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Snails in Hyperbolic Plane

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ABSTRACT

The properties of the limaçon of Pascal in the Euclidean plane are well known. The aim of this paper is to obtain the curves in the hyperbolic plane having the similar properties. That curves are named hyperbolic snails and defined as the circle pedal curves.
It is shown that all of them are circular quartics, while some of them are entirely circular.

Key words: limaçon of Pascal, hyperbolic plane, entirely circular 4-order curves

MSC 2010: 51M09, 51M10, 51M15

Puževi u hiperboličkoj ravnini

SAŽETAK

Dobro su poznata svojstva krivulje euklidske ravnine zvane Pascalov puž. U ovom se radu u hiperboličkoj ravnini konstruiraju krivulje sa sličnim svojstvima. Te su krivulje nazvane hiperboličkim puževima i definirane kao nožišne krivulje kružnica.
Pokazuje se da su svi hiperbolički puževi cirkularne kvartike, a neke od njih su čak potpuno cirkularne.

Ključne riječi: Pascalov puž, hiperbolička ravnin, potpuno cirkularne krivulje 4. reda

1 Introduction

The properties of the limaçon of Pascal in the Euclidean plane are well known. It is a bicircular curve of fourth order, that can be obtained as a circle pedal curve, ([5], pp. 133–134). The pedal point (pole) P is a node, cusp or isolated double point of this pedal curve depending on whether it is outside, on or inside the circle. The limaçon has cusps at the apsolute points and a singular focus that coincides with the midpoint of the segment OP, where O is the center of the circle. There are the limaçones of Pascal that have singular foci lying on them.
The limaçon of Pascal can also be obtained by the inversion of a conic if its focus coincides with the pole of the inversion, ([5], p. 122). The limaçon of Pascal possesses a node, cusp or isolated double point depending on whether the generating conic is a hyperbola, parabola or ellipse, respectively.
The limaçon of Pascal possesses an axis of symmetry.
One can ask himself if there is a curve in the hyperbolic plane having the similar properties.

Let a be the absolute conic for the Cayley-Klein model of the hyperbolic plane (H-plane) represented by a circle of classical Euclidean plane. The interior points of the absolute conic are called *real points*, exterior points are *ideal points* and points of the absolute conic are called *absolute points*, [4].

A perpendicularity in the H-plane is defined by the absolute polarity. This means that two lines are *perpendicular* iff one passes through the absolute pole of the other. The *pedal* of a given curve with respect to a point P is the locus of the foot of the perpendicular from the point P to the tangent line to the given curve, [4].
A curve in the H-plane is *circular* if it touches the absolute conic at least at one point. If a curve possesses a common tangent with the absolute conic (isotropic asymptote) at each intersection point, it is *entirely circular*, [6].
A curve having two touching points with the absolute conic, possess a singular focus defined as an intersection of isotropic tangent lines at the absolute points, [3].

2 Hyperbolic snail

Definition 1 *A hyperbolic snail (H-snail) is a circle pedal curve.*

There are three classes of circles in the H-plane. Therefore, different types of H-snail can be expected. The circles are classified into the hypercycles, cycles and horocycles depending on whether they touch the absolute conic at two different real points, at a pair of imaginary points, or whether their four absolute points coincide, respectively, [4].

Theorem 1 *H-snail is a fourth order curve touching the absolute conic at two points.*

Proof. Let us construct the pedal curve k^4 of a circle c_2. Let the given circle c_2 be e.g. a hypercycle with the center O and absolute touching points O_1 and O_2 and let P be the pole of the pedal transformation, Figure 1.
The construction should be made in the following way: The connecting line PT, where T is the absolute pole of the tangent line t of the circle c_2, intersects t in a point T_N lying on the required curve.
Absolute touching points O_1, O_2 obviously lie on the curve k^4 since they are the feet of the perpendiculars from the point P to the isotropic asymptotes OO_1 and OO_2.
Through each of the intersection points A_1, A_2 of the absolute conic a and the polar line p of the pole P with respect to a, pass two tangent lines to the hypercycle. Consequently, A_1, A_2 are double points of the required curve. According to Chasles correspondence principal, [9], k^4 is fourth order curve as it is the result of (1, 2)-correspondence between the first order pencil of lines (P) and the second order pencil (c_2) of the tangent lines of the conic c_2.
Two tangent lines to the hypercycle c_2 pass through the point P. Their poles are located on the polar line p. The connecting lines of those points with the point P are the tangent lines of k^4 at its double point P.
The constructed curve k^4 is the fouth order curve touching the absolute conic at O_1, O_2. It has three double points O, A_1, A_2, and a quadruple focus O.

