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Event oriented system analysis

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Abstract

This article presents an attempt towards a probabilistic event oriented system analysis in engineering. Engineering systems are represented as either complete or incomplete systems of events and as compounds of various subsystems of events. The event oriented system analysis investigates important subsystems in engineering systems, such as operational modes and failure modes and their interrelations. The analysis is also applicable to engineering systems with various relations among the sets of events, such as mutually exclusive and inclusive sets. Further, the systems and subsystems are subjected to probability and uncertainty analysis. The system uncertainty analysis is based on entropy. General relations among the probability, uncertainty of the system and uncertainties of the subsystems are derived by using information theory. Specific mathematical aspects and available methods in the uncertainty modelling of systems and subsystems are summarised. Numerical examples confirm the relevance of the event oriented system analysis and indicate potential improvements in system design.

Keywords: Engineering; Mechanics; System analysis; System of events; Probability; Reliability; Entropy; Information; Uncertainty

1. Introduction

Each object can be viewed as a system in different ways. An object is often a part or subsystem of a more complex system. The object itself can consist of many components and possibly of more subsystems. Systems in engineering are often viewed as objects of many discrete interacting components with uncertain capabilities. Moreover, systems are subjected to uncertain external demands. Sometimes, the components are grouped into a number of subsystems, each of them pertaining to some specific characteristic function of the system. Subsequently, many systems can be assumed to depend only on the current states of their components.

Complex engineering systems can be subjected to service modes and effects analysis in order to identify the events that can occur at the component, subsystem and system levels. The goals of such an analysis are to determine the effects of known operational and failure modes on the overall behaviour of the systems [3,8,11]. In addition to operational modes and effects analysis, semiquantitative and quantitative methods can be applied to predict the probabilities of safe operation or the accidents [10]. Redundancies [5,6] and robustness [4] can also be considered. Service modes and effects analysis is an essential step towards understanding complex systems without which reliability and uncertainty analysis cannot be performed.

The procedures presented in this article are applied in addition to the traditional system analysis for the solution of practical numerical examples in engineering. The aim is to demonstrate the usefulness of the event oriented system analysis as a tool for assessment of system performances and improvements of engineering system design with respect to system uncertainties.

2. Uncertainty modelling

There are many uncertainties concerning systems and various methods of analysis. However, in the usually adopted random variable model, the uncertainties of the components are due to statistically uncertain capabilities regarding the geometry, the material properties, the workmanship, different uncertain demands, operational conditions and loads, as well as modelling and subjective uncertainties. The probabilistic system analysis is based on the application of probability theory to basic stochastic events, which can be defined by random design variables. Such an analysis provides system reliability or system failure probability using the usually set algebra.

The traditional probabilistic approach to discrete engineering systems (mechanical, structural, electrical, aerospace, nuclear, marine, etc.), with uncertain capabilities and operating under uncertain conditions, takes into account the random physical and technical characteristics of the components of the system and the stochastic environmental effects.

Moreover, systems may be considered at another level. In the event oriented system analysis, a system is defined not

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Nomenclature

A_i, E_i	Random events in general
$H(\cdot)$	Entropy of a system of events
$H^1(\cdot)$	Entropy of a system of events of order one
N, n	Numbers of systems and of subsystems
N_o, N_f	Number of operational events and of failure events
$p(\cdot)$, p_i	Probabilities of random events, (sub)systems
$\mathscr{S}$	System of events in general
$\mathscr{S}'$	System of subsystems of events
$\mathscr{S}_i$	Subsystem of events in general
$\mathscr{O}$, $\mathscr{F}$	Subsystems of operational and of failure events

only by its physical and/or technical components, but also by its all, or at least known or important states. Primarily, the states of the engineering system can be considered as operational or inoperational. The majority of the states are observable, but there may be some unobservable, undefined, unknown or less important states as well. Some of the states can be in common with several system features. The states of a system are represented by a system and by subsystems of random events in different relations and on various levels.

The word "uncertainty" in the context of engineering system of events should be given a more precise meaning [14]. The notion of uncertainty, applied to a system of events, is an uncertainty in the objective sense due to the fact that actually several events are possible. It is not the uncertainty in the mind of observers concerning the outcomes of an experiment [13]. The uncertainty arises from the number of events and the unpredictability of the events or subsystems.

In connection with the notion of uncertainty, the concept of information has to be mentioned. The uncertainty diminishes with the reception of relevant information. The uncertainty with respect to outcome may be considered equal to the information furnished by the occurrence of this outcome. Thus, uncertainty can also be measured. Terminology often alternates. The concept of entropy in the information theory was first applied to transmission of various information. Later, it was extended to the probability theory and engineering systems, providing more comprehensive definitions of system characteristics by introducing system uncertainties. A strong connection also exists between the notion of entropy in thermodynamics and the information theory.

The basic idea in this article is to make use of Shannon's entropy or information [12] to assess the uncertainty of systems and subsystems. In addition to the basic definition of entropy given by Shannon and Weaver for complete systems, Renyi's entropy [13] can be used to assess the uncertainty of incomplete system of events. The entropy can be considered both as the measure of the uncertainty, which prevailed before the experiment was accomplished, and as a measure of the information expected from an experiment. The theorem about the entropy associated with the mixture of distributions [13], and the theorem about dependent systems [9] can be used to assess the uncertainty of subsystems.

Possible ambiguities in a complex system reliability assessment can be resolved by uncertainty analysis. Such problems arouse when, for example, systems with the same reliability but different probability distributions of a number of operational and failure modes are evaluated.

3. Uncertainty measures

The uncertainty of a single stochastic event A with a known probability $p = p(A) \neq 0$ plays a fundamental role in the information theory. The entropy of a single stochastic event is defined as $E = H_1(p) = -\log_2 p(A)$ [15], and can be interpreted either as a measure of how unexpected the event was, or as a measure of the information yielded by the event [1].

More important than single stochastic events are the systems of events. Events are considered as abstract concepts and the relations among events are characterised axiomatically.

The algebraic structure of the set of event turns to the Boolean algebra [13].

A system of events: $E_1, E_2, \ldots, E_n$ is called a complete system of events if the following relations in the events space hold:

$$E_k \neq \emptyset \qquad (k = 1, 2, \ldots, n) \tag{1}$$

$$E_j E_k = \emptyset \qquad (\text{for } j \neq k) \tag{2}$$

$$E_1 + E_2 + \cdots + E_n = I \tag{3}$$

- The "$\emptyset$" in Eqs. (1) and (2) means an impossible event.
- The fact that E_j and E_k are exclusive is expressed in Eq. (2).
- Eq. (3) denotes that at least one of the events E_k, $k = 1, 2, \ldots, n$, occurs.
- The I denotes a definite event.

The definitions of complete and incomplete systems of events in probability space imply the following:

- A system of events E_k, $k = 1, 2, \ldots, n$, is said to be complete if for $i \neq j$, $A_i A_j = \emptyset$, and if the occurrence of an event E_k is "almost sure", i.e. if it has the property $p(\sum_k E_k) = \sum_k p(E_k) = 1$.

The definition also involves that one and only one event must occur in each trial.

- If some outcomes of an 'experiment' are not known, or their probabilities cannot be determined, or if only some of all possible events are taken into account because only they are observable, an incomplete probability distribution p_k can be considered. If for $p_k(p_k > 0;\ k = 1, 2, \ldots, n)$ there is $\sum_k p_k < 1$ and not necessarily $\sum_k p_k = 1$, the system is incomplete.

3.1. Uncertainty associated with a complete system of events

For a quantitative analysis of a system of events, e.g. in mathematics, the presentation of a system of events only in terms of probability space, i.e. by the distribution of probabilities of events, is sufficient. However, for more complex qualitative analysis of systems, as is usually the case in engineering, the systems and the subsystems of events can be presented both by the notion of events and by the appropriate probabilities associated with each of the events, denoted as a finite scheme [9]. Let us consider a system $\mathscr{S}$, constituted by the events E_i, $i = 1, 2, \ldots, n$, and with the appropriate probabilities associated with each of the events $p_i = p(E_i)$, presented as a finite scheme:

$$\mathscr{S} = \begin{pmatrix} E_1 & E_2 & \ldots & E_n \\ p_1 & p_2 & \ldots & p_n \end{pmatrix}$$

The entropy of the complete system of events $\mathscr{S}$ [12] is supposed to depend only on the probability distribution of considered events $\mathscr{P} = (p_1, p_2, \ldots, p_n)$ and can be denoted in different equivalent ways, as shown:

$$H(\mathscr{S}) = H_n(\mathscr{S}) = H_n(p_1, p_2, \ldots, p_n) = H_n(\mathscr{P})$$
$$= -\sum_{i=1}^{n} p_i \log p_i = \sum_{i=1}^{n} p_i \log \frac{1}{p_i} \qquad (4)$$

The quantity (4) is called the entropy of the complete probability distribution or the entropy of the complete system of events. It is known as Shannon's entropy or Shannon's information.

