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Rectifiable Oscillations and Singular Behaviour of Solutions of Second-Order Linear Differential Equations

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Abstract

Rectifiable oscillations on the finite interval $I = (0, 1)$ of all solutions of the equation $(P): y'' + f(x)y = 0$, $x \in I$, has been recently introduced and studied in Pašić [6] and Wong [12], where $f(x)$ is of Euler type coefficient. It has been continued in [7], where $f(x)$ is of Riemann-Weber type and in [4], where $f(x)$ is of more general type, that is, $f(x)$ satisfies the so-called Hartman-Wintner asymptotic condition near $x = 0$. These results show that the infiniteness of arclength of any solution curve y of (P) depends on asymptotic behavior of $f(x)$ near $x = 0$. In this paper we do not suppose Hartman-Wintner asymptotic condition on f . Instead, we impose certain growth conditions on f and we study connection of oscillatory solutions of (P) and singular behavior of its solutions y near $x = 0$. In a sense, it generalizes results and methods presented in [6]. Finally, we define some open questions.

Mathematics Subject Classification: 34B05, 34C10

Keywords: Linear equations; Oscillations; Graph; Rectifiability

1 Introduction and statement of the main results

Let $I = (0, 1)$. A real function $y = y(x)$ is *oscillatory* (respectively *nonoscillatory*) on I if it has an infinite (respectively finite) number of zeros on I . A linear differential equation $y'' + f(x)y = 0$ is said to be *oscillatory* (respectively *nonoscillatory*) on I if all its nontrivial solutions are oscillatory (respectively nonoscillatory) on I . A fundamental result from the theory of oscillations for second order linear differential equations says that for $\lambda > \frac{1}{4}$ (respectively

$\lambda \leq \frac{1}{4}$) all nontrivial solutions of Euler differential equation $y'' + \lambda x^{-2}y = 0$ are oscillatory (respectively nonoscillatory) on I , see for instance in [10].

The graph of y is determined as usual by $G(y) := \{(t, y(t)) : 0 \leq t \leq 1\} \subseteq \mathbf{R}^2$, and the length of graph $G(y)$ is defined by

$$\text{length}(G(y)) := \sup \sum_{i=1}^m \|(t_i, y(t_i)) - (t_{i-1}, y(t_{i-1}))\|_2,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \dots < t_m = b$ of the interval \bar{I} and where $\|\cdot\|_2$ denotes euclidian norm in \mathbf{R}^2 . The graph $G(y)$ is said to be *rectifiable curve* in \mathbf{R}^2 if it is satisfied $\text{length}(G(y)) < +\infty$. Otherwise, $G(y)$ is said to be *unrectifiable curve* in \mathbf{R}^2 . On rectifiable and unrectifiable curves in \mathbf{R}^2 see in [3], [5], [9], and [11].

Definition 1.1 An oscillatory function y is said to be *rectifiable* (respectively *unrectifiable*) *oscillatory* on I if its graph $G(y)$ is a rectifiable (respectively unrectifiable) curve in \mathbf{R}^2 . A linear differential equation $y'' + f(x)y = 0$ is said to be *rectifiable* (respectively *unrectifiable*) *oscillatory* on I if all its nontrivial solutions are rectifiable (respectively unrectifiable) oscillatory on I .

An essential example for the linear differential equation so that the rectifiable and unrectifiable oscillations of its solutions only depend on a parameter, is the following generalization of famous Euler linear differential equation,

$$(E_\alpha) : y'' + \frac{\lambda}{x^\alpha}y = 0, \quad x \in I,$$

where $\lambda > 0$ and $\alpha > 2$. If $\alpha = 2$ and $\lambda > 1/4$, we get the Euler linear differential equation which is rectifiable oscillatory on I , which is not difficult to check since its fundamental system of all solutions is explicitly given. If $\alpha = 4$, then equation (E_α) admits also explicit solutions, which are unrectifiable oscillatory on I . If $\alpha > 2$, the following boundary-layer conditions have been separately proposed in [6],

$$(B_1) : \begin{cases} \text{there are } c > 0 \text{ and } d \in I \text{ both depending on } y \text{ such that} \\ |y'(x)| \leq \frac{c}{x^{\alpha/4}}, \text{ for all } x \in (0, d), \end{cases}$$

$$(B_2) : \begin{cases} \text{there are } c > 0 \text{ and } d \in I \text{ both depending on } y \text{ such that} \\ |y(x)| \leq cx^{\alpha/4}, \text{ for all } x \in (0, d). \end{cases}$$

Including (B_1) and (B_2) in equation (E_α) , the rectifiable oscillations of (E_α) was examined in [6, Theorem 1.4 and Theorem 1.5] from this point of view.

