beta,\rho_1\rho_2\dots\rho_k} := \frac{\partial}{\partial(\partial_{\rho_k})}\dots\frac{\partial}{\partial(\partial_{\rho_2})}\frac{\partial}{\partial(\partial_{\rho_1})}\phi^{\alpha}_{\beta}$$

and we use the extension of this notation to more complicated expressions, e.g. $(ab)_{,\rho} = a_{,\rho}b + ab_{,\rho}$ is the derivative of the product ab with respect to ∂_{ρ} .

24. Lemma. Let $\hat{x}_1, \ldots, \hat{x}_n$ be a basis of \mathfrak{g} . For any ϕ as above,

The proof is an obvious induction, using the chain rule.

Using the Leibniz rule we can rewrite the formula (11) as a sum of terms for which every derivative operator $\frac{\partial}{\partial(\partial_{\rho})}$ is applied only to a single ϕ -series, rather than to products. Indeed, it is clear that $\frac{\partial}{\partial(\partial_{\rho_1})}$ applies only to $\phi_{\alpha_1}^{\mu}$, then $\frac{\partial}{\partial(\partial_{\rho_2})}$ applies either to $\phi_{\alpha_1}^{\mu}$ or $\phi_{\alpha_2}^{\rho_1}$, and in general, $\frac{\partial}{\partial(\partial_{\rho_s})}$ applies to $\phi_{\alpha_p}^{\rho_{p-1}}$ where $1 \leq p \leq s$ and $\rho_0 := \mu$. This means that we have (k-1)! summands. For example for k = 4 we have 6 summands:

$$\begin{split} & \phi^{\mu}_{\alpha_{1},\rho_{1}} \phi^{\rho_{1}}_{\alpha_{2},\rho_{2}} \phi^{\rho_{2}}_{\alpha_{3},\rho_{3}} \phi^{\rho_{3}}_{\alpha_{4}} + \phi^{\mu}_{\alpha_{1},\rho_{1}} \phi^{\rho_{1}}_{\alpha_{2},\rho_{2}\rho_{3}} \phi^{\rho_{2}}_{\alpha_{3}} \phi^{\rho_{3}}_{\alpha_{4}} + \phi^{\mu}_{\alpha_{1},\rho_{1}\rho_{3}} \phi^{\rho_{1}}_{\alpha_{2},\rho_{2}} \phi^{\rho_{3}}_{\alpha_{4}} \phi^{\rho_{3}}_{\alpha_{4}} \\ & + \phi^{\mu}_{\alpha_{1},\rho_{1}\rho_{2}} \phi^{\rho_{1}}_{\alpha_{2}} \phi^{\rho_{3}}_{\alpha_{3}} + \phi^{\mu}_{\alpha_{1},\rho_{1}\rho_{2}} \phi^{\rho_{1}}_{\alpha_{2},\rho_{3}} \phi^{\rho_{3}}_{\alpha_{3}} \phi^{\rho_{3}}_{\alpha_{4}} + \phi^{\mu}_{\alpha_{1},\rho_{1}\rho_{2}\rho_{3}} \phi^{\rho_{1}}_{\alpha_{3}} \phi^{\rho_{1}}_{\alpha_{3}} \phi^{\rho_{1}}_{\alpha_{2}} \phi^{\rho_{3}}_{\alpha_{4}} \\ \end{split}$$

I will call this expansion "expansion 1".

25. We now specialize to the case of the series corresponding to the symmetric ordering

$$\phi^{\alpha}_{\beta,\rho_1,\ldots,\rho_s} = \sum_{N=s}^{\infty} (-1)^N \frac{B_N}{N!} (\mathbb{C}^N)^{\alpha}_{\beta,\rho_1\ldots,\rho_s}$$

The sum over $N \geq k$ for each ϕ in the form of expansion 1, will be called expansion 2. By applying the Leibniz rule again, we notice that $(\mathbb{C}^N)_{,\rho_1\dots\rho_s}$ is a sum of N!/(N-s)! summands, each of which is monomial which is a product of $N - s \mathbb{C}$ -s and s C-s. This is the expansion 3. Performing consequently expansions 1,2 and 3, the commutator in (11) becomes a multiple sum of terms which are labelled by certain class of attributed planar trees and each summand is certain contraction of several \mathbb{C} -tensors and several C-tensors with k + 1 external indices $\mu, \alpha_1, \dots, \alpha_k$, and with some pre-factor involving (products of) Bernoulli numbers and factorials. To describe the details, we introduce several "classes" of planar rooted trees and their "semantics".

26. Class \mathcal{T} consists of all planar rooted trees with two kinds of nodes, white and black, where black nodes may only be leaves.

We will draw the trees in \mathcal{T} with the root on the top. 'Planar' implies that the (left to right) order of child branches of every node matters. If $t \in \mathcal{T}$, then $w(t) \geq 0$ and $b(t) \geq 0$ are the number of white and black nodes in t respectively. Class \mathcal{T} is graded in obvious way $\mathcal{T} = \coprod_{P=1}^{\infty} \mathcal{T}_P$ by the total number of nodes P, and bigraded by the numbers b and w of black and white nodes: $\mathcal{T} = \coprod_{w+b>0} \mathcal{T}_{w,b}$. Clearly $\mathcal{T}_P = \coprod_{w+b=P} \mathcal{T}_{w,b}$.

Class \mathcal{T}^{ord} consists of pairs (t, l) where $t \in \mathcal{T}$ and l is a numeration (with values $1, \ldots, w$) on the set of white nodes of t which is descending in the sense that white children nodes are always assigned greater values than their parent nodes. Let \mathcal{T}_P^{ord} and $\mathcal{T}_{w,b}^{ord}$ be the sets of all pairs $(t, l) \in T^{ord}$ such that $t \in \mathcal{T}_P$ and $t \in \mathcal{T}_{w,b}$ respectively. Given $s \in \mathcal{T}^{ord}$ and $t \in \mathcal{T}$ we say $s \in t$ if s = (t, l) for some numeration l. This means that we identify t with the set of all pairs of the form (t, l).