3.2. Uncertainty associated with incomplete systems of events

Another measure of uncertainty is Renyi's entropy of order α, which is defined for $\alpha \neq 1$, as:

$$H_n^{\alpha}(\mathscr{S}) = H^{\alpha}(\mathscr{S}) = \frac{1}{1-\alpha} \log_2\left(\sum_{i=1}^{n} p_i^{\alpha} / \sum_{i=1}^{n} p_i\right) \qquad (5)$$

The quantity (5) may also be viewed as a measure of the amount of uncertainty corresponding to either complete or incomplete distribution of probabilities pertaining to the system of events $\mathscr{S}$.

The entropy of an incomplete system of events $\mathscr{S}$ can be viewed as the limiting case of (5) for $\alpha \to 1$ and can be viewed as the arithmetic mean (expected value) of the single entropies $-\log p_i$ with weights p_i. Renyi himself denoted this quantity as the Shannon's entropy of order one [13]:

$$H_n^{1}(\mathscr{S}) = H^{1}(\mathscr{S}) = \left(-\sum_{i=1}^{n} p_i \log p_i\right) / \sum_{i=1}^{n} p_i \qquad (6)$$

Note that the entropy defined by Eq. (6) is often denoted as Renyi's entropy of order one [1]. This paper uses the notation Renyi's/Shannon's entropy of order one.

3.3. Properties of uncertainty measures

Some definitions and properties of entropy important in engineering are summarised.

- The definition of the unit of uncertainty is not more and not less arbitrary than the choice of the unit of some physical quantity. For example, if the logarithm applied in Eqs. (4)–(6) is of base two, the unit of entropy is denoted as one "bit". One bit is the uncertainty of a system of two equally probable events. If the natural logarithm is applied, the unit is denoted as one nit.
- Outcomes with zero probability do not change the uncertainty. By convention, $0 \log 0 = 0$.
- The entropy $H_n(\mathscr{S})$ is equal to zero, when the state of the system $\mathscr{S}$ can be surely predicted, i.e. no uncertainty exists at all. This occurs when one of the probabilities of events p_i, $i = 1, 2, \ldots, n$ is equal to one, let us say p_k, and all the other probabilities are equal to zero, $p_i = 0$, $i \neq k$.
- The entropy is maximal when all events are equally probable, and the probability of failure is equal to $p_i = 1/n$, for $i = 1, 2, \ldots, n$, and it amounts to $H_n(\mathscr{S})_{\max} = \log n$ [7]. Hartley's entropy corresponds to the Renyi's entropy of order 0 [1].
- The entropy increases as the number of events increase.
- The entropy does not depend on the sequence of events: $H_n(p_1, p_2, \ldots, p_n) = H_n(p_{k(1)}, p_{k(2)}, \ldots, p_{k(n)})$, where k is an arbitrary permutation on $(1, 2, \ldots, n)$.
- The entropy is the only function appropriate for the uncertainty measure (the uniqueness theorem) [2,9].

There are other important properties of the entropy concerning composite events.

- For two independent systems of events, $\mathscr{A} = (A_1, A_2, \ldots, A_m)$ and $\mathscr{B} = (B_1, B_2, \ldots, B_n)$, where the probability of the occurrence of two states of the systems is defined by $p(A_i \cap B_j) = p(A_i)p(B_j)$, the entropy of a system which is called the direct product of distributions denoted as $\mathscr{A}\mathscr{B}$, is defined as follows (additivity of entropy):

$$H(\mathscr{A}\mathscr{B}) = H(\mathscr{A}) + H(\mathscr{B}) \qquad (7)$$

- If the systems of events, $\mathscr{A} = (A_1, A_2, \ldots, A_m)$ and $\mathscr{B} = (B_1, B_2, \ldots, B_n)$, are dependent, where the probability of the occurrence of two states is defined by $p(A_i \cap B_j) = p(A_i)p(B_j/A_i)$, the entropy of the compound system denoted as $\mathscr{A}\mathscr{B}$ [9], is defined as follows:

$$H(\mathscr{A}\mathscr{B}) = H(\mathscr{A}) + H(\mathscr{B}/\mathscr{A}) \tag{8}$$

$H(\mathscr{B}/\mathscr{A})$ in Eq. (8) is the average entropy of the system $\mathscr{B}$ with respect to the system $\mathscr{A}$ and represents the conditional entropy of system $\mathscr{B}$ with respect to the system $\mathscr{A}$:

$$H(\mathscr{B}/\mathscr{A}) = \sum_i p(A_i)\cdot H(\mathscr{B}/A_i) \tag{9}$$

$H(\mathscr{B}/A_i)$ in Eq. (9) is the conditional entropy of system $\mathscr{B}$ with respect to the event A_i in system $\mathscr{A}$.

4. Subsystems of events

Let us consider a system $\mathscr{S}$ of N disjoint events E_{ij}, with appropriate probabilities $p(E_{ij})$:

$$\mathscr{S} = \begin{pmatrix} E_{11} & \dots & E_{1m_1} & \dots & E_{i1} & \dots & E_{im_i} & \dots & E_{n1} & \dots & E_{nm_n} \\ p(E_{11}) & \dots & p(E_{1m_1}) & \dots & p(E_{i1}) & \dots & p(E_{im_i}) & \dots & p(E_{n1}) & \dots & p(E_{nm_n}) \end{pmatrix}$$

The events of system $\mathscr{S}$ can be grouped regardless of the ordering of events into subsystems of events $\mathscr{S}_i$, $i = 1, 2, \dots, n$, each containing E_{ij}, $j = 1, 2, \dots, m_i$ elements, as presented:

$$\mathscr{S}_i = \begin{pmatrix} E_{i1} & \dots & E_{ij} & \dots & E_{im_i} \\ p(E_{i1}) & \dots & p(E_{ij}) & \dots & p(E_{im_i}) \end{pmatrix}$$

The probability $p(\mathscr{S}_i)$ associated with each of the subsystems $\mathscr{S}_i$, $i = 1, 2, \dots, n$, is as follows:

$$p(\mathscr{S}_i) = \sum_{j=1}^{m_i} p(E_{ij}) \tag{10}$$

Neither the probability nor the entropy depends on the sequence of events within the subsystem $\mathscr{S}_i$. Note that the subsystems $\mathscr{S}_i$ can be either exclusive or inclusive, having some common events.

The system $\mathscr{S}$ can be in general presented as a union of subsystems of events $\mathscr{S}_i$, as follows:

$$\mathscr{S} = (\mathscr{S}_1 \cup \dots \cup \mathscr{S}_i \cup \dots \cup \mathscr{S}_n)$$

For strictly disjoint subsystems of events $\mathscr{S}_i$, the system $\mathscr{S}$ can be presented as:

$$\mathscr{S} = (\mathscr{S}_1 + \dots + \mathscr{S}_i + \dots + \mathscr{S}_n)$$

The probability associated with system $\mathscr{S}$ is defined as:

$$p(\mathscr{S}) = p\Big(\bigcup_{i=1}^{n} \mathscr{S}_i\Big) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} p(E_{ij}) \tag{11}$$

The probability distribution $p(E_{ij})$ associated to a subsystem $\mathscr{S}_i$ is considered as a partial distribution of probabilities of the entire system $\mathscr{S}$. To every partial probability distribution, there can be assigned an ordinary probability distribution by the substitution of $p(E_{ij}/\mathscr{S}_i)$ instead of $p(E_{ij})$, which may be interpreted as a complete conditional distribution of probabilities with respect to the condition that the system $\mathscr{S}_i$ occurs with probability $p(\mathscr{S}_i)$. It is obvious that $p(E_{ij}/\mathscr{S}_i)\cdot p(\mathscr{S}_i) = p(E_{ij})$.

Each subsystem $\mathscr{S}_i$ can also be viewed under the condition that in the system of all known modes $\mathscr{S}$, only the subsystem $\mathscr{S}_i$ occurs. The conditional probability of system $p(\mathscr{S}/\mathscr{S}_i)$ depends only on the probability of subsystem itself, i.e. $p(\mathscr{S}/\mathscr{S}_i) = p(\mathscr{S}_i)$. Obviously, $p(\mathscr{S}_i/\mathscr{S}_i) = 1$.