Theorem A. *Let $\lambda > 0$ when $\alpha \geq 2$ and $\lambda > 1/4$ when $\alpha = 2$. Then we have:*
(i) *for $\alpha \in [2, 4)$, the linear problem $(E_\alpha) \& (B_1)$ is rectifiable oscillatory on I ;*

(ii) for $\alpha = 4$, the linear problem $(E_\alpha)\&(B_1)$ is unrectifiable oscillatory on I ;
 (iii) for $\alpha > 4$, the linear problem $(E_\alpha)\&(B_1)$ is also unrectifiable oscillatory on I provided it admits the existence of two linearly independent solutions.
 Furthermore, the conclusions (i), (ii), and (iii) are also true for the linear problem $(E_\alpha)\&(B_2)$.

In this paper, instead of equation (E_α) , we study the following class of second order linear differential equations,

$$y'' + f(x)y = 0, \quad x \in I, \tag{1}$$

where $y \in C(\bar{I}) \cap C^2(I)$ and the coefficient $f(x)$ is positive, continuous on $(0, 1]$ and singular at $x = 0$.

Very recently in [12], the results from Theorem A have been also achieved for the equation (E_α) , where the boundary-layer conditions (B_1) and (B_2) are excluded. Moreover, it has been done for equation (1) under the following two asymptotic conditions on $f(x)$ at $x = 0$: Hartman-Wintner type,

$$f^{-\frac{1}{4}}[f^{-\frac{1}{4}}]'' \in L^1(I), \tag{2}$$

and Euler type,

$$f(x) \sim \lambda x^{-\alpha} \text{ near } x = 0, \text{ where } \alpha > 2. \tag{3}$$

It was stated in the following result, see [12, Theorem 1].

Theorem B. *Let $f \in C^2((0, 1])$, $f(x) > 0$ on I , and let $f(x)$ satisfy (2) and (3). Then Eq. (1) is rectifiable oscillatory on I provided $2 < \alpha < 4$ and unrectifiable oscillatory on I provided $\alpha \geq 4$.*

The asymptotic condition (3) can be generalized to some general asymptotic behaviour of $f(x)$ near $x = 0$, which is based on the integrability of $\sqrt[4]{f(x)}$ on I as follows, see [4, Theorem 1.4].

Theorem C. *Let $f \in C^2((0, 1])$, $f(x) > 0$ on I , $\lim_{x \rightarrow 0} f(x) = \infty$, and let $f(x)$ satisfy the Hartman-Wintner asymptotic condition (2). The linear differential equation (1) is rectifiable oscillatory on I provided $\sqrt[4]{f(x)} \in L^1(0, 1)$ and unrectifiable oscillatory on I provided $\sqrt[4]{f(x)} \notin L^1(0, 1)$.*

Thus, in Theorem B and Theorem C, the rectifiable oscillations of (1) on I is characterized by asymptotic behaviours of the coefficient $f(x)$ near $x = 0$. In this paper, this class of oscillations can be studied without taking asymptotic

conditions (2) and (3). That is, for an arbitrarily given positive real number θ , it is supposed that there are β and $\varepsilon \in I$ such that $0 < \beta < \theta$ and

$$f(x) > \frac{\beta^2}{x^{2\beta+2}} + \frac{1-\beta^2}{4x^2}, \quad x \in (0, \varepsilon). \quad (4)$$

Next, instead of (B_1) we consider the following boundary-layer condition,

$$\left\{ \begin{array}{l} \text{there are } c > 0 \text{ and } d \in I \text{ both depending on } y \text{ such that} \\ |y'(x)| \leq cx^{-\frac{\theta+1}{2}} \text{ for all } x \in (0, d). \end{array} \right. \quad (5)$$

It is clear that the problem $(E_\alpha) \& (B_1)$ is a particular case of the problem (1) & (5) in particular for $f(x) = \lambda x^{-\alpha}$ and $\theta = \alpha/2 - 1$.

The rectifiable oscillations of (1) will be derived in the dependence of the values of $\theta > 0$, which appears in the boundary-layer condition (5). It is the subject of the first main result of the paper.