27. (Example: counting trees in \mathcal{T}^{ord}) Let s_w be the cardinality of $\mathcal{T}_{w,0}^{ord}$, that is the number of distinct numerated planar rooted trees with descending numeration and only white nodes. We suggest reader to check that $s_1 = s_2 = 1$, $s_3 = 3$ an $s_5 = 15$. It is easy to derive a recursion for s_w . The trees in $\mathcal{T}_{w+1,0}^{ord}$ have a root node with at most w numerated branches which are themselves planar rooted trees with labels. The exact labelling

is determined by first choosing the set of labels of each branch, and then choosing a descending numeration on the labels within each branch. For the whole process w labels are available, regarding that the root branch is mandatory labelled with 1. Thus we obtain the recursion

$$s_{w+1} = \sum_{k=1}^{w} \sum_{w_1+w_2+\ldots+w_k=w} \frac{w!}{w_1!w_2!\cdots w_k!} s_{w_1}s_{w_2}\cdots s_{w_k}, \quad w \ge 1.$$

The solution of this recursion is $s_w = (2w - 3)!! = 1 \cdot 3 \cdot 5 \cdots (2w - 3)$. Cardinality of $\mathcal{T}_{b,w}^{ord}$ may be determined similarly: for $w \ge 0$,

$$s_{w+1,b} = \sum_{k=1}^{w} \sum_{\substack{w_1 + \ldots + w_k = w \\ b_1 + \ldots + b_k = b}} \frac{w!}{w_1! w_2! \cdots w_k!} s_{w_1,b_1} s_{w_2,b_2} \cdots s_{w_k,b_k}.$$

28. Suppose now \mathfrak{g} and its basis $\hat{x}_1, \ldots, \hat{x}_n$ are fixed; and hence the dual basis $\partial^1, \ldots, \partial^n$ and the structure constants C_{jk}^i . Given $t \in \mathcal{T}_{w,b}^{ord}$ and labels $1 \leq \mu, \alpha_1, \ldots, \alpha_w \leq n$, we define $\operatorname{ev}(t)_{\alpha_1,\ldots,\alpha_w}^{\mu} \in S(\mathfrak{g}^*)$ as follows. We first replace the numeration labels $1, \ldots, w$ on white nodes with $\alpha_1, \ldots, \alpha_w$. Then label arbitrarily the inner lines by distinct new variables $\rho_1, \ldots, \rho_{w+b-1}$, and attach a new external incoming line to the root node and label it with label μ . To form an expression $\operatorname{ev}(t)_{\alpha_1,\ldots,\alpha_w}^{\mu}$ apply the Feynman-like rules: to each white node with label α_k , incoming node ρ_l and outgoing nodes $\rho_{v_1}, \ldots, \rho_{v_s}$ assign value $(-1)^s \frac{B_s}{s!} C_{\alpha_k \rho_{v_1}}^* C_{*\rho_{v_2}}^* \cdots C_{*\rho_{v_s}}^{\rho_l}$, where * replaces the intermediate labels of summation which agree pairwise: superscript * of a preceding C with the first lower subscript of subsequent C. If s = 0, i.e. the white node is a leaf, the value is the Kronecker delta $\delta_{\alpha}^{\rho_l}$. To each black leaf assign $\partial^{\rho_l} \in \mathfrak{g}^* \subset S(\mathfrak{g}^*)$. Multiply so assigned values of all nodes and sum over labels of internal lines. Example:

$$\frac{B_2}{2!} (C^*_{\alpha_1 \rho_1} \partial^{\rho_1}) C^{\mu}_{* \rho_2} \delta^{\rho_2}_{\alpha_2} = \frac{1}{12} \mathbb{C}^*_{\alpha_1} C^{\mu}_{* \alpha_2} \qquad (12)$$

Clearly $\operatorname{ev}(t)_{\alpha_1,\ldots,\alpha_w}^{\mu}$ are components of some tensor which will be of course denoted $\operatorname{ev}(t) \in \mathfrak{g} \otimes T^n(\mathfrak{g}^*)$. In this notation,

$$\left[\dots\left[\left[\hat{\partial}^{\mu}, \hat{x}_{\alpha_{1}}\right], \hat{x}_{\alpha_{2}}\right], \dots, \hat{x}_{\alpha_{w}}\right] = \sum_{b=0}^{\infty} \sum_{t \in \mathcal{T}_{w,b}^{ord}} \operatorname{ev}(t)_{\alpha_{1},\dots,\alpha_{w}}^{\mu}$$
(13)

29. For a tree $t \in \mathcal{T}_{w,b}^{ord}$ one defines its full evaluation

$$\operatorname{fev}(t)^{\mu} := \frac{1}{w!} \partial^{\alpha_1} \cdots \partial^{\alpha_w} \otimes \operatorname{ev}(t)^{\mu}_{\alpha_1, \dots, \alpha_w},$$

and for $s \in \mathcal{T}$ one defines

$$fev(s)^{\mu} := \sum_{t \in s, t \in \mathcal{T}^{ord}} fev(t)^{\mu}.$$

30. (Basic selection rule) Suppose a tree $t \in \mathcal{T}$ has at least one white node y such that its most left child branch is a white leaf. Then for all μ ,

$$fev(t)^{\mu} = 0.$$

Proof. Once the Feynman rules are applied the fact is rather obvious. Namely, suppose that white node has s child branches, its label is k and of its most left child branch is l (then l > k). Then the Feynman rules for $ev(t)^{\mu}_{\alpha_1,\dots,\alpha_k,\dots,\alpha_l,\dots,\alpha_w}$ assign to the white node y the factor $(-1)^s \frac{B_s}{s!} C^*_{\alpha_1\rho_1} C^*_{*\rho_2} \cdots C^{\rho_0}_{*\rho_s}$ if the incoming line to y is labelled by ρ_0 and outgoing from left to right by ρ_1,\dots,ρ_s . The white leaf contributes by a factor $\delta^{\rho_1}_{\alpha_2}$. Thus we get a factor of the type $C^*_{\alpha_k\rho_1}\delta^{\rho_1}_{\alpha_l} = C^*_{\alpha_k\alpha_l}$ which is antisymmetric in lower indices. To obtain fev $(t)^{\mu}$ contract $1 \otimes ev(t)^{\mu}_{\alpha_1,\dots,\alpha_k,\dots,\alpha_l,\dots,\alpha_w}$ with the symmetric tensor $\frac{1}{w!}\partial^{\alpha_1}\cdots\cdots\partial^{\alpha_w}\otimes 1$ what vanishes by symmetry reasons. Q.E.D.