If the system $\mathscr{S}$ is not a complete system of events, i.e. $p(\mathscr{S}) < 1$, an incomplete system of events has to be considered. The appropriate uncertainty of the incomplete system $\mathscr{S}$ should be presented by the Renyi's/Shannon's entropy of order one as follows:

$$H_N^1(\mathscr{S}) = H_N(\mathscr{S})/p(\mathscr{S}) \tag{12}$$

The system $\mathscr{S}$ can also be considered under the condition that only the observable events are of interest. The conditional entropy of system $\mathscr{S}$ can be obtained as follows:

$$H_N(\mathscr{S}/\mathscr{S}) = -\sum_{i=1}^{N} \frac{p(E_i)}{p(\mathscr{S})} \log \frac{p(E_i)}{p(\mathscr{S})} = H_N^1(\mathscr{S}) + \log p(\mathscr{S}) \tag{13}$$

The summa applied in Eq. (12) is calculated as shown:

$$H_N(\mathscr{S}) = -\sum_{i=1}^{n} \sum_{j=1}^{m_i} p(E_{ij}) \log p(E_{ij}) = \sum_{i=1}^{n} H_{m_i}(\mathscr{S}_i) \tag{14}$$

The maximal attainable entropy of systems $\mathscr{S}$ for either complete or incomplete systems, is obtained for $N = \sum_{i=1}^{n} m_i$ equally probable events in amount of $H_N^1(\mathscr{S})_{\max} = \log(N/p(\mathscr{S}))$.

4.1. Uncertainty associated with subsystems of events

The theorem about the entropy associated with mixture of distributions [13] can be applied to assess the uncertainty of a subsystem of events. A mixing distribution may be reinterpreted in terms of subsystems. Subsystems of events are considered to be associated with a mixture of partial distributions [16]. The uncertainty of the subsystem $\mathscr{S}_i$, regardless of its exclusiveness or inclusiveness with respect to other subsystems, can be expressed as the Shannon's entropy applied only to the partial probability distribution of the system $\mathscr{S}$ considered under the condition that the subsystem $\mathscr{S}_i$ occurs. Such a condition entropy does not depend on the system probability $p(\mathscr{S})$, being independent of whether the system $\mathscr{S}$ is complete or incomplete.

According to the definition of the entropy (4), and using conditional probabilities of the occurrence of subsystems, it follows:

$$H_{m_i}(\mathscr{S}/\mathscr{S}_i) = -\sum_{j=1}^{m_i} \frac{p(E_{ij})}{p(\mathscr{S}_i)} \log \frac{p(E_{ij})}{p(\mathscr{S}_i)} \tag{15}$$

The entropies of the subsystems in Eq. (15) depend only on the states of the subsystems $\mathcal{S}_i$ itself, not on any other states of the system $\mathcal{S}$ and on the system $\mathcal{S}$ itself.

The following relation expresses the loss of the entropy of a subsystem $\mathcal{S}_i$ considered as a partial distribution, with respect to the entropy $H^1(\mathcal{S}_i)$ of a system $\mathcal{S}_i$ viewed as an incomplete system:

$$H_{m_i}(\mathcal{S}/\mathcal{S}_i) = H^1_{m_i}(\mathcal{S}_i) + \log p(\mathcal{S}_i) \tag{16}$$

The Renyi's/Shannon's entropy of order one $H^1_{mi}(\mathcal{S}_i)$ in Eq. (16) correspond to the entropy of incomplete system of event defined as:

$$H^1_{m_i}(\mathcal{S}_i) = H_{m_i}(\mathcal{S}_i)/p(\mathcal{S}_i) \tag{17}$$

The partial summa $H_{mi}(\mathcal{S}_i)$ in Eqs (14) and (17) within the system $\mathcal{S}$ corresponds to the partial probability distribution of subsystem of events $\mathcal{S}_i$, and can be calculated as:

$$H_{m_i}(\mathcal{S}_i) = -\sum_{j=1}^{m_i} p(E_{ij}) \log p(E_{ij}) \tag{18}$$

The maximal attainable conditional entropy of the subsystem $\mathcal{S}_i$ is obtained for m_i equally probable events and amounts as:

$$H_{m_i}(\mathcal{S}/\mathcal{S}_i)_{\max} = \log m_i \tag{19}$$

4.2. The relation of the uncertainties of the system and of disjoint subsystems

The system of $\mathcal{S}$ can be viewed also as a compound of n subsystems $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$, denoted as $\mathcal{S}'$:

$$\mathcal{S}' = (\mathcal{S}_1, \dots, \mathcal{S}_i, \dots, \mathcal{S}_n) = \begin{pmatrix} \mathcal{S}_1 & \dots & \mathcal{S}_i & \dots & \mathcal{S}_n \\ p(\mathcal{S}_1) & \dots & p(\mathcal{S}_i) & \dots & p(\mathcal{S}_n) \end{pmatrix}$$

The probability associated with system $\mathcal{S}'$ is $p(\mathcal{S}') = \sum_{i=1}^{n} p(\mathcal{S}_i) = p(\mathcal{S}) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} p(E_{ij})$.

The maximal attainable entropy of system $\mathcal{S}'$ for either complete or incomplete systems, is obtained for n equality probable events as $H^1_n(\mathcal{S}')_{\max} = \log(n/p(\mathcal{S}'))$.

If the system $\mathcal{S}'$ is not complete systems of events, i.e. $p(\mathcal{S}') < 1$, then incomplete systems of events have to be considered. The appropriate uncertainty of the incomplete system $\mathcal{S}'$ should be presented by the Renyi's/Shannon's entropy of order one as follows:

$$H^1_n(\mathcal{S}') = H_n(\mathcal{S}')/p(\mathcal{S}') \tag{20}$$

The system $\mathcal{S}'$ can also be considered under the condition that only the observable subsystems of events are of interest. The conditional entropy of system $\mathcal{S}'$ can be obtained as follows:

$$H_n(\mathcal{S}'/\mathcal{S}') = -\sum_{i=1}^{n} \frac{p(\mathcal{S}_i)}{p(\mathcal{S})} \log \frac{p(\mathcal{S}_i)}{p(\mathcal{S})} = H^1_n(\mathcal{S}') + \log p(\mathcal{S}) \tag{21}$$

The summa applied in Eq. (20) is calculated as:

$$H_n(\mathcal{S}') = -\sum_{i=1}^{n} p(\mathcal{S}_i) \log p(\mathcal{S}_i) \tag{22}$$

The general relation of the probabilities and conditional entropies of complete or incomplete system and those of the disjoint subsystems, can be derived by taking the weighted summa of all the conditional entropies of subsystems [16], as follows:

$$\sum_{i=1}^{n} p(\mathcal{S}_i)\cdot H_{m_i}(\mathcal{S}/\mathcal{S}_i) = p(\mathcal{S})\cdot[H_N(\mathcal{S}/\mathcal{S}) - H_n(\mathcal{S}'/\mathcal{S}')] = p(\mathcal{S})\cdot[H^1_N(\mathcal{S}) - H^1_n(\mathcal{S}')] \tag{23}$$

The expression (23) may be considered as a straightforward application of the theorem about the entropy associated with a system of disjoint subsystems. The average of the entropy of the subsystems $\mathcal{S}_i$, with weights equal to the associated probabilities $p(\mathcal{S}_i)$, is equal to the entropy of the system of events $\mathcal{S}$, reduced for the entropy of the system of subsystems $\mathcal{S}'$.

The reduction in the entropy of system $\mathcal{S}$ is a consequence of the knowledge about its partitioning into subsystems. The relation (23) does not depend on whether the systems $\mathcal{S}$ and $\mathcal{S}'$ are complete systems, i.e. $p(\mathcal{S}) = p(\mathcal{S}') = 1$, $H_N(\mathcal{S}) = H^1_N(\mathcal{S})$ and $H_n(\mathcal{S}') = H^1_n(\mathcal{S}')$, or incomplete systems due to $p(\mathcal{S}) = p(\mathcal{S}') < 1$, $H_N(\mathcal{S}) = H^1_N(\mathcal{S}) + \log p(\mathcal{S})$ and $H_n(\mathcal{S}') = H^1(\mathcal{S}') + \log p(\mathcal{S}')$.