Theorem 1.2 *Let $f \in C^2((0, 1])$, $f(x) > 0$ on I , and let f satisfy (4). Let there be two linearly independent solutions of (1) which satisfy (5).*

- (i) *If $0 < \theta < 1$, then equation (1) is rectifiable oscillatory on I .*
- (ii) *If $\theta \geq 1$ and $\beta \geq \frac{\theta+1}{2}$, then equation (1) is unrectifiable oscillatory on I .*

This type of results will be also shown for equation (1) with solutions y satisfying the following boundary-layer condition:

$$\left\{ \begin{array}{l} \text{there are } c > 0 \text{ and } d \in I \text{ both depending on } y \text{ such that} \\ |y(x)| \leq cx^{\frac{\theta+1}{2}} \text{ for all } x \in (0, d), \end{array} \right. \quad (6)$$

where it is supposed the existence of real numbers γ , $M > 0$, and $\varepsilon \in I$ such that $0 < \beta < \theta \leq \gamma$ and

$$f(x) < \frac{M^2\gamma^2}{x^{2\gamma+2}} + \frac{1-\gamma^2}{4x^2}, \quad x \in (0, \varepsilon). \quad (7)$$

It is clear that $(E_\alpha) \& (B_2)$ is a particular cases of (1) & (6) especially for $f(x) = \lambda x^{-\alpha}$ and $\theta = \alpha/2 - 1$. The second main result of the paper is the following.

Theorem 1.3 *Let $f \in C^2((0, 1])$, $f(x) > 0$ on I , and let f satisfy (4) and (7). Let there be two linearly independent solutions of (1) which satisfy (6).*

- (i) *If $0 < \theta < 1$ and $\gamma < \frac{\theta+1}{2}$, then (1) is rectifiable oscillatory on I .*
- (ii) *If $\theta \geq 1$ and $\gamma - \beta \leq \frac{\theta-1}{2}$, then (1) is unrectifiable oscillatory on I .*

The proofs of conclusions (i) and (ii) from previous theorems will be proved in Section 3 and Section 4 respectively. It is worth to mention that in Pašić [6] was asked a question: whether all solutions of equation (E_α) satisfy the boundary-layer conditions (B_1) and (B_2) ? The answer has been yes, and it has been presented in Wong [12]. In this paper, we formulate the following open problems:

First open problem. *Find a function $f(x)$ (or a class of functions $f(x)$) which does not satisfy the Hartman-Wintner condition (2) but all solutions of corresponding equation (1) satisfy the boundary-layer condition (5) (respectively (6)) for some $\theta > 0$.*

Second open problem. *Find a function $f(x)$ (or a class of functions $f(x)$) which does not satisfy the Hartman-Wintner condition (2), which satisfies (4) and (7), but all solutions of corresponding equation (1) satisfy the boundary-layer condition (5) (respectively (6)) for some $\theta > 0$.*

If the second problem could be solved, Theorem 1.2 and Theorem 1.3 would take an important place in analysis of oscillations of solutions of ode's like Theorem B and Theorem C stated above. In order to provide a further insight into this problem, we present some known necessary conditions such that the Hartman-Wintner condition (2) holds, see [4, Lemma 2.2] and [8, Lemma 2.3].

Lemma 1.4 *Let $f \in C^2((0, 1])$, $f(x) > 0$ on I , $\lim_{x \rightarrow 0} f(x) = \infty$, and let f satisfy (2). Then we have:*

$$f^{\frac{1}{2}} \notin L^1(I), \quad (8)$$

$$\lim_{x \rightarrow 0} f^{-\frac{3}{2}}(x) f'(x) = 0, \quad (9)$$

$$[f^{-\frac{3}{2}} f']' \in L^1(I). \quad (10)$$

Thus, concerning the first open problem, in order to prove that $f(x)$ does not satisfy (2), by Lemma 1.4 it suffices to show that $f(x)$ does not satisfy only one of the conditions (8), (9), and (10). This, however, would only be the first step in understanding the relationship between Hartmann-Wintner conditions and boundary-layer conditions like (B_1) and (B_2) . On the other hand, answer to the question posed in the second open problem requires construction of function f which satisfies (4) and (7) (and therefore (8)). The latter fact represents the key property of $f(x)$ in our consideration.

Finally, the essential example for the linear differential equation in which coefficient $f(x)$ satisfies the growth conditions (4) and (7), and all its solutions

satisfy the boundary-layer conditions (5) and (6), is the equation

$$y''(x) + \left(\frac{\delta^2}{x^{2\delta+2}} + \frac{1-\delta^2}{4x^2}\right)y(x) = 0 \text{ on } I,$$

where $\delta = \theta$. Its general solutions are given in the explicit form $y(x) = x^{(\delta+1)/2}(c_1 \cos \frac{1}{x^\delta} + c_2 \sin \frac{1}{x^\delta})$.