Notice that this selection rule holds for fev but not for ev (the latter does not involve symmetrization). The subset of trees which are not excluded in calculation of fev by the basic selection rules are called (fev)-**contributing trees** and the correspoding subclasses are distinguished with superscript c, e.g. $\mathcal{T}_{w,b}^c \subset \mathcal{T}_{w,b}$.

By similar symmetry reasons, the following result holds:

31. Lemma. Let $\hat{x}_1, \ldots, \hat{x}_n$ be a basis of \mathfrak{g} . If $\xi : S(\mathfrak{g}) \to U(\mathfrak{g})$ is the coexponential map, then for $w \geq 2$,

$$\sum_{\sigma \in \Sigma(w)} [\dots [[\hat{\partial}^{\mu}, \hat{x}_{\sigma\alpha_1}], \hat{x}_{\sigma\alpha_2}], \dots, \hat{x}_{\sigma\alpha_w}](1) = 0,$$

where on the left hand side the evaluation at unit element ("vacuum") is in the sense of the action of the Weyl algebra on the usual symmetric algebra $S(\mathfrak{g})$. The evaluation at vacuum simply kills all the strictly positive powers of ∂ -s, hence only the terms coming from trees in $\mathcal{T}_{w,0}^{ord}$ survive. Thus the lemma may be restated as

$$\sum_{\sigma \in \Sigma(k)} \sum_{t \in \mathcal{T}_{w,0}^{ord}} \operatorname{ev}(t)^{\mu}_{\sigma \alpha_1 \cdots \sigma \alpha_w} = 0.$$

The proof in the latter form is obvious: applying the Feynman rules to a graph with w nodes and w - 1 internal lines produces a tensor which is proportional to some contracted product of w - 1 copies of the structure constants tensor C, w - 1 contractions, w lower external labels and one upper external label μ . In particular at least one pair of labels α_i, α_j will be attached as lower labels of the same C-tensor. By the antisymmetry in subscripts of C, after symmetrization of $\alpha_1, \ldots, \alpha_w$ we obtain zero.

32. Corollary. In the symmetric ordering (if ξ is the coexponential map), the formula for the derivatives of $(\hat{a})^p = (a^\beta \hat{x}_\beta)^p$ is of the classical (undeformed) shape, i.e.

$$\frac{1}{s!}\hat{\partial}^{\alpha_1}\hat{\partial}^{\alpha_2}\cdots\hat{\partial}^{\alpha_s}(\hat{a}^p) = \binom{p}{s}a^{\alpha_1}a^{\alpha_2}\dots a^{\alpha_s}\hat{a}^{p-s}, \quad p \ge s.$$

This follows by an induction on k; the induction step involves applying the case k = 1. For k = 1, the formula follows from (10) for $\hat{f} = 1$ after noticing that $a^{\alpha_1}a^{\alpha_2}\cdots a^{\alpha_k}$ in (10) is symmetric under permutations of α_1,\ldots,α_k , hence by **31** the only term which survives is the top degree term which is of classical shape.

33. Up to the fourth order in total derivative, or equivalently, third order in *C*-s one gets the following

$$\begin{aligned} \Delta \hat{\partial}^{\mu} &= 1 \otimes \hat{\partial}^{\mu} + \hat{\partial}^{\mu} \otimes 1 + \frac{1}{2} C^{\mu}_{\alpha\beta} \hat{\partial}^{\alpha} \otimes \hat{\partial}^{\beta} + \frac{1}{12} C^{\star}_{\alpha\beta} C^{\mu}_{\star\gamma} (\hat{\partial}^{\alpha} \otimes \hat{\partial}^{\beta} \hat{\partial}^{\gamma} + \hat{\partial}^{\beta} \hat{\partial}^{\gamma} \otimes \hat{\partial}^{\alpha}) \\ &- \frac{1}{24} C^{*}_{\alpha\beta} C^{*}_{\star\gamma} C^{\mu}_{\star\delta} \hat{\partial}^{\alpha} \hat{\partial}^{\gamma} \otimes \hat{\partial}^{\beta} \hat{\partial}^{\delta} + O(C^{4}) \end{aligned}$$

where we sum on pairs of repeated indices (including *, where on two consecutive ones).

34. Theorem. If ξ is the coexponential map, the coproduct is given by

$$\Delta \hat{\partial}^{\mu} = 1 \otimes \hat{\partial}^{\mu} + \hat{\partial}^{\alpha} \otimes [\hat{\partial}^{\mu}, \hat{x}_{\alpha}] + \frac{1}{2} \hat{\partial}^{\alpha} \hat{\partial}^{\beta} \otimes [[\hat{\partial}^{\mu}, \hat{x}_{\alpha}], \hat{x}_{\beta}] + \dots$$

or, in symbolic form,

$$\Delta \hat{\partial}^{\mu} = \exp(\hat{\partial}^{\alpha} \otimes \operatorname{ad} (-\hat{x}_{\alpha}))(1 \otimes \hat{\partial}^{\mu})$$
(14)

and in the tree expansion form, using the notation from 29,

$$\Delta \hat{\partial}^{\mu} = \sum_{t \in \mathcal{T}^{ord}} \text{fev}(t)^{\mu}.$$
 (15)

Of course, each $\operatorname{ad}(-\hat{x}_{\alpha})$ in (14) has to be applied to $\hat{\partial}^{\mu}$ before applying the whole expression on the elements in $S(\mathfrak{g}) \otimes S(\mathfrak{g})$ (for the Leibniz rule for the star product) or on the elements in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ (for the Leibniz rule for the usual noncommutative product).