The expression (23), can be also be rewritten in terms of entropies of incomplete systems as follows:

$$\sum_{i=1}^{n} p(\mathcal{S}_i)\cdot H^1_{m_i}(\mathcal{S}_i) = p(\mathcal{S})\cdot H^1_N(\mathcal{S}) \tag{24}$$

4.2.1. The relation of the uncertainties of k out of n disjoint subsystems

In more general terms, the relation among any k out of n disjoint subsystems of events can be derived as:

$$\sum_{i=1}^{k} p(\mathcal{S}_i)\cdot H(\mathcal{S}/\mathcal{S}_i) = \sum_{i=1}^{k} p(\mathcal{S}_i)\cdot\{H[\mathcal{S}/(\mathcal{S}_1 + \mathcal{S}_2 + \cdots + \mathcal{S}_k)] - H[\mathcal{S}/(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k)]\}$$
$$= \sum_{i=1}^{k} p(\mathcal{S}_i)\cdot[H^1(\mathcal{S}_1 + \mathcal{S}_2 + \cdots + \mathcal{S}_k) - H^1(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k)]$$
$$= H(\mathcal{S}_1 + \mathcal{S}_2 + \cdots + \mathcal{S}_k) - H(\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k) \tag{25}$$

The terms in Eq. (25) are defined as follows:

$$H(\mathscr{S}/\mathscr{S}_1+\mathscr{S}_2+\cdots+,\mathscr{S}_k)$$

$$=-\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{p(E_{ij})}{\sum_{m=1}^{k}p(\mathscr{S}_m)}\log\frac{p(E_{ij})}{\sum_{m=1}^{k}p(\mathscr{S}_m)}$$

$$=H^1(\mathscr{S}_1+\mathscr{S}_2+\cdots+\mathscr{S}_k)+\log\sum_{i=1}^{k}p(\mathscr{S}_i) \quad (26)$$

$$H^1(\mathscr{S}_1+\mathscr{S}_2+\cdots+\mathscr{S}_k)=\frac{H(\mathscr{S}_1+\mathscr{S}_2+\cdots+\mathscr{S}_k)}{\sum_{i=1}^{k}p(\mathscr{S}_i)}$$

$$=\frac{\sum_{i=1}^{k}H(\mathscr{S}_i)}{\sum_{i=1}^{k}p(\mathscr{S}_i)} \quad (27)$$

$$H(\mathscr{S}_1+\mathscr{S}_2+\cdots+\mathscr{S}_k)=-\sum_{i=1}^{k}\sum_{j=1}^{m_i}p(E_{ij})\log p(E_{ij})$$

$$=\sum_{i=1}^{k}H(\mathscr{S}_i) \quad (28)$$

$$H(\mathscr{S}/\mathscr{S}_1,\mathscr{S}_2,\ldots,\mathscr{S}_k)=-\sum_{i=1}^{k}\frac{p(\mathscr{S}_i)}{\sum_{m=1}^{k}p(\mathscr{S}_m)}\log\frac{p(\mathscr{S})}{\sum_{m=1}^{k}p(\mathscr{S}_m)}$$

$$=H^1(\mathscr{S}_1,\mathscr{S}_2,\ldots,\mathscr{S}_k)+\log\sum_{i=1}^{k}p(\mathscr{S}_i) \quad (29)$$

$$H^1(\mathscr{S}_1,\mathscr{S}_2,\ldots,\mathscr{S}_k)=\frac{H(\mathscr{S}_1,\mathscr{S}_2,\ldots,\mathscr{S}_k)}{\sum_{i=1}^{k}p(\mathscr{S}_i)} \quad (30)$$

$$H(\mathscr{S}_1,\mathscr{S}_2,\ldots,\mathscr{S}_k)=-\sum_{i=1}^{k}p(\mathscr{S}_i)\log p(\mathscr{S}_i) \quad (31)$$

The weighted summa on the left-hand side in Eq. (25) does not depend on whether $\mathscr{S}$ and $\mathscr{S}'$ are complete or incomplete systems.

The relation (25) can be also be rewritten in terms of entropies of incomplete systems as follows:

$$\sum_{i=1}^{k}[p(\mathscr{S}_i)\cdot H^1_{m_i}(\mathscr{S}_i)]=\left[\sum_{i=1}^{k}p(\mathscr{S}_i)\right]\cdot H^1_k(\mathscr{S}_1+\mathscr{S}_2+\cdots+\mathscr{S}_k) \quad (32)$$

4.2.2. The relation of the uncertainties of the disjoint subsystems on arbitrary partitioning level

Some may be interested in uncertainties associated with subsystems at an optional level of system or subsystem partitioning. Let us suppose that any of the subsystems $\mathscr{S}_i$ of n_i events E can be built up of m_i disjoint sub-subsystems. $\mathscr{S}_{ij}$, each consisting of n_{ij}, $j=1,2,\ldots,n_i$, appropriate basic events E as defined earlier. Such a subsystem can be presented as a subsystem of subsystem $\mathscr{S}'_i$ as follows:

$$\mathscr{S}'_i=\begin{pmatrix}\mathscr{S}_{i1} & \ldots & \mathscr{S}_{ij} & \ldots & \mathscr{S}_{im_i}\\ p(\mathscr{S}_{i1}) & \ldots & p(\mathscr{S}_{ij}) & \ldots & p(\mathscr{S}_{im_i})\end{pmatrix}$$

The uncertainty of the sub-subsystem $\mathscr{S}_{ij}$ can be expressed as the Shannon's entropy applied only to the partial distribution of the subsystem $\mathscr{S}_i$ considered under the condition that the subsystem $\mathscr{S}_{ij}$ occurs. Such a conditional entropy does not depend on the probability $p(\mathscr{S}_i)$, also being independent of whether the system $\mathscr{S}$ is complete or incomplete. According to Eq. (4), it follows:

$$H(\mathscr{S}_i/\mathscr{S}_{ij})=-\sum_{j=1}^{n_{ij}}\frac{p(E_{ij})}{p(\mathscr{S}_{ij})}\log\frac{p(E_{ij})}{p(\mathscr{S}_{ij})}$$

$$=H^1(\mathscr{S}_{ij})+\log p(\mathscr{S}_{ij})=\frac{H(\mathscr{S}_{ij})}{p(\mathscr{S}_{ij})}+\log p(\mathscr{S}_{ij}) \quad (33)$$

The relation among the uncertainties of sub-subsystems can be derived analogously to the relation (23), and it represents the conditional entropy of subsystem $\mathscr{S}_i$ with respect to subsystem $\mathscr{S}'_i$ as:

$$H(\mathscr{S}_i/\mathscr{S}'_i)=\sum_{j=1}^{m_i}p(\mathscr{S}_{ij})\cdot H(\mathscr{S}_i/\mathscr{S}_{ij})$$

$$=p(\mathscr{S}_i)[H^1(\mathscr{S}_i)-H^1(\mathscr{S}'_i)]$$

$$=p(\mathscr{S}_i)\cdot[H(\mathscr{S}/\mathscr{S}_i)-H(\mathscr{S}'/\mathscr{S}'_i)] \quad (34)$$

The relation (34) for subsystem $\mathscr{S}_i$, as a partial distribution of a known complete or incomplete distribution, does not depend on the other states of the system $\mathscr{S}$ which are not in $\mathscr{S}_i$.

In general, the uncertainty of the subsystems of events at any level, considered under the condition that only the events constituting the subsystem occur, depends only on the events pertaining to the subsystem itself, and not on the other events of the system. The relation (34) can also be rewritten in terms of entropies of incomplete systems as:

$$\sum_{j=1}^{m_i} p(\mathscr{S}_{ij})\cdot H^1(\mathscr{S}_{ij}) = p(\mathscr{S}_i)\cdot H^1(\mathscr{S}_i) \tag{35}$$

The Renyi's/Shannon's entropy of order one $H^1(\mathscr{S})$ can be applied instead of the Shannon's entropy $H(\mathscr{S})$, for both complete and incomplete systems. For $p(\mathscr{S}) = 1$, since all the relations are the same, regardless of whether the considered system is either a complete or an incomplete one.

4.2.3. Uncertainty associated with dependent systems of events

Another interpretation can be given to the relation of the entropy of system and disjoint subsystems. Consider systems $\mathscr{S}' = (\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_n)$ and $\mathscr{S} = (\mathscr{S}_1 + \mathscr{S}_2 + \cdots + \mathscr{S}_n)$ as dependent.

The entropy of two dependent systems can be obtained according to Eq. (8) as follows:

$$H(\mathscr{S}'\mathscr{S}) = H(\mathscr{S}') + H(\mathscr{S}/\mathscr{S}') \tag{36}$$

The conditional entropy $H(\mathscr{S}/\mathscr{S}')$ of system $\mathscr{S}$ with respect to system $\mathscr{S}'$ in Eq. (36) is on the basis of Eq. (9), equals to the following term:

$$\begin{aligned} H(\mathscr{S}/\mathscr{S}') &= p(\mathscr{S}_1)\cdot H(\mathscr{S}/\mathscr{S}_1) + p(\mathscr{S}_2)\cdot H(\mathscr{S}/\mathscr{S}_2) + \cdots \\ &\quad + p(\mathscr{S}_n)\cdot H(\mathscr{S}/\mathscr{S}_n) \\ &= \sum_{i=1}^{n} p(\mathscr{S}_i)\cdot H(\mathscr{S}/\mathscr{S}_i) \end{aligned} \tag{37}$$

In Eq. (37), the terms $H(\mathscr{S}/\mathscr{S}_i)$, $i = 1, 2, \ldots, n$ are the entropy of system $\mathscr{S}$ under the condition that subsystems $\mathscr{S}_i$ occur. In the case when the states of system $\mathscr{S}'$ are entirely defined by states of system $\mathscr{S}$, as it is in these considerations, the following relation holds: $H(\mathscr{S}'\mathscr{S}) = H(\mathscr{S})$.