2 Some preliminaries

Our analysis relies essentially on two well-known comparison principles for solutions of linear ODE's.

Theorem 2.1 (Sturm's comparison principle I) *Let $x_0, x_1 \in I$, $x_0 < x_1$, be two consecutive zero points of any solution $y = y(x)$ of the linear equation $y'' + f(x)y = 0$, $x \in I$. Let $z = z(x)$ be any solution of the linear equation $z'' + g(x)z = 0$, $x \in I$. If $f(x) < g(x)$ on (x_0, x_1) and $f, g \in C([x_0, x_1])$, then there is at least one zero point $x_2 \in J$ of z such that $x_0 < x_2 < x_1$.*

Theorem 2.2 (Sturm's comparison principle II) *Let $y = y(x)$ and $z = z(x)$ be two linearly independent solutions of the linear equation $y''(x) + f(x)y(x) = 0$, $x \in I$. Let $x_0, x_1 \in I$ be two consecutive zero points of y . Then there exists exactly one zero point $x_2 \in I$ of the solution z such that $x_0 < x_2 < x_1$.*

An immediate consequence of these comparison principles is the following lemma which describes elementary behavior of y and its zeros in the neighborhood of 0.

Lemma 2.3 *Suppose that f satisfies (4). Let y be a solution of (1). Then we have:*

- (i) *on any open interval $J \subseteq I$ such that $0 \notin \bar{J}$ there exists at most finitely many zero points of y ;*
- (ii) *there exists a decreasing sequence (a_k) of consecutive zero points of y such that $a_k \searrow 0$, i.e., $a_{k+1} < a_k$, $y(a_k) = 0$, $y(x) \neq 0$ for $x \in (a_{k+1}, a_k)$, and $\lim_{k \rightarrow \infty} a_k = 0$.*
- (iii) *For any given $k \in \mathbf{N}$ the solution y is either convex or concave on (a_{k+1}, a_k) . Moreover, for each $k \in \mathbf{N}$ there exists exactly one stationary point $s_k \in (a_{k+1}, a_k)$ such that $y'(s_k) = 0$, $yy'(x) > 0$ for $x \in (a_{k+1}, s_k)$, and $yy'(x) < 0$ for $x \in (s_k, a_k)$.*

Proof. To prove (i), we argue as follows: since $f \in C(\bar{J})$ for every open interval J such that $J \subseteq I, 0 \notin \bar{J}$, there exists $M = M(J) > 0$ such that there holds

$$f(x) < M, \text{ for every } x \in \bar{J}.$$

Set $z(x) := \sin \sqrt{M}x$. Then $z = z(x)$ is a non-zero solution of the equation $z'' + Mz = 0$ on J . At this point we apply Theorem 2.1 to the solutions y and z , so that between two consecutive zero points of y there exists at least one zero point of z . Hence, if we assume that y has infinitely many zero points on J , we arrive at contradiction, since z obviously has at most finitely zero points on J . Now we turn our attention to proving (ii). Assumption $\beta > 0$ implies that there exists an open interval $J_0 \subseteq I, 0 \in \bar{J}_0$ such that there holds

$$\frac{\beta^2}{x^{2\beta+2}} + \frac{1 - \beta^2}{4x^2} > \frac{1}{2x^2} \text{ for every } x \in J_0.$$

According to Theorem 2.1, applied to the solutions y and respectively $w(x) := \sqrt{x} \sin(2\ln x)$ of the equations (1) and respectively

$$w''(x) + \frac{1}{2x^2}w(x) = 0,$$

it follows that between two consecutive zero points of w in J , denoted by x_k and x_{k+1} (for sufficiently large k), there exists at least one zero point of y , denoted by a_k , such that $a_k \in J$ and $x_{k+1} < a_k < x_k$. Since $x_k = e^{-4k\pi}$, the sequence (a_k) has desired property (ii). Finally, the claim (iii) is an immediate consequence of the fact that $f(x) > 0$ on I and including also the sign of y on (a_{k+1}, a_k) in the equation (1). \diamond

Lemma 2.4 *Let $y = y(x)$ be a non-zero solution of (1) and let (a_k) be the decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. If f satisfies (7), then there exists $c_1 > 0$ and $k_0 \in \mathbf{N}$ such that for every $k > k_0$ there holds*

$$|y'(a_{k+1})|a_{k+1}^{\gamma+1} \leq c_1|y(s_k)|, \tag{11}$$

where $s_k \in (a_{k+1}, a_k)$ such that $y'(s_k) = 0$.