If the given circle c_2 is cycle or horocycle, it is similar, Figure 2. The cycle touches the absolute conic at the pair of imaginary points, and the same holds for its pedal curve k^4. The horocycle pedal curve hyperosculates the absolute conic a at the touching point $O = O_1 = O_2$ with the intersection multiplicity 4. □

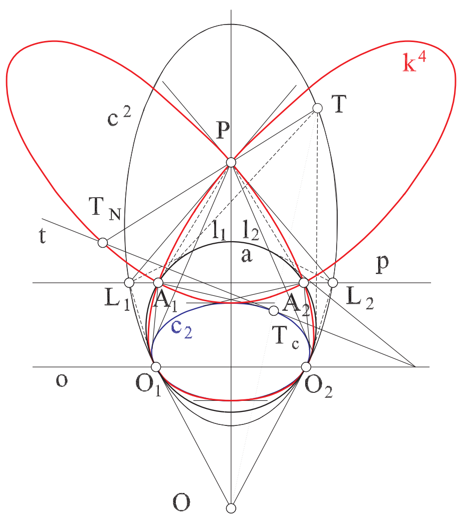

Figure 1

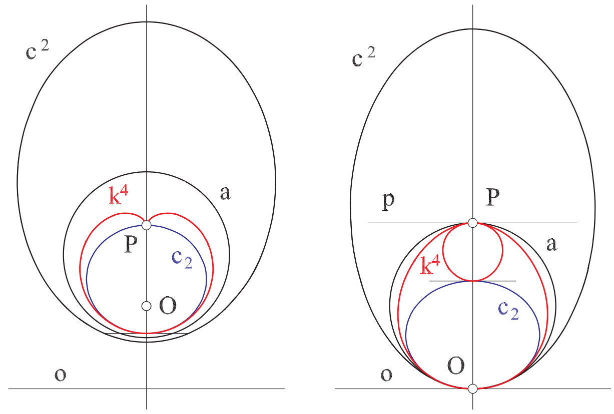

Figure 2

Corollary 1 *If the pole P of the pedal transformation is an absolute point, H-snail is an entirely circular fourth order curve.*

Proof. If the pole P lies on the absolute conic, three double points P, A_1, A_2 of H-snail coincide with the point P at which both tangents coincide with the line p, Figure 2.

Generally, an entirely circular quartic in the H-plane possesses six quadruple foci. The isotropic tangent lines OO_1, OO_2 of k^4 intersect the twice counted isotropic tangent line p at the points F_1, F_2, respectively. Therefore, entirely circular H-snail k^4 possesses two quadruple foci O, P, and two eightfold foci F_1, F_2. □

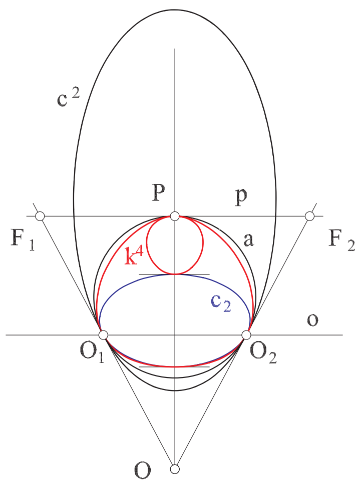

Figure 3

Remark 1 If the pedal point P is a real point, k^4 has two imaginary double points on the absolute conic. These are imaginary contact points of the absolute conic and tangents passing through the point P.
The pole P is a node, cusp or isolated double point depending on whether it is outside, on or inside the circle c_2, respectively.

Remark 2 Let us denote by c^2 the reciprocal curve of the circle c_2 in the absolute polarity, Figure 1. It's easy to see that the pedal curve k^4 of the circle c_2 is the inverse curve of the circle c^2 with respect to the same pole P, [8]. This fact simplifies derivation of some constructive and synthetic conclusions about H-snails.

The limaçon of Pascal possesses an axis of symmetry. The analogous statement holds in the hyperbolic plane.

Theorem 2 *Let the H-snail k^4 be the pedal curve of the circle c_2 with the center O with respect to the pedal point P. The line $s = OP$ is an axis of symmetry of the H-snail.*

Proof. The fact that every circle of the H-plane is a collinear image of the absolute conic a, [3], will be applied in the proof.
Let T_N be a point on k^4, obtained as the foot of the perpendicular from the pole P to the tangent line t, Figure 4. Let q be the line through T_N perpendicular to s. The connecting line $z = ST$ intersects the circle c^2 in the point T', being the pole of the tangent line t' and inverse of the point T'_N. By A and B the intersection points of line s with lines z and q are denoted.
A circle is a symmetric curve with respect to every diameter. In other words, the points T and T' are equally distanced from the line s and equality $(SA, TT') = -1$ holds. After connecting these four points with the pole P a harmonic quadruple of lines is obtained, [1]. Accordingly, $(SB, T_N T'_N) = -1$. This finishes the proof. □

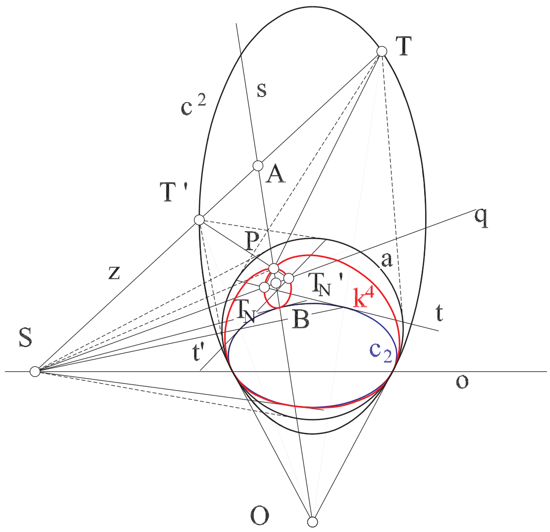

Figure 4

For further studies of the entirely circular quartics in the hyperbolic plane the papers [2], [7] and [8] can be consulted.

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