Finally, using the theorem about dependent systems according to Eqs. (8) and (9), the same result for the relation of the uncertainties of the system and of the subsystems is obtained by using the theorem about mixture of distributions for complete systems of events in Eq. (23), as given next:

$$\begin{aligned} H(\mathscr{S}/\mathscr{S}') &= p(\mathscr{S}_1)\cdot H(\mathscr{S}/\mathscr{S}_1) + p(\mathscr{S}_2)\cdot H(\mathscr{S}/\mathscr{S}_2) + \cdots \\ &\quad + p(\mathscr{S}_n)\cdot H(\mathscr{S}/\mathscr{S}_n) \\ &= H(\mathscr{S}) - H(\mathscr{S}') \end{aligned} \tag{38}$$

The term (38) can also be viewed as a conditional entropy of a partial system $(\mathscr{S}_1 + \mathscr{S}_2 + \cdots \mathscr{S}_k)$ with respect to the system $(\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_k)$, and can also be denoted as $H[(\mathscr{S}_1 + \mathscr{S}_2 + \cdots + \mathscr{S}_k)/(\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_k)]$.

4.3. Uncertainties of inclusive subsystems of events

Let us suppose that the system of events $\mathscr{S}$ contains two subsystems with some common events. The first subsystem is denoted as $\mathscr{S}_p$, containing elements E_{pl}, $l = 1, 2, \ldots, m_p$ and the second is denoted as $\mathscr{S}_q$, containing elements E_{qk}, $k = 1, 2, \ldots, m_q$, as presented next:

$$\mathscr{S}_p = \begin{pmatrix} E_{p1} & \ldots & E_{pl} & \ldots & E_{pm_p} \\ p(E_{p1}) & \ldots & p(E_{pl}) & \ldots & p(E_{pm_p}) \end{pmatrix}$$

$$\mathscr{S}_q = \begin{pmatrix} E_{q1} & \ldots & E_{ql} & \ldots & E_{qm_q} \\ p(E_{q1}) & \ldots & p(E_{ql}) & \ldots & p(E_{qm_q}) \end{pmatrix}$$

The system $\mathscr{S}$ can be presented more generally as a union of more subsystem of events as shown:

$$\mathscr{S} = (\mathscr{S}_1 \cup \ldots \cup \mathscr{S}_p \cup \mathscr{S}_q \cup \ldots \cup \mathscr{S}_n)$$

The probabilities $p(\mathscr{S}_p)$ and $p(\mathscr{S}_q)$ associated with the subsystems $\mathscr{S}_p$ and $\mathscr{S}_q$, are calculated as

$$p(\mathscr{S}_p) = \sum_{l=1}^{m_p} p(E_{pl}) \qquad \text{and} \qquad p(\mathscr{S}_q) = \sum_{l=1}^{m_q} p(E_{ql}).$$

The conditional entropies of the system $\mathscr{S}$ with respect to the subsystems $\mathscr{S}_p$ and $\mathscr{S}_q$, are calculated according to Eq. (15) as follows:

$$H_{m_p}(\mathscr{S}/\mathscr{S}_p) = -\sum_{l=1}^{m_p} \frac{p(E_{pl})}{p(\mathscr{S}_p)} \log \frac{p(E_{pl})}{p(\mathscr{S}_p)} \tag{39}$$

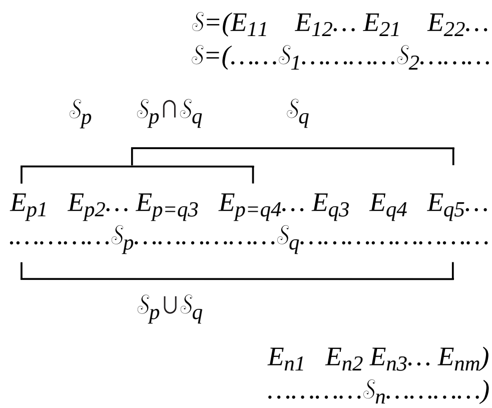
$$H_{m_q}(\mathscr{S}/\mathscr{S}_q) = -\sum_{l=1}^{m_q} \frac{p(E_{ql})}{p(\mathscr{S}_q)} \log \frac{p(E_{ql})}{p(\mathscr{S}_q)} \tag{40}$$

The conditional entropies of the system $\mathscr{S}$ with respect to the subsystems $\mathscr{S}_p$ and $\mathscr{S}_q$ in Eqs. (39) and (40) depend only on the states of the subsystems, and not on any other state of the system.

Let us suppose that there are $r = 1, 2, \ldots, m_{pq}$ common events denoted as $E_{ps(r)} = E_{qt(r)}$, where s and t are appropriate selections of common events. The subsystem containing common events is a subsystem too, and is to be considered as an intersection of two events as follows:

$$\mathscr{S}_p \cap \mathscr{S}_q = \begin{pmatrix} E_{ps(1)} = E_{qt(1)} & \ldots & E_{ps(l)} = E_{qt(l)} & \ldots & p(E_{ps(m_{ij})} = E_{qt(m_{ij})}) \\ p(E_{ps(1)} = E_{qt(l)}) & \ldots & p(E_{ps(l)} = E_{qt(l)}) & \ldots & E_{ps(m_{ij})} = E_{qt(m_{ij})} \end{pmatrix}$$

The system $\mathcal{S}$ can be presented as a summa of disjoint subsystems of events as follows:



The probability of the intersection of two subsystems $\mathcal{S}_p$ and $\mathcal{S}_q$, can be defined as

$$p(\mathcal{S}_p \cap \mathcal{S}_q) = \sum_{l=1}^{m_{ij}} p(E_{ps(l)} = E_{qt(l)})$$

The conditional entropy of the system $\mathcal{S}$ with respect to the intersection of two subsystems of events $\mathcal{S}_p$ and $\mathcal{S}_q$, is calculated based on Eq. (15) as follows:

$$H_{m_{pq}}(\mathcal{S}/\mathcal{S}_p \cap \mathcal{S}_q) = -\sum_{l=1}^{m_{pq}} \frac{p(E_{ps(l)} = E_{qt(l)})}{p(\mathcal{S}_p \cap \mathcal{S}_q)} \log \frac{p(E_{ps(l)} = E_{qt(l)})}{p(\mathcal{S}_p \cap \mathcal{S}_q)} \quad (41)$$

It is of interest to find out the entropy of a union of subsystems considered as a system of $m = m_p + m_q - m_{pq}$ elements. The probability of the union of two subsystems can be defined as

$$p(\mathcal{S}_p \cup \mathcal{S}_q) = p(\mathcal{S}_p) + p(\mathcal{S}_q) - p(\mathcal{S}_p \cap \mathcal{S}_q).$$

By taking the weighted summa of all the conditional entropies of system $\mathcal{S}$ with respect to the subsystems $\mathcal{S}_p$ and $\mathcal{S}_q$, as well as their intersection, the following relation to the probability and uncertainty of the union of events can be derived:

$$\begin{aligned} &p(\mathcal{S}_p)\cdot[H_{m_p}(\mathcal{S}/\mathcal{S}_p) - \log p(\mathcal{S}_p)] + p(\mathcal{S}_q)\cdot[H_{m_q}(\mathcal{S}/\mathcal{S}_q) \\ &\quad - \log p(\mathcal{S}_q)] - p(\mathcal{S}_p \cap \mathcal{S}_q)\cdot[H_{m_{pq}}(\mathcal{S}/\mathcal{S}_p \cap \mathcal{S}_q) \\ &\quad - \log p(\mathcal{S}_p \cap \mathcal{S}_q)] \\ &\quad = p(\mathcal{S}_p \cup \mathcal{S}_q)\cdot[H_{m_{pq}}(\mathcal{S}/\mathcal{S}_p \cup \mathcal{S}_q) - \log p(\mathcal{S}_p \cup \mathcal{S}_q)] \end{aligned} \quad (42)$$

The next term may be denoted as the conditional entropy of the system of subsystem denoted as $\mathcal{S}'$ with respect to the union of the system of subsystems $\mathcal{S}_p$ and $\mathcal{S}_q$:

$$\begin{aligned} H_2(\mathcal{S}'/\mathcal{S}_p \cup \mathcal{S}_q) &= \frac{-p(\mathcal{S}_p)\cdot\log p(\mathcal{S}_p) - p(\mathcal{S}_q)\cdot\log p(\mathcal{S}_q)}{p(\mathcal{S}_p \cup \mathcal{S}_q)} \\ &\quad \frac{+p(\mathcal{S}_p \cup \mathcal{S}_q) + p(\mathcal{S}_p \cap \mathcal{S}_q)\cdot\log p(\mathcal{S}_p \cap \mathcal{S}_q)}{p(\mathcal{S}_p \cup \mathcal{S}_q)} \\ &\quad + \log p(\mathcal{S}_p \cup \mathcal{S}_q) \end{aligned} \quad (43)$$