Proof. By multiplying (1) by y' , and by integrating it from a_{k+1} to s_k , we can write

$$-\int_{a_{k+1}}^{s_k} (y^2)'(t)dt = \int_{a_{k+1}}^{s_k} f(t)(y^2)'(t)dt,$$

which leads to

$$(y')^2(a_{k+1}) = \int_{a_{k+1}}^{s_k} f(t)(y^2)'(t)dt.$$

By (7) for sufficiently large $k \in \mathbf{N}$ we estimate

$$(y')^2(a_{k+1}) \leq \int_{a_{k+1}}^{s_k} 2 \frac{M^2 \gamma^2}{t^{2\gamma+2}} (y^2)'(t) dt \leq 2 \frac{M^2 \gamma^2}{a_{k+1}^{2\gamma+2}} \int_{a_{k+1}}^{s_k} (y^2)'(t) = 2 \frac{M^2 \gamma^2}{a_{k+1}^{2\gamma+2}} (y^2)(s_k),$$

which amounts to (11). \diamond

Lemma 2.5 *Let $y = y(x)$ be a non-zero solution of (1) and let (a_k) be the decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$.*

(i) *If f satisfies (4), then for every $k \in \mathbf{N}$ there holds*

$$|a_k - a_{k+1}| \leq C_0 a_k^{\beta+1}, \quad \text{where } C_0 := \frac{\pi}{\beta}. \tag{12}$$

(ii) *If f satisfies (4), then there exists $c_0 > 0$ and $k_0 \in \mathbf{N}$ such that for every $k > k_0$ there holds*

$$1 \leq \frac{a_k}{a_{k+1}} \leq c_0. \tag{13}$$

(iii) *If f satisfies (4) and (7), then there exists $c_1 > 0$ and $k_0 \in \mathbf{N}$ such that for every $k > k_0$ there holds*

$$|y'(a_{k+1})| a_k^{\gamma+1} \leq c_1 |y(s_k)|, \tag{14}$$

where $s_k \in (a_{k+1}, a_k)$ such that $y'(s_k) = 0$.

Proof. Firstly, we prove (i). Consider $m := \min_{x \in [a_{k+1}, a_k]} \frac{\beta^2}{x^{2\beta+2}}$. By (4) for every $x \in [a_{k+1}, a_k]$ there holds $f(x) > m$. We define $u(x) := \sin(\sqrt{m}(x - a_{k+1}))$. Then for every $x \in I$ there holds $u''(x) + mu(x) = 0$. It is clear that a_{k+1} and $a_{k+1} + \frac{\pi}{\sqrt{m}}$ are consecutive zero points of u . We set $J := (a_{k+1}, a_k)$. Let us suppose that there holds

$$a_{k+1} + \frac{\pi}{\sqrt{m}} < a_k. \tag{15}$$

By assumption (15) it is possible to apply Theorem 2.1 to the interval J and to equations in u and y . It follows that between two zero points $x_1 := a_{k+1} + \frac{\pi}{\sqrt{m}}$ and $x_0 := a_{k+1}$ of u there exists at least one zero point $x_2 \in J$ of y such that there holds $x_0 < x_2 < x_1$. Therefore, we get $a_{k+1} < x_2 < a_k$, which contradicts the fact that (a_k) is a sequence of consecutive zero points of y . Thus, we conclude that the assumption (15) was false. Since $m = \beta^2/a_k^{2\beta+2}$, we derive (12). To obtain (ii), we use (12), so that there holds

$$\left| 1 - \frac{a_{k+1}}{a_k} \right| \leq \frac{\pi}{\beta} a_k^\beta, \quad \lim_{k \rightarrow +\infty} \frac{a_{k+1}}{a_k} = 1.$$

Hence, we deduce that there exists $c_1 > 0$ and $k_0 \in \mathbf{N}$ such that for every $k \geq k_0$ we have $\frac{a_{k+1}}{a_k} \geq c_1$. Therefore, we get $1 \leq \frac{a_k}{a_{k+1}} \leq c_0$, where $c_0 := \frac{1}{c_1}$. Finally, we prove (iii). In light of (13), inequality $|y'(a_{k+1})|a_{k+1}^{\gamma+1} \leq \sqrt{2}\gamma|y(s_k)|$ of Lemma 2.4 clearly implies (14). \diamond

Remark 2.6 It is important to note that even simple variants of ODE (1) may not possess property (13). Such is, for example, the Riemann-Weber version of the Euler equation, $y'' + \frac{1}{x^2}(\frac{1}{4} + \frac{\lambda}{\log^2 x})y = 0$. For details see in [7].