Proof. It is well known that the expressions of the form $(\hat{a})^p$ where $\hat{a} = \sum_{\alpha} a^{\alpha} \hat{x}_{\alpha}$ with varying $a = (a^{\alpha})$ span $U(\mathfrak{g})$. Thus it is sufficient to show that for all a, all $\hat{f} \in U(\mathfrak{g})$ and all p the twisted Leibniz rule

$$\hat{\partial}^{\mu}(\hat{a}^{p}\hat{f}) = \sum_{w=0}^{p} \frac{1}{w!} \sum_{\alpha_{1},\dots,\alpha_{w}} \hat{\partial}^{\alpha_{1}} \cdots \hat{\partial}^{\alpha_{w}}(\hat{a}^{p}) [[\dots [\hat{\partial}^{\mu}, \hat{x}_{\alpha_{1}}], \dots], \hat{x}_{\alpha_{w}}](\hat{f}).$$

holds. This follows by comparing the Corollary 32 which holds for symmetric ordering only with the formula (10) which holds for general ordering.

35. Let $\partial^{abc} = \partial^a \partial^b \partial^c$ and so on. Recall $\phi^{\mu}_{\nu} = \phi^{\mu}_{\nu}(\partial) = [\hat{\partial}^{\mu}, \hat{x}_{\nu}]$. **Corollary.** In symmetric ordering, for any \hat{f}, \hat{g} in $U(\mathfrak{g})$,

$$\phi^{\mu}_{\nu}(\partial)(\hat{f}\hat{g}) = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{k=1}^{N} \partial^{i_1 \cdots i_{k-1} i_{k+1} \cdots i_N} \phi^{i_k}_{\nu}(\partial)(\hat{f}) \cdot [[\dots [\partial^{\mu}, \hat{x}_{i_1}], \dots, \hat{x}_{i_N}](\hat{g})$$

Notice that the last sum is from 1, not 0. Summation over repeated indices understood. This formula is equivalent to giving the formula deformed coproduct for the argument $\Delta([\hat{\partial}^{\mu}, \hat{x}_{\nu}]) = \Delta(\phi^{\mu}_{\nu})$. For the proof, calculate $\hat{\partial}^{\mu}((\hat{x}_{\nu}\hat{f})\hat{g})$ using the twisted Leibniz rule from the theorem **34**, and subtract similarly $\hat{x}_{\nu}\hat{\partial}^{\mu}(\hat{f}\hat{g})$ and group the terms and commutators appropriately.

36. Let $\tau : S(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*) \to S(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$ be the standard flip interchanging the tensor factors (in the completed tensor product).

Theorem. Let $s_{1,p}$ be the unique tree in $\mathcal{T}_{1,p}^{\text{ord}}$. Then for all μ ,

$$\tau(\text{fev}(s_{1,p})^{\mu}) = (-1)^{p+1} \sum_{(t,l) \in \mathcal{T}_{p,1}^{ord}} \text{fev}(t)^{\mu}$$
(16)

or more explicitly

$$\operatorname{ev}(s_{1,p})^{\mu}_{\beta} \otimes \partial^{\beta} = (-1)^{p+1} \sum_{t \in \mathcal{T}_{p,1}^{\operatorname{ord}}} \frac{1}{p!} \partial^{\alpha_{1}} \cdots \partial^{\alpha_{p}} \otimes \operatorname{ev}(t)^{\mu}_{\alpha_{1},\dots,\alpha_{p}}$$
(17)



The diagrams above show $s_{1,4}$ and the 8 diagrams $t_1, \ldots, t_8 \in \mathcal{T}_{4,1}^c$. *Proof.* For p = 1 the assertion is a tautology. Let us prove the assertion for p > 1.

By the Feynman rules, the LHS of (17) equals

$$(-1)^{p}\frac{B_{p}}{p!}C^{*}_{\beta\rho_{1}}C^{*}_{*\rho_{2}}\cdots C^{\mu}_{*\rho_{p}}\partial^{\rho_{1}}\partial^{\rho_{2}}\cdots \partial^{\rho_{p}}\otimes \partial^{\beta} = (-1)^{p}\frac{B_{p}}{p!}(\mathbb{C}^{p})^{\mu}_{\beta}\otimes \partial^{\beta}.$$

Therefore it is sufficient and we will show by induction that

$$\sum_{t \in \mathcal{T}_{p,1}^{ord}} \frac{1}{p!} \partial^{\alpha_1} \cdots \partial^{\alpha_p} \otimes \operatorname{ev}(t)_{\alpha_1,\dots,\alpha_p}^{\mu} = (-1)^{p+1} \frac{B_p}{p!} (\mathbb{C}^p)_{\beta}^{\mu} \otimes \partial^{\beta}.$$

Bournulli numbers and hence this expression are zero for odd p > 1 and nonzero for even p > 1.

By the basic selection rule **30**, the only trees $t \in \mathcal{T}_{p,1}$ (no labelling) which may give a nonzero contribution are those who have no leftmost white leafs, and regarding that there is only one black node in our case, only one white node may have a leftmost leaf (which is black). That means that every contributing tree in $\mathcal{T}_{p,1}^c$ is composed as follows: start with a vertical chain made of $r + 1 \leq p$ white nodes ending with a black node on the bottom and on this white chain there are attached $(p - r - 1) \geq 0$ right-hand side leafs (to some among the white nodes of the vertical chain), but no branches of length ≥ 2 are attached.