After the substitution of Eq. (43) in Eq. (42), the following simplified relation is derived:

$$\begin{aligned} &p(\mathcal{S}_p)\cdot H_{m_p}(\mathcal{S}/\mathcal{S}_p) + p(\mathcal{S}_q)\cdot H_{m_q}(\mathcal{S}/\mathcal{S}_q) \\ &\quad - p(\mathcal{S}_p \cap \mathcal{S}_q)\cdot H_{m_{pq}}(\mathcal{S}/\mathcal{S}_p \cap \mathcal{S}_q) \\ &\quad = p(\mathcal{S}_p \cup \mathcal{S}_q)\cdot[H_{m_{pq}}(\mathcal{S}/\mathcal{S}_p \cup \mathcal{S}_q) - H_2(\mathcal{S}'/\mathcal{S}_p \cup \mathcal{S}_q)] \end{aligned} \quad (44)$$

The term (42) can also be rewritten in terms of entropies of incomplete systems as follows:

$$\begin{aligned} &p(\mathcal{S}_p)\cdot H^1_{m_p}(\mathcal{S}_p) + p(\mathcal{S}_q)\cdot H^1_{m_q}(\mathcal{S}_q) \\ &\quad - p(\mathcal{S}_p \cap \mathcal{S}_q)\cdot H^1_{m_{pq}}(\mathcal{S}_p \cap \mathcal{S}_q) \\ &\quad = p(\mathcal{S}_p \cup \mathcal{S}_q)\cdot H^1_{m_{pq}}(\mathcal{S}_p \cup \mathcal{S}_q) \end{aligned} \quad (45)$$

5. Engineering systems of events

Let us suppose that there is a number of, let us say, n_c physical or technical components of an engineering system. The observable outcomes associated with the component can be denoted as basic events or modes. Even inclusive events or common cause events are random events, and have to be identified as basic events. The basic event may happen, when denoted A_i, or not, when denoted $\bar{A}_i$, $i = 1, 2, \ldots, n_e$, which also represents an event, sometimes called simple alternative. The n_e is the total number of basic events, not necessarily equal to the number of components n_c. The quantitative methods of system analysis require component operational data about basic events, such as the probability of proper operation $R_i = p(A_i)$ or the probability to fail $P_{f,i} = p(\bar{A}_i) = 1 - p(A_i)$. A system $\mathcal{E}_j$ of two events representing only one of the states of a component can be represented as

$$\mathcal{E}_j = \begin{pmatrix} A_j & \bar{A}_j \\ p(A_j) & p(\bar{A}_j) \end{pmatrix}.$$

The uncertainty that a single state of a component is operational or fails can be expressed as the entropy of the system of two events $\mathcal{E}_j$, as: $H(\mathcal{E}_j) = -p(A_j) \log p(A_j) - p(\bar{A}_j) \log p(\bar{A}_j)$. The maximal entropy for two equally probable events amounts $H_2(\mathcal{E}_j)_{\max} = \log_2 2 = 1$.

According to an operational modes and effects analysis, all possible or at least all-relevant and observable events E_i of a system could be determined using basic events. Methods such as enumeration, event-tree and fault-tree analysis are at disposal. Some of the events E_i can be regarded as operational modes, denoted as E_i^o (status = O), whilst some events can be regarded as failure modes, denoted with E_i^f (status = F). The probabilities of possible modes can hopefully be calculated using quantitative methods, and will be denoted $p(E_i), i = 1, 2, \ldots, N$. N is the total number of all known, or at least all observable possible events constituting system of event $\mathscr{S}$.

The system $\mathscr{S}$ can also be presented as a summa of operational and failure subsystem as shown:

$$\mathscr{S} = \begin{pmatrix} E_1 & E_2 & \ldots & E_N \\ p(E_1) & p(E_2) & \ldots & p(E_N) \end{pmatrix} = (\mathscr{O} + \mathscr{F})$$

Consider also in more detail the two important subsystems of the system $\mathscr{S}$. The first, denoted $\mathscr{O}$, comprises all of the events E, denoted E_i^o, $i = 1, 2, \ldots, N_o$, when the system is operating. The second, denoted $\mathscr{F}$, consists of events E_i^f, $i = N_o + 1, \ldots, N_o + N_f$ when the system fails:

$$\mathscr{O} = \begin{pmatrix} E_1^o & E_2^o & \ldots & E_{N_o}^o \\ p(E_1^o) & p(E_2^o) & \ldots & p(E_{N_o}^o) \end{pmatrix}$$

$$\mathscr{F} = \begin{pmatrix} E_{N_o+1}^f & E_{N_o+2}^f & \ldots & E_{N_o+N_f}^f \\ p(E_{N_o+1}^f) & p(E_{N_o+2}^f) & \ldots & p(E_{N_o+N_f}^f) \end{pmatrix}$$

The total number of events is equal to $N_o + N_f = N$. It may be also noted that the sequence of events within the system or within the subsystems is irrelevant with respect to intended reliability and uncertainty considerations.

The overall reliability of the system corresponds to all of the outcomes when the system is operating, and can be calculated as the probability of the subsystem of operational modes $p(\mathscr{O})$:

$$R(\mathscr{S}) = p(\mathscr{O}) = \sum_{i=1}^{N_o} p(E_i^o) \tag{46}$$

The appropriate failure probability of the system corresponds to all of the outcomes when the system fails and can be calculated as the probability of the subsystem of failure modes $p(\mathscr{F})$:

$$P_f(\mathscr{S}) = p(\mathscr{F}) = \sum_{i=N_o+1}^{N_o+N_f} p(E_i^f) \tag{47}$$

In any case, either for complete systems or for incomplete systems, the next relation holds:

$$P(\mathscr{S}) = p(\mathscr{O}) + p(\mathscr{F}) = \sum_{i=1}^{N} p(E_i) \tag{48}$$

A system $\mathscr{S}'$ of two subsystems of events, each considered as a compound event denoted $\mathscr{O}$ for the operating system and denoted $\mathscr{F}$ for the failed system, can be defined as follows:

$$\mathscr{S}' = (\mathscr{O}, \mathscr{F}) = \begin{pmatrix} \mathscr{O} & \mathscr{F} \\ p(\mathscr{O}) & p(\mathscr{F}) \end{pmatrix}$$

The event oriented system analysis may be applied to any relation of sets of events or subsystems, such as exclusive or inclusive sets, as well as dependent and independent events, under the condition of proper partitioning of the system of events to a basic set of disjoint events. Such a partitioning can be provided, for example, by the well-known exclusion–inclusion expansion of union of events.

5.1. Uncertainty associated with engineering systems and subsystems

The system $\mathscr{S}$ under the condition that it is operational $\mathscr{O}$ or failed $\mathscr{F}$ can be presented respectively, as follows:

$$\mathscr{S}/\mathscr{O} = \begin{pmatrix} E_1^o/\mathscr{O} & E_2^o/\mathscr{O} & \ldots & E_{N_o}^o/\mathscr{O} \\ \dfrac{p(E_1^o)}{p(\mathscr{O})} & \dfrac{p(E_2^o)}{p(\mathscr{O})} & \ldots & \dfrac{p(E_{N_o}^o)}{p(\mathscr{O})} \end{pmatrix},$$

$$\mathscr{S}/\mathscr{F} = \begin{pmatrix} E_{N_o+1}^f/\mathscr{F} & E_{N_o+2}^f/\mathscr{F} & \ldots & E_{N_o+N_f}^f/\mathscr{F} \\ \dfrac{p(E_{N_o+1}^f)}{p(\mathscr{F})} & \dfrac{p(E_{N_o+2}^f)}{p(\mathscr{F})} & \ldots & \dfrac{p(E_{N_o+N_f}^f)}{p(\mathscr{F})} \end{pmatrix}$$

Shannon's entropy of system $\mathscr{S}$, under the condition that the system is operating $\mathscr{O}$, is shown as:

$$H_{N_o}(\mathscr{S}/\mathscr{O}) = -\sum_{i=1}^{N_o} \frac{p(E_i^o)}{p(\mathscr{O})} \cdot \log \frac{p(E_i^o)}{p(\mathscr{O})}, \tag{49}$$

Shannon's entropy of system $\mathscr{S}$ under the condition that the system is failing $\mathscr{F}$, is shown as:

$$H_{N_f}(\mathscr{S}/\mathscr{F}) = -\sum_{i=N_o+1}^{N_o+N_f} \frac{p(E_i^f)}{p(\mathscr{F})} \cdot \log \frac{p(E_i^f)}{p(\mathscr{F})} \tag{50}$$

The entropy of the operational modes in Eq. (49) and of the failure modes in Eq. (50) depends only on the states of the subsystem of operational and failure modes, and not on any other state of the system.