3 Rectifiable oscillations

In this section, we analyze the case of rectifiable oscillations for (1). It will be largely based on the following elementary result about rectifiable oscillations for smooth convex-concave real functions.

Lemma 3.1 *Let $y = y(x)$ be a real function on I such that $y \in C(\bar{I}) \cap C^2(I)$. Let (s_k) be decreasing sequence of consecutive points of local extremes of y on I . Then there holds*

$$\text{length}(G(y)) \leq 2 \sum_k |y(s_k)| + b. \tag{16}$$

In particular, if the series $\sum_k |y(s_k)|$ is convergent, then y is rectifiable oscillatory on I .

Proof. Set $l_k := \text{length}(G(y^k))$, where by y^k we denote restriction of y to interval $[s_{k+1}, s_k]$. Consider arbitrary partition σ of $[s_{k+1}, s_k]$ generated by finite sequence of points (t_i^k) such that $t_{i-1}^k < t_i^k$ for every i . Consider piecewise affine continuous function $y_\sigma^k : [s_{k+1}, s_k] \rightarrow \mathbf{R}$ through the points $(t_i^k, y(t_i^k))$. We define $l_k(\sigma) := \text{length}(y_\sigma^k)$, $\Delta x_i^k := t_i^k - t_{i-1}^k$, $\Delta y_i^k := y(t_i^k) - y(t_{i-1}^k)$ and $\Delta_{i,k}(\sigma) := \text{length}(y_{\sigma,i}^k)$, where $y_{\sigma,i}^k$ stands for restriction of y_σ^k to interval $[t_{i-1}^k, t_i^k]$. Then there holds $\Delta_{i,k}(\sigma) \leq \Delta x_i^k + \Delta y_i^k$, which (as we sum on both sides of inequality over all i) gives $l_k(\sigma) \leq s_k - s_{k+1} + |y(s_k)| + |y(s_{k+1})|$. Since by definition of l_k there holds $l_k = \sup_\sigma l_k(\sigma)$, where supremum is taken over all such partitions σ , we get

$$l_k = \sup_\sigma \sum_i \Delta_{i,k}(\sigma) \leq s_k - s_{k+1} + |y(s_k)| + |y(s_{k+1})|. \tag{17}$$

By taking the sum over k in (17), we obtain (16). \diamond

In order to apply this general result, we require to estimate the density of zeros of solutions of ODE (1).

Lemma 3.2 *Let $y = y(x)$ be a non-zero solutions of the equation (1) and let (a_k) be the decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. If f satisfies (7), then there exists $k_0 \in \mathbf{N} \cup \{0\}$ such that for every $k > k_0$ there holds*

$$a_k \leq C \left(\frac{1}{k - k_0} \right)^{\frac{1}{\gamma}}, \quad \text{where } C := \left(\frac{M}{\pi} \right)^{\frac{1}{\gamma}}. \tag{18}$$

Proof. Consider the sequence of real numbers $(x_{k,\delta})$ such that

$$x_{k,\delta} = \left(\frac{M}{k\pi} \right)^{\frac{1}{\delta}}, \quad \text{where } k \in \mathbf{N} \text{ and } \delta > 0.$$

We set $z_\delta(x) := x^{(\delta+1)/2} \sin \frac{1}{x^\delta}$, $x \in I$. It is easy to verify that there holds

$$z_\delta'' + \left(\frac{M^2 \delta^2}{x^{2\delta+2}} + \frac{1 - \delta^2}{4x^2} \right) z_\delta = 0, \quad x \in I. \tag{19}$$

Since $x_{k,\gamma} \searrow 0$ and $a_k \searrow 0$, there exists $k_0 = k_0(\gamma) \in \mathbf{N}$ such that for every $k \geq k_0(\gamma)$ there holds $x_{k,\gamma} \in J_\gamma$ and $a_k \in J_\gamma$. By an application of Theorem 2.1 to the ODE's (1) and (19) when $\delta = \gamma$, by (7) it follows that for two consecutive zero points a_{k_0+1} and a_{k_0} of y there exists a zero point of z_γ , which we denote by $x_{i_0,\gamma}$ such that $a_{k_0+1} < x_{i_0,\gamma} < a_{k_0}$. We can apply the same reasoning to conclude that for every $j \in \mathbf{N}$ there exists zero point of z_γ , denoted by $x_{i_0+j-1,\gamma}$, such that $a_{k_0+j} < x_{i_0+j-1,\gamma}$. Set $k := k_0 + j$. Then we can write

$$a_k < x_{i_0+k-k_0-1,\gamma} < x_{k-k_0,\gamma},$$

where $k > k_0$. Hence, we recover (18). \diamond

Now we are ready to prove the main result of this section.