Notice that each $t \in \mathcal{T}_{p,1}^c$ for p > 1 may be also composed alternatively starting with the top white node, attaching the left-most branch $t' \in \mathcal{T}_{r,1}^c$ and p - r - 1 leafs, $r \ge 0$. We group the trees by the number $0 \le r < p$. Let us now consider the ordered trees $t \in \mathcal{T}_{p,1}^{c,\text{ord}}$. To the top node we must assign label 1, then we may choose any r remaining numbers β_1, \ldots, β_r to distribute them within t' branch according to the usual ordering rules within t' and distribute the remaining p - r - 1 labels $\gamma_1, \ldots, \gamma_{p-r-1}$ to the white leafs in any order. Other way around, given t with labels, if t' as a branch of t, then its labels are renumerated as 1 to r in the same order. For example, labels 2, 5, 7, 8, 3 of white nodes in t' as a branch will be replaced by the position labels 1, 3, 4, 5, 2 in t' as an independent tree. Thus for a given ordering

$$\operatorname{ev}_{1,\dots,r}^{\mu}(t) = (-1)^{p} \frac{B_{p}}{p!} \sum_{\rho} C_{1,\rho}^{*} C_{*\gamma_{1}}^{*} \cdots C_{*\gamma_{p-r-1}}^{\mu} \operatorname{ev}_{\beta_{1},\dots,\beta_{r}}^{\rho}(t')$$

(of course each *i* has to be replaced by α_i). Now we need to count all ordering and combine into fev. The ordering constraints described above give some

combinatorial factors, as well as 1/n! in the definition of fev. We obtain

$$\sum_{t \in \mathcal{T}_{1,p}^{c}} \operatorname{fev}(t)^{\mu} = \frac{1}{p!} \sum_{r} (-1)^{p-r+1} \frac{B_{p-r}}{(p-r)!} {p-1 \choose r} r! \operatorname{fev}(t')^{\rho} (p-r-1)! ((\mathbb{C}^{p-r})^{\mu}_{\rho} \otimes 1).$$

Notice here an additional sign from the first C-factor (by antisymmetry of lower indices): $C^*_{\alpha_1\rho}\partial^{\alpha_1} = -\mathbb{C}^*_{\rho}$.

By the induction hypothesis, $\text{fev}(t')^{\rho} = (-1)^r \frac{B_r}{r!} (\mathbb{C}^r)^{\rho}_{\beta} \otimes \partial^{\beta}$, hence,

$$\sum_{t\in\mathcal{T}_{1,p}^c} \text{fev}(t)^{\mu} = \frac{1}{p}(-1)^p \sum_r \frac{B_{p-r}}{(p-r)!} \frac{B_r}{r!} ((\mathbb{C}^p)^{\mu}_{\beta} \otimes \partial^{\beta})$$

Regarding that, for p > 1, B_{p-r} and B_r on the right are simultaneously nonzero if and only if r and p-r are both even, the proof finishes by applying the well known identity for Bernoulli numbers

$$\sum_{s=1}^{l} \frac{B_{2s}}{(2s)!} \frac{B_{2l-2s}}{(2l-2s)!} = \frac{-B_{2l}}{(2l-1)!} + \frac{1}{4}\delta_{l,1}, \qquad l > 0.$$

37. In these terms we state the following conjecture on the star product

In our notation we will often not distinguish any more ∂ from ∂ ; with the convention that when we write $[\partial, \hat{x}]$ where $\hat{x} \in U(\mathfrak{g})$ we mean $\hat{\partial}$; as well as when we apply $\hat{\partial}(\hat{f})$ with $\hat{f} \in U(\mathfrak{g})$; however when we apply $\partial(f)$ with $f \in S(\mathfrak{g})$ we mean the usual (undeformed) Fock representation. In any case Δ is deformed and Δ_0 undeformed coproduct: $\Delta_0(\partial^\mu) = 1 \otimes \partial^\mu + \partial^\mu \otimes 1$.

38. Conjecture.

$$f \star g = \sum_{i_1, i_2, \dots, i_n \ge 0} \frac{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}}{i_1! \cdots i_n!} m\left(\left(\prod_{l=1}^n (\Delta - \Delta_0) ((\partial^l)^{i_l}) \right) (f \otimes g) \right), \quad (18)$$

where $f, g \in S(\mathfrak{g})$ and m is the commutative multiplication of polynomials $S(\mathfrak{g}) \otimes S(\mathfrak{g}) \to S(\mathfrak{g})$. Notice that for any concrete f and g, the summation on the right has only finitely many nonzero terms. This formula is proved in some special cases ([8]). In general, if f is a first order monomial and g arbitrary, this formula boils down to our main formula of article [3].

Formula (18) can be expressed via normal ordered exponential :exp(): (here x-s to the left, ∂ -s to the right)

$$f \star g = m : \exp(x_{\alpha}(\Delta - \Delta_0)(\partial^{\alpha})) : (f \otimes g)$$

and m is the usual product.

39. In articles [8, 6] for a particular Lie algebra, the case of "kappadeformed Eucledean space" the conjecture has been verified for general ϕ .

40. Theorem. For symmetric ordering the conjecture holds for all g.

In fact we can prove it more general, for those ϕ which are obtained using certain procedure of twisting basis by a wide class inner automorphisms.

41. (The recursive form of Campbell-Hausdorff series.) Given $X, Y \in \mathfrak{g}$ where \mathfrak{g} is finite-dimensional with a norm inducing the standard topology. The series H(X, Y) is uniquely defined by

$$\exp(X)\exp(Y) = \exp(H(X,Y))$$

and it converges in such norm. Then $H(X,Y) = \sum_{N=0}^{\infty} H_N(X,Y)$ where "Dynkin's Lie polynomials" $H_N = H_N(X,Y)$ are defined recursively by $H_1 = X + Y$ and

$$(N+1)H_{N+1} = \frac{1}{2}[X-Y,H_N] + \sum_{r=0}^{\lfloor N/2-1 \rfloor} \frac{B_{2r}}{(2r)!} \sum_{s} [H_{s_1},[H_{s_2},[\dots,[H_{s_{2r}},X+Y]\dots]]$$

where the sum over s is the sum over all 2r-tuples $s = (s_1, \ldots, s_{2r})$ of strictly positive integers whose sum $s_1 + \ldots + s_{2r} = N$. This identity is well-known and we do not reprove it here.