The maximal attainable entropy of system $\mathscr{S}$ under the condition that the system is operating is:

$$H_{N_o}(\mathscr{S}/\mathscr{O})_{\max} = \log N_o \tag{51}$$

The maximal attainable entropy of system $\mathscr{S}$ under the condition that the system is failing is:

$$H_{N_f}(\mathscr{S}/\mathscr{F})_{\max} = \log N_f \tag{52}$$

The entropy of the complete system of all known or all observable possible events $\mathscr{S}$, as well as the maximally attainable entropy, can be obtained from Eq. (5) and from

Hartley's formula:

$$H_N(\mathscr{S}) = H_N(\mathscr{O} + \mathscr{F}) = -\sum_{i=1}^{N} p(E_i)\cdot\log p(E_i) \quad (53)$$

$$H_N(\mathscr{S})_{\max} = \log N \quad (54)$$

The entropy of the complete system $\mathscr{S}'$ of $\mathscr{O}$ and $\mathscr{F}$, as well the maximally attainable entropy are:

$$H_2(\mathscr{S}') = H_2(\mathscr{O}, \mathscr{F}) = -p(\mathscr{O})\cdot\log p(\mathscr{O}) - p(\mathscr{F})\cdot\log p(\mathscr{F}) \quad (55)$$

$$H_2(\mathscr{S}')_{\max} = \log 2 \quad (56)$$

The operational and failure modes are of utmost interest for the engineering system designers and for the system users. The subsystems of operational and failure modes can also be considered on different levels of the hierarchical representation of the basic events with respect to their importance in system design.

5.2. Relations of the uncertainties of engineering systems and subsystems

The weighted summa of entropies of the subsystems $\mathscr{O}$ and $\mathscr{F}$ of a complete system $\mathscr{S}$, in terms of the theorem about mixture of distributions, can be represented according to Eq. (23), as shown:

$$p(\mathscr{O})\cdot H_{N_o}(\mathscr{S}/\mathscr{O}) + p(\mathscr{F})\cdot H_{N_f}(\mathscr{S}/\mathscr{F}) = H_N(\mathscr{O} + \mathscr{F}) - H_2(\mathscr{O}, \mathscr{F}) \quad (57)$$

Eq. (57) represents the relation of the probabilities and entropies of subsystems of operational and failure events to the entropies of the system and subsystems of all possible events.

Consider again the systems $\mathscr{S} = (\mathscr{O} + \mathscr{F})$and $\mathscr{S}' = (\mathscr{O}, \mathscr{F})$ in terms of the theorem about dependent systems. According to Eqs. (8) and (9), the entropy of two dependent systems can be obtained by:

$$H(\mathscr{S}\,\mathscr{S}') = H(\mathscr{S}') + H(\mathscr{S}/\mathscr{S}') \quad (58)$$

The term in Eq. (58) is calculated as follows:

$$H(\mathscr{S}/\mathscr{S}') = p(\mathscr{O})\cdot H(\mathscr{S}/\mathscr{S}) + p(\mathscr{F})\cdot H(\mathscr{S}/\mathscr{F}) \quad (59)$$

The term (59) represents the conditional entropy of system $\mathscr{S}$ with respect to system $\mathscr{S}'$.

$H(\mathscr{S}/\mathscr{O})$ and $H(\mathscr{S}/\mathscr{F})$ are called the conditional entropies of system $\mathscr{S}$ under the condition that subsystems $\mathscr{O}$ and $\mathscr{F}$ occur, respectively. When the states of system $\mathscr{S}'$ are entirely defined by the states of the system $\mathscr{S}$, as it is the case in the present consideration, the following relation holds:

$$H(\mathscr{S}\,\mathscr{S}') = H(\mathscr{S}) \quad (60)$$

Finally, using the theorem about dependent system of events, the same result for the relation of the subsystem's entropies is obtained, as by using the theorem about mixture of distributions:

$$p(\mathscr{O})\cdot H(\mathscr{S}/\mathscr{S}) + p(\mathscr{F})\cdot H(\mathscr{S}/\mathscr{F}) = H(\mathscr{S}) - H(\mathscr{S}') \quad (61)$$

If the system $\mathscr{S}$ is not a complete system of events, i.e. $p(\mathscr{S}) < 1$, an incomplete system of events is considered. The Renyi's/Shannon's entropy of order one in Eq. (6), can be applied for system uncertainty assessment $H^1(\mathscr{S})$, for either complete or incomplete systems of events, as follows:

$$H_N^1(\mathscr{S}) = H_N^1(\mathscr{O} + \mathscr{F}) = \frac{H_2(\mathscr{S})}{p(\mathscr{S})} \quad (62)$$

Renyi's/Shannon's entropy of order one $H^1(\mathscr{S}')$ of the systems of subsystems of events $\mathscr{S}'$ can be used to assess the uncertainty of the system built from subsystems based on Eq. (6) as shown:

$$H_2^1(\mathscr{S}') = H_2^1(\mathscr{O}, \mathscr{F}) = \frac{H_2(\mathscr{S}')}{p(\mathscr{S}')} \quad (63)$$

In Eqs. (60)–(63), $H(\mathscr{S})$ and $H(\mathscr{S}')$ are defined in Eqs. (53) and (55).

The systems $\mathscr{S}$ and $\mathscr{S}'$ can also be considered under the condition that only the observable events are of interest. The conditional entropies can be obtained as presented in Eqs. (13) and (21).

The following relations within the system $\mathscr{S}$ considered as a set of subsystem $\mathscr{O}$ and $\mathscr{F}$, can be obtained on the basis of Eq. (12) as follows:

$$p(\mathscr{O})\cdot H_{N_o}(\mathscr{S}/\mathscr{O}) + p(\mathscr{F})\cdot H_{N_f}(\mathscr{S}/\mathscr{F})$$
$$= p(\mathscr{S})\cdot[H_N^1(\mathscr{S}) - H_2^1(\mathscr{S}')]$$
$$= p(\mathscr{S})\cdot[H_N(\mathscr{S}/\mathscr{S}) - H_n(\mathscr{S}'/\mathscr{S}')]$$
$$= H_N(\mathscr{S}) - H_2(\mathscr{S}') = H_N(\mathscr{O} + \mathscr{F}) - H_2(\mathscr{O}, \mathscr{F}) \quad (64)$$

The uncertainties of operational and failure modes and their relations can be applied in the assessment of system performances. Following guidelines can be intuited:

- Higher entropy of operational modes is a consequence of a more uniform distribution of probabilities of operational modes and can indicate the increase of the system's operational abundance.
- Higher entropy of failure modes is a consequence of more uniform distribution of probabilities of failure modes and can be related to the increase of the system endurance to failures.

6. Example a vertically loaded foundation supported by piles

An event oriented system analysis procedure is demonstrated on a system reliability and system uncertainty analysis of vertically loaded foundations supported by vertical piles [10]. Variation in system configurations and different

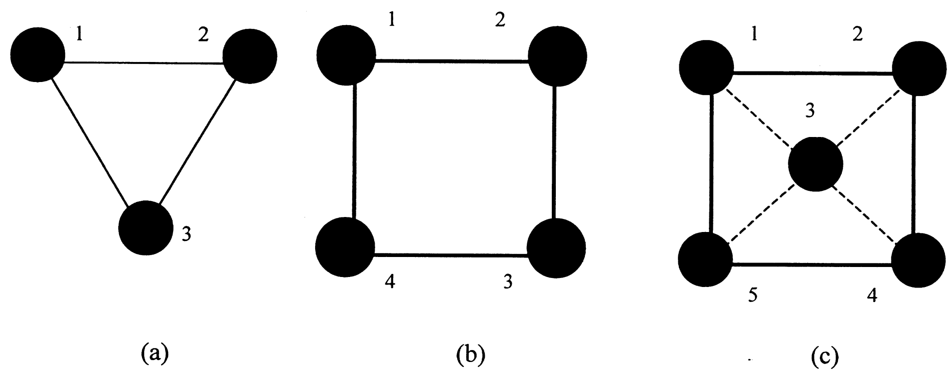


Fig. 1. Foundations supported by three, four and five piles

component probability distributions are considered (Fig. 1(a)–(c)).

6.1. System service modes and effects analysis

First, a system service modes and effects analysis is performed in order to identify all the modes and appropriate probabilities of occurrence.