Proofs of the conclusion (i) of Theorem 1.2 and Theorem 1.3. At the first, let y be a non-zero solution of (1)&(5). By Lemma 2.3, we have that y is oscillatory on I . Since $\theta \in (0, 1)$ and by means of (5), we obtain that $y' \in L^1(0, d)$ which implies that y is rectifiable oscillatory on $(0, d)$. Here, the following fact have been used: the graph $G(y)$ is rectifiable curve in \mathbf{R}^2 if, and only if, $y' \in L^1$ (see [1, Theorem 1, p.217]). Now, rectifiability of $G(y)$ on entire I follows from (i) and (iii) of Lemma 2.3.

Next, let y be a non-zero solution of (1)&(6). We begin by noting that boundary-layer condition (6) in particular implies that for sufficiently large k there holds

$$|y(s_k)| \leq c_0 s_k^{\frac{\theta+1}{2}} \leq c_0 a_k^{\frac{\theta+1}{2}},$$

where (s_k) are chosen as in the claim (iii) of Lemma 2.3. By (18) there exists constant $c_1 > 0$ such that for sufficiently large k there holds

$$|y(s_k)| \leq c_1 \left(\frac{1}{k - k_0} \right)^{\frac{\theta+1}{2\gamma}}.$$

Since $\gamma < \frac{\theta+1}{2}$, it results that the series

$$\sum_{k > k_0} \left(\frac{1}{k - k_0} \right)^{\frac{\theta+1}{2\gamma}}$$

is convergent. In effect, the series $\sum_k |y(s_k)|$ is convergent in this case, and by Lemma 3.1, we obtain that y is rectifiable oscillatory on I . This observation furnishes the proof. \diamond

4 Unrectifiable oscillations

In this section, we pay attention to the case $\theta \geq 1$, which guarantees unrectifiable oscillations of the linear problems (1)&(5) and (1)&(6). The proof is based on the following lemmas. At the first, we recall [6, Proposition 4.2].

Lemma 4.1 *Let $y = y(x)$ be a real function, $y \in C(\bar{I})$, and let $a_k \in I$ be a decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$. If y is rectifiable oscillatory on I , then for any sequence $b_k \in (a_{k+1}, a_k)$, $k \in \mathbf{N}$,*

$$\sum_{k=1}^n |y(b_k)| \leq \text{length}(G(y)), \quad \text{for each } n \in \mathbf{N}. \tag{20}$$

Lemma 4.2 *Let f satisfy (4). Let $y = y(x)$ be a non-zero solution of (1) and let (a_k) be the decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$.*

(i) *For every $k \in \mathbf{N}$ there holds*

$$a_k \geq C \left(\frac{1}{k + k_0} \right)^{\frac{1}{\beta}}, \quad \text{where } C := \left(\frac{1}{\pi} \right)^{\frac{1}{\beta}}. \tag{21}$$

(ii) *In particular, for any $p \in (0, \beta]$, the series $\sum_k a_k^p$ is divergent.*

Proof. First, we present the proof of (21). Consider the sequence of real numbers $(x_{k,\beta})$ and the function $z_\beta(x)$ all the same as in the proof of Lemma 3.2, where $M = 1$ and $\delta = \beta$. Since $x_{k,\beta} \searrow 0$ and $a_k \searrow 0$, for sufficiently large $k \geq k_0 = k_0(\beta)$, the zero points $x_{k,\beta}$ and a_k obviously belong to the

interval J_β . By (4) and Theorem 2.1 applied to the ODE's (1) and (19) as $\delta = \beta$ and $M = 1$, between two consecutive zero points $x_{k_0+1,\beta}$ and $x_{k_0,\beta}$ of (19), we deduce the existence of a zero point of y , denoted by a_{i_0} , such that $x_{k_0+1,\beta} < a_{i_0} < x_{k_0,\beta}$. Similarly, there exists a sequence of numbers indexed by $k \in \mathbf{N}$, denoted by a_{i_0+k-1} , such that $x_{k_0+k,\beta} < a_{i_0+k-1} < a_k$, $k \in \mathbf{N}$, which proves (21).