42. (Linear parts in either X or Y) The linear part in X of the Hausdorff series is $H_{1,\star}(X,Y) = \sum_{N=0}^{\infty} (-1)^N \frac{B_N}{N!} [Y, [\dots, [Y,X]]]$ where N is the degree of Y in the Lie polynomial involved. Similarly, the linear part in Y is $H_{\star,1}(X,Y) = \sum_{N=0}^{\infty} \frac{B_N}{N!} [X, [\dots, [X,Y]]]$ where N is the degree of Y in the Lie polynomial involved.

43. (Symmetries of Hausdorff series) Identity $e^X e^Y = (e^{-Y}e^{-X})^{-1}$ implies H(-Y, -X) = -H(X, Y). Dynkin's polynomials are of fixed total degree, hence the change $(X, Y) \mapsto (-Y, -X)$ does not mix them and $H_P(-Y, -X) = -H_P(X, Y)$ for all P > 0. We refine the degree grading on a free Lie algebra on two generators by a bigrading which induces a decomposition $H_P(X, Y) = \sum_{w+b=P} H_{w,b}(X, Y)$ where $H_{w,b}$ is the sum of all Lie polynomials in $H_P(X, Y)$ of degree w in X and degree b in Y. Clearly, knowing H_P determines $H_{w,b}$ for all w, b with w + b = P.

44. Proposition. The following w-recursion and b-recursion hold

$$(w+1)H_{w+1,b} = \frac{1}{2}[X, H_{w,b}] + \sum_{r=0}^{\lfloor w/2 - 1 \rfloor} \frac{B_{2r}}{(2r)!} \sum_{w_i, b_i} [H_{w_1, b_1}, [\dots, [H_{w_{2r}, b_{2r}}, X] \dots]]$$

$$bH_{w+1,b} = -\frac{1}{2}[Y, H_{w,b}] + \sum_{r=0}^{\lfloor b/2 - 1 \rfloor} \frac{B_{2r}}{(2r)!} \sum_{w_i, b_i} [H_{w_1, b_1}, [\dots, [H_{w_{2r}, b_{2r}}, Y] \dots]]$$

where in the sum on the RHS $\sum_i w_i = w$ and $\sum_i b_i = b$ for the w-recursion and $\sum_i w_i = w + 1$ and $\sum_i b_i = b - 1$ for the b-recursion.

Proof. For the purpose of the proof we introduce two new sets of Lie polynomials. The first set will have members $H_{w,b}^W$ and the latter $H_{w,b}^B$ where $w \ge 0, b \ge 0, w + b > 0$. For w = 0 we set $H_{w,b}^W = H(w,b)$ what is 0 unless b = 1 when $H_{0,1}^W = X$; similarly for b = 0 we set $H^B(w,b) = H(w,b)$. Also set $H_{1,0}^W = Y$ and $H_{0,1}^B = X$, regarding that (0,0) point is undefined. By definition, w-recursion is used to define $H_{w,b}^{W}$ at all other pairs (w,b)and similarly the *b*-recursion is used to define $H^B_{w,b}$. E.g. for *w*-recursion we first use the recursion at the line b = 0, increasing from w = 1 on, then at the line b = 1, increasing from w = 1, and so on. Clearly each recursion relation is used exactly once to determine one new value and all instances of relations are used. Notice that on the line b = 0, the w + 1 =w + b + 1 = P + 1, hence the w-recursion gives the same values on this line as the standard recursion for $H_{w,b}$. In that manner we notice that the initial values (line b = 0 and (0, 1)) given to H^B agree with the value of H^W and H obtained by w-recursion and the standard recursion. The initial values hence also satisfy the symmetries $H_{w,b}(X,Y) = -H_{b,w}(-Y,-X)$ in both cases. We want to prove that the values within the quadrant agree as well, not only the conditions on the boundary. But, the *b*-recursion may be obtained from *w*-recursion also by the same symmetry operation! Regarding that the symmetry holds for initial values and also for the recursion, than this is true for each pair of new points to which the two recursions assign the values. Conclusion: $H^B = H^W$. Therefore we can now safely combine two recursions without being afraid of nonconsistency. But adding up the w-recursion and b-recursion we clearly get the standard recursion. Regarding that the initial value w + b = P = 1 for standard recursion is checked and that the standard recursion is the consequence, and also that the values H_P determine $H_{w,b}$, we conclude $H = H^B = H^W$.

45. (Recursive formula for D = D(k,q)) Let $\hat{x}_1, \ldots, \hat{x}_n$ be a basis of \mathfrak{g} , $i = \sqrt{-1}, X = ik^a \hat{x}_a, Y = iq^a \hat{x}_a$ and $H(X,Y) = iD^a(k,q)\hat{x}_a$, where $k = (k^1, \ldots, k^n), q = (q^1, \ldots, q^n)$; let also $D = D(k,q) = (D^1(k,q), \ldots, D^n(k,q))$. Then $D^{\mu}(k,q) = \sum_{N=0}^{\infty} D^{\mu}_n(k,q)$ where $D^{\mu}_1(k,q) = k^{\mu} + q^{\mu}$ and the recursion

$$(N+1)D_{N+1}^{\mu} = \frac{1}{2}(k^{a} - q^{a})(E_{N})_{a}^{\mu} + \sum_{r=1}^{\lfloor N/2 - 1 \rfloor} \frac{B_{2r}}{(2r)!} \sum_{s} (k^{a} + q^{a})(E_{s_{1}} \cdots E_{s_{2r}})_{a}^{\mu}$$
(19)

holds where

$$(E_P)^{\mu}_{\nu} := \sum_{\sigma} i C^{\mu}_{\nu\sigma} D^{\sigma}_P, \quad P \ge 1,$$

are the components of a matrix E_P , and the product of matrices on the right is via the convention that the superscript is the *row* index. The sum over *a* on the right is understood and the sum over *s* is again over 2r-tuples of positive integers adding up to *N*.

46. For the coexponential map ξ , the equality $\xi(\exp(ik^a x_a)) = \exp(ik^a \hat{x}_a)$ holds. Therefore the star product $f \star g = \xi^{-1}(\xi(f) \cdot \xi(g))$ reduces to calculations with Hausdorff series. Namely if $f(x) = \exp(ik^a x_a)$, $g(x) = \exp(iq^a x_a)$, then $(f \star g)(x) = \exp(iD^a(k, q)x_a)$. For general f and g, it is convenient to expand f and g in Fourier components (reasoning understood in the sense of 8) $f(x) = \int \frac{d^n k}{(2\pi)^n} (Ff)(k) \exp(ik^a x_a)$ and, by bilinearity, we obtain

$$(f \star g)(x) = \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} (Ff)(k)(Fg)(q) \exp(iD^a(k,q)x),$$

or alternatively,

 $(f \star g)(x) = m \exp(iz_a(D^a(-i\partial \otimes 1, -i \otimes \partial) + i\partial^a \otimes 1 + i \otimes \partial^a))(f \otimes g)(x)|_{z_a = x_a}$ where $\partial = (\partial^1, \dots, \partial^n)$. Now notice that $D_1^a(-i\partial \otimes 1, -i \otimes \partial) = -i\partial^a \otimes 1 - i \otimes \partial^a$, hence

$$(f \star g)(x) = m \exp(i z_a (D^a - D_1^a)(-i\partial \otimes 1, -i \otimes \partial))(f \otimes g)(x)|_{z_a = x_a}$$

Notice that $iD_1^a(-i\partial \otimes 1, -i \otimes \partial) = \Delta_0(\partial^a)$. In fact, using the filtration by the total degree, we see that the previous theorem is equivalent to

47. Theorem. Let $\Delta_P(\partial^a)$ be the summand in $\Delta(\partial^a)$ consisting of terms of total homogeneity $P \ge 1$. Then for every $P \ge 1$,

$$iD_P^a(-i\partial \otimes 1, -i \otimes \partial) = \Delta_P(\partial^a)$$

The theorem will be proved by induction on P. In other words, we have to prove the corresponding recursion for Δ_P . We use two tools: Fourier transform (this is only heuristic term here, strictly speaking we use the denseness of the linear span of all exponential series $\exp(a^{\alpha}x_{\alpha})$ in $\widehat{S(\mathfrak{g})}$ and do not require the existence of the imaginary unit, cf. **12**) and the combinatorics of the trees whose Feynman rule contribution is involved here. Every degree in homogeneity corresponds to a node. The new node can always be assumed the top node, and, in particular white. This gives in the same way as the counting of trees in **27**), but with weights, the w-recursion formula (in Fourier transformed form) which agrees with the w-recursion formula for Hausdorff series. The initial conditions for w-recursion are the same as for the w-recursion of the Hausdorff series, as calculated in Theorem **36**. Therefore the equality.

48. The classical case of Moyal noncommutative space, where the deformation is given by an antisymmetric matrix $\theta_{\mu,\nu}$ and the commutation relations are given by $[x_{\mu}, x_{\nu}] = \theta_{\mu,\nu}$ can be treated as special case of this framework by multiplying $\theta_{\mu,\nu}$ by a central element c. Then one calculates the star product and obtains the classical formula, after setting back c to 1. In the classical case also one has the formula $f \star g = mF(f \otimes g)$ where F is a Drinfeld twist. For this to make sense in general we need to have some Hopf structure, Our "normally ordered exponential" formula for the star product should be rewritten by means of some element in $H \otimes H$ where H is some (completed) Hopf algebra containing $S(\mathfrak{g}^*)$, but we also need some combinations involving x_i -s as seen from the formula. Hence H is strictly bigger than $S(\mathfrak{g}^*)$ and smaller than the full Weyl algebra (the latter has no Hopf structure extending the one on $S(\mathfrak{g}^*)$). Twisted Leibniz rules for some operators involving both x_i -s and ∂^i -s are studied in the literature and maybe a good step toward understanding which extension of $S(\mathfrak{g}^*)$ could be the place where F lives (e.g. for "kappa-deformed space" the answer is known).

49. By the Hausdorff formula, using the notation from (8),

$$\begin{split} \xi(\exp(ikx))\xi(\exp(iqx)) &= & \exp(iK(k)\hat{x})\exp(iK(q)\hat{x}) \\ &= & \exp(iD(K(k),K(q))\hat{x}) \\ &= & \xi(\exp(iK^{-1}(D(K(k),K(q)))x)) \end{split}$$

where we wrote the contractions with surpressed indices. If we denote

$$D_{\phi}(k,q) := K^{-1}(D(K(k), K(q))), \quad K = K_{\phi}$$

then we write this as $\xi(\exp(ikx))\xi(\exp(iqx)) = \xi(\exp(iD_{\phi}(k,q)x))$ or equivalently

$$\exp(ikx) \star_{\phi} \exp(iqx) = \exp(iD_{\phi}(k,q)x)$$

In physics papers (e.g. [7, 8]) $\xi(\exp(ikx))$ is usually written as ϕ -ordered exponential : $\exp(ik\hat{x}) :_{\phi}$. Similar expressions one can write for the deformed coproducts (in Fourier harmonics picture).

$$iD^{\mu}_{\phi}(k,q) \exp(iD_{\phi}(k,q)x) = \partial^{\mu}(\exp(iD_{\phi}(k,q)x))$$

= $\partial^{\mu}(\exp(ikx) \star_{\phi} \exp(iqx))$
= $m_{\phi}(\Delta_{\phi}(\partial^{\mu})(\exp(ikx) \otimes \exp(iqx)))$
= $\Delta^{\mu}_{\phi}(ik,iq)(\exp(ikx) \star_{\phi} \exp(iqx))$
= $\Delta^{\mu}_{\phi}(ik,iq) \exp(iD_{\phi}(k,q)x)$

where $\Delta^{\mu}_{\phi}(ik, iq)$ is obtained from $\Delta_{\phi}(\partial_{\mu})$ by substituting $\partial^{\alpha} \mapsto k^{\alpha}$ or q^{α} depending on the tensor factor and multiplying. Thus $iD^{\mu}_{\phi}(k,q) = \Delta^{\mu}_{\phi}(ik,iq)$.