Second, to prove (ii), we note that (21) provides

$$a_k^p \geq \left[\frac{1}{(k + k_0)\pi} \right]^{\frac{p}{\beta}}, \quad k \in \mathbf{N}.$$

Since $0 < p \leq \beta$, we have $\frac{p}{\beta} \leq 1$ and thus (ii) follows. \diamond

Besides the claim (ii) of Lemma 4.2, the following lemma will play essential role in the proof of the unrectifiable oscillations for the linear problems (1)&(5) and (1)&(6).

Lemma 4.3 *Suppose that $\theta > 0$ and the problem (1)&(5) (respectively (1)&(6)) admits two linearly independent solutions. Let f satisfy (4) and (7). If y is a non-zero solution to problem (1)&(5) (respectively (1)&(6)) and if (a_k) is the decreasing sequence of consecutive zero points of y such that $a_k \searrow 0$, then there exists constant $c_0 > 0$, a natural number k_0 , and a sequence (b_k) such that for each $k > k_0$ there holds $b_k \in (a_{k+1}, a_k)$ and*

$$|y(b_k)| \geq c_0 b_k^{(\theta+1)/2}, \quad (\text{respectively } |y(b_k)| \geq c_0 b_k^{[\gamma+1-(\theta+1)/2]}). \quad (22)$$

Proof. Consider arbitrary non-zero solution y of (1). If y_1 and y_2 are two linearly independent solutions of (1)&(5) (respectively (1)&(6)), then y is linearly independent with respect to either y_1 or y_2 . Let y_1 be such a solution. By Theorem 2.2 we obtain the sequence of consecutive zero points of y_1 , denoted by (b_k) , such that $a_{k+1} < b_k < a_k$, for every $k \in \mathbf{N}$.

It is known, see for instance [2] that for any two linearly independent solutions y and z of (1) the Wronskian $W(y, z)(x) = y'(x)z(x) - z'(x)y(x) = c \neq 0$ for all $x \in I$. Hence, for linearly independent solutions y and y_1 there exists a constant $c \in \mathbf{R} \setminus \{0\}$ which satisfies $y(x)y_1'(x) - y'(x)y_1(x) = c$ for all $x \in I$. In particular, for $x := b_k$ we have

$$|y(b_k)||y_1'(b_k)| = |c|. \quad (23)$$

Since the condition (5) gives us $|y_1'(b_k)| \leq |c_1|b_k^{-\frac{\theta+1}{2}}$, from equality (23) we get (22), in this case. However, under the condition (6), by means of (iii) of Lemma 2.5, we observe the following calculation:

$$|y_1'(b_k)| \leq c_1 b_k^{-(\gamma+1)} |y_1(s_k)| \leq c_2 b_k^{-(\gamma+1)} b_k^{\frac{\theta+1}{2}} = c_2 b_k^{\frac{\theta+1}{2} - (\gamma+1)},$$

where $s_k \in (b_{k+1}, b_k)$ is the sequence of stationary points of the solutions y_1 . Putting this inequality in (23) we get (22) in this case too. \diamond

Finally, we state the main result of this section.

Proofs of the conclusion (ii) of Theorem 1.2 and Theorem 1.3. Since $\theta \geq 1$, it is possible to find $\beta > 0$ and $\gamma > \beta$ satisfying supposed inequality $\gamma - \beta \leq (\theta + 1)/2$, that is, $\gamma + 1 - (\theta + 1)/2 \leq \beta$. In this direction, let $p = (\theta + 1)/2$, (respectively $p = \gamma + 1 - (\theta + 1)/2$). It is clear that $0 < p \leq \beta$ in both cases of p . Next, consider a non-zero solution y to (1)&(5) (respectively (1)&(6)). By Lemma 2.3 the solution y is oscillatory, and so there exists the decreasing sequence (a_k) of consecutive zero points of a_k such that $a_k \searrow 0$. Let us suppose, for a moment, that y is rectifiable oscillatory on I . Then by Lemma 4.1, for every sequence (b_k) such that $b_k \in (a_{k+1}, a_k)$, $k \in \mathbf{N}$ and every $n \in \mathbf{N}$ there holds $\sum_{k=1}^n |y(b_k)| \leq \text{length}(G(y))$. Combining this fact by (22), where b_k is given in Lemma 4.3, for arbitrary $n \in \mathbf{N}$, $n > k_0$, we obtain

$$\sum_{k=1}^n a_{k+1}^p \leq \sum_{k=1}^n b_k^p \leq c_1 \sum_{k=1}^n |y(b_k)| \leq c_1 \text{length}(G(y)) < +\infty .$$

It contradicts the claim (ii) of Lemma 4.2 for both cases of p . Therefore, y is unrectifiable oscillatory. \diamond

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