50. Let $M_{\tau} := C^{\lambda}_{\tau\mu} x_{\lambda} \partial^{\mu}$. The correspondence $\hat{x}_{\tau} \mapsto M_{\tau}$ is a homomorphism of Lie algebras $\mathfrak{g} \to \operatorname{Lie}(A_{n,k})$ – if we corestrict to the image $\mathfrak{g}^{M} = \operatorname{Span}_{k} \{M_{1}, \ldots, M_{n}\}$ and restrict the action of ∂ -s to $\mathfrak{g} \subset S(\mathfrak{g})$, then this is precisely the adjoint representation. On the other hand, the $\mathfrak{g}^{M} \oplus \mathfrak{g}^{*} \subset A_{n,k}$ is closed under the bracket (obviously: $[M_{\tau}, \partial^{\rho}] = -C^{\rho}_{\tau,\mu}\partial^{\mu}$, hence $\mathfrak{g} \cong \mathfrak{g}^{M}$ acts on \mathfrak{g}^{*} here by the coadjoint representation).

51. Theorem. Let $f \in \widehat{S(\mathfrak{g}^*)}$. Then (in symmetric ordering)

$$M_{\mu}(x_{\nu} \star f) - x_{\nu} \star M_{\mu}f = M_{\mu}(x_{\nu})f + M_{\tau}\chi^{\tau}_{\mu\nu}f$$
(20)

where for every $1 \leq \tau, \mu, \nu \leq n, \ \chi^{\tau}_{\mu\nu} \in \widehat{S(\mathfrak{g}^*)}$ and

$$\chi_{\mu\nu}^{\tau} = \sum_{N=1}^{\infty} (-1)^{N} \frac{B_{N}}{N!} \left[C_{\mu\alpha}^{\tau} (\mathbb{C}^{N-1})_{\nu}^{\alpha} - (\mathbb{C}^{N-1})_{\nu,\alpha}^{\tau} \mathbb{C}_{\mu}^{\alpha} \right]$$

where $\mathbb{C}^{\alpha}_{\beta} := C^{\alpha}_{\beta\rho}\partial^{\rho}$, $(\mathbb{C}^{N-1})^{\tau}_{\nu,\alpha} := \frac{\partial}{\partial(\partial^{\alpha})}(\mathbb{C}^{N-1})^{\tau}_{\nu}$, and $M_{\mu}(x_{\nu}) = C^{\lambda}_{\mu\nu}x_{\lambda}$.

Proof. Write $x_{\nu} \star f = x_{\alpha} \phi_{\nu}^{\alpha} f$ hence $M_{\mu}(x_{\nu} \star f) = C_{\mu\rho}^{\lambda} x_{\lambda} \partial^{\rho} x_{\alpha} \phi_{\nu}^{\alpha} f = C_{\mu\rho}^{\lambda} x_{\lambda} (\delta_{\alpha}^{\rho} + x_{\alpha} \partial^{\rho}) \phi_{\nu}^{\alpha} f = x_{\nu} \star M_{\mu} f + C_{\mu\alpha}^{\lambda} x_{\lambda} \phi_{\nu}^{\alpha} f - x_{\alpha} C_{\mu\rho}^{\lambda} \partial^{\rho} \phi_{\nu,\lambda}^{\alpha} f$, relabel the indices in the last term to obtain

$$\begin{aligned} M_{\mu}(x_{\nu} \star f) - x_{\nu} \star M_{\mu}f &= x_{\tau}(C^{\tau}_{\mu\alpha}\phi^{\alpha}_{\nu} - \mathbb{C}^{\lambda}_{\mu}\phi^{\tau}_{\nu,\lambda})f \\ &= \sum_{N=0}^{\infty} (-1)^{N} \frac{B_{N}}{N!} x_{\tau}[C^{\tau}_{\mu\alpha}(\mathbb{C}^{N})^{\alpha}_{\nu} - (\mathbb{C}^{N})^{\tau}_{\nu,\lambda}\mathbb{C}^{\lambda}_{\mu}]f. \end{aligned}$$

For N = 0 only the summand $C^{\tau}_{\mu\alpha}(\mathbb{C}^N)^{\alpha}_{\nu} = C^{\tau}_{\mu\nu}$ survives within the brackets. For N > 1 both summands survive, and within the second summand use the Leibniz rule for $\frac{\partial}{\partial(\partial^{\lambda})}$ in the form $(\mathbb{C}^N)^{\tau}_{\nu,\lambda} = \mathbb{C}^{\tau}_{\rho}(\mathbb{C}^{N-1})^{\rho}_{\nu,\lambda} + C^{\tau}_{\rho\lambda}(\mathbb{C}^{N-1})^{\rho}_{\nu}$. In the rightmost summand so obtained, use the Jacobi identity, in the form $-C^{\tau}_{\rho\lambda}\mathbb{C}^{\lambda}_{\mu} = -C^{\tau}_{\mu\lambda}\mathbb{C}^{\lambda}_{\rho} + \mathbb{C}^{\tau}_{\lambda}C^{\lambda}_{\mu\rho}$, contracted with $(\mathbb{C}^{N-1})^{\rho}_{\nu}$, and after a cancelation of one summand, accounting for the signs and for the antisymmetry in lower indices, and reassembling the M_{τ} , one obtains the formula above.

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