6.1.1. The foundation on three piles

The foundation on three piles is stable if and only if all three piles can carry loads (Fig. 1(a)). There are $N = 2^3 = 8$ outcomes. By inspection, all possible operational and failure modes are determined. The subsystem of operational modes $\mathscr{O}$ consist only of one, $N_o = 1$, fully operational mode:

$$p(E_1^o) = p(A_1)p(A_2)p(A_3);\quad O : 3/3$$

The subsystem of failure modes $\mathscr{F}$ consists of $N_f = 7$ modes with different failure seriousness:

$$p(E_2^f) = p(\bar{A}_1)p(A_2)p(A_3);\quad F : 1/3$$

$$p(E_3^f) = p(A_1)p(\bar{A}_2)p(A_3);\quad F : 1/3$$

$$p(E_4^f) = p(A_1)p(A_2)p(\bar{A}_3);\quad F : 1/3$$

$$p(E_5^f) = p(\bar{A}_1)p(\bar{A}_2)p(A_3);\quad F : 2/3$$

$$p(E_6^f) = p(\bar{A}_1)p(A_2)p(\bar{A}_3);\quad F : 2/3$$

$$p(E_7^f) = p(A_1)p(\bar{A}_2)p(\bar{A}_3);\quad F : 2/3$$

$$p(E_8^f) = p(\bar{A}_1)p(\bar{A}_2)p(\bar{A}_3);\quad F : 3/3$$

6.1.2. The foundation on four piles

The foundation supported by four piles is stable if three or more piles can carry a load in following configurations: [1,2,3,4], [1,2,3], [2,3,4], [3,4,1] and [4,1,2] (Fig. 1(b)). There are $N = 2^4 = 16$ possible outcomes. By enumeration, all possible operational and failure modes are determined. The subsystem of operational modes $\mathscr{O}$ consists of $N_o = 5$ modes with different operational capacities, as follows:

$$p(E_1^o) = p(A_1)p(A_2)p(A_3)p(A_4);\quad O : 4/4$$

$$p(E_2^o) = p(\bar{A}_1)p(A_2)p(A_3)p(A_4);\quad O : 3/4$$

$$p(E_3^o) = p(A_1)p(\bar{A}_2)p(A_3)p(A_4);\quad O : 3/4$$

$$p(E_4^o) = p(A_1)p(A_2)p(\bar{A}_3)p(A_4);\quad O : 3/4$$

$$p(E_5^o) = p(A_1)p(A_2)p(A_3)p(\bar{A}_4);\quad O : 3/4$$

The subsystem of failure modes $\mathscr{F}$ consists of $N_f = 11$ events with different failure seriousness as given next:

$$p(E_6^f) = p(\bar{A}_1)p(\bar{A}_2)p(A_3)p(A_4);\quad F : 2/4$$

$$p(E_7^f) = p(\bar{A}_1)p(A_2)p(\bar{A}_3)p(A_4);\quad F : 2/4$$

$$p(E_8^f) = p(\bar{A}_1)p(A_2)p(A_3)p(\bar{A}_4);\quad F : 2/4$$

$$p(E_9^f) = p(A_1)p(\bar{A}_2)p(\bar{A}_3)p(A_4); F : 2/4$$

$$p(E_{10}^f) = p(A_1)p(\bar{A}_2)p(A_3)p(\bar{A}_4);\quad F : 2/4$$

$$p(E_{11}^f) = p(A_1)p(A_2)p(\bar{A}_3)p(\bar{A}_4);\quad F : 2/4$$

$$p(E_{12}^f) = p(\bar{A}_1)p(\bar{A}_2)p(\bar{A}_3)p(A_4);\quad F : 3/4$$

$$p(E_{13}^f) = p(\bar{A}_1)p(\bar{A}_2)p(A_3)p(\bar{A}_4);\quad F : 3/4$$

$$p(E_{14}^f) = p(\bar{A}_1)p(A_2)p(\bar{A}_3)p(\bar{A}_4);\quad F : 3/4$$

$$p(E_{15}^f) = p(A_1)p(\bar{A}_2)p(\bar{A}_3)p(\bar{A}_4);\quad F : 3/4$$

$$p(E_{16}^f) = p(\bar{A}_1)p(\bar{A}_2)p(\bar{A}_3)p(\bar{A}_4);\quad F : 4/4$$

6.1.3. The foundation on five piles

The foundation supported by five piles is stable if three or

Table 1
Uncertainties for the given system target reliability $p(\mathscr{O}) = 0.99$ for various system configurations of vertically loaded foundations with three, four and five piles

Piles No. and Reliability	Modes		$H(\mathscr{O}+\mathscr{F})$ *bits* Eq. (53)	H($\mathscr{O}$, $\mathscr{F}$) *bits* Eq. (55)	H($\mathscr{S}/\mathscr{O}$) *bit* Eq. (49)	H($\mathscr{S}/\mathscr{F}$) *bits* Eq. (50)	$H(\mathscr{O}+\mathscr{F})$ − H($\mathscr{O}$, $\mathscr{F}$) *bits* Eq. (57)
	N_o	N_F					
Maximal values			3.	1.	0.	2.8073	3.
3 0.996655	1	7	0.0969	0.0808	0.0000	1.6173	0.0161
Maximal values			4.	1.	2.3219	3.4594	4.
4 0.958002	5	11	1.0055	0.0808	0.9062	2.7578	0.9147
Maximal values			5.	1.	3.8073	4.1699	1.
5 0.933016	14	18	1.7728	0.0808	1.6836	2.5234	1.6720

more piles can carry a load with the exception of the combinations [1,3,5] and [2,3,4] (Fig. 1(c)). There are $N = 2^5 = 32$ possible outcomes. By the use of minimal path method, all possible $N_o = 14$ operational modes with different operational capacities are determined and constitute the subsystem of operational modes $\mathscr{O}$.

$$p(E_1^o) = p(A_1)p(A_2)p(A_3)p(A_4)p(A_5); \quad O: 5/5$$

$$p(E_2^o) = p(A_1)p(A_2)p(A_3)p(A_4)p(\bar{A}_5); \quad O: 4/5$$

$$p(E_3^o) = p(A_1)p(A_2)p(A_3)p(\bar{A}_4)p(A_5); \quad O: 4/5$$

$$p(E_4^o) = p(A_1)p(A_2)p(A_3)p(\bar{A}_4)p(\bar{A}_5); O: 3/5$$

$$p(E_5^o) = p(A_1)p(A_2)p(\bar{A}_3)p(A_4)p(A_5); \quad O: 4/5$$

$$p(E_6^o) = p(A_1)p(A_2)p(\bar{A}_3)p(A_4)p(\bar{A}_5); \quad O: 3/5$$

$$p(E_7^o) = p(A_1)p(A_2)p(\bar{A}_3)p(\bar{A}_4)p(A_5); \quad O: 3/5$$

$$p(E_8^o) = p(A_1)p(\bar{A}_2)p(A_3)p(A_4)p(A_5); \quad O: 4/5$$

$$p(E_9^o) = p(A_1)p(\bar{A}_2)p(A_3)p(A_4)p(\bar{A}_5); \quad O: 3/5$$

$$p(E_{10}^o) = p(A_1)p(\bar{A}_2)p(\bar{A}_3)p(A_4)p(A_5); \quad O: 3/5$$

$$p(E_{11}^o) = p(\bar{A}_1)p(A_2)p(A_3)p(A_4)p(A_5); \quad O: 4/5$$

$$p(E_{12}^o) = p(\bar{A}_1)p(A_2)p(A_3)p(\bar{A}_4)p(A_5); \quad O: 3/5$$

$$p(E_{13}^o) = p(\bar{A}_1)p(A_2)p(\bar{A}_3)p(A_4)p(A_5); \quad O: 3/5$$

$$p(E_{14}^o) = p(\bar{A}_1)p(\bar{A}_2)p(A_3)p(A_4)p(A_5); \quad O: 3/5$$

Employing minimal cat set method, all $N_f = 18$ failure modes of different failure seriousness are determined, and constitute the subsystem of failure modes $\mathscr{F}$, as follows:

$$p(E_{15}^f) = p(\bar{A}_1)p(A_2)p(\bar{A}_3)p(A_4)p(\bar{A}_5); \quad F: 3/5$$

$$p(E_{16}^f) = p(\bar{A}_1)p(A_2)p(\bar{A}_3)p(\bar{A}_4)p(\bar{A}_5); \quad F: 4/5$$

$$p(E_{17}^f) = p(\bar{A}_1)p(\bar{A}_2)p(\bar{A}_3)p(A_4)p(\bar{A}_5); \quad F: 4/5$$

$$p(E_{18}^f) = p(\bar{A}_1)p(\bar{A}_2)p(\bar{A}_3)p(\bar{A}_4)p(\bar{A}_5); \quad F: 5/5$$

$$p(E_{19}^f) = p(A_1)p(\bar{A}_2)p(\bar{A}_3)p(\bar{A}_4)p(A_5); \quad F: 3/5$$

$$p(E_{20}^f) = p(A_1)p(\bar{A}_2)p(\bar{A}_3)p(\bar{A}_4)p(\bar{A}_5); \quad F: 4/5$$

$$p(E_{21}^f) = p(\bar{A}_1)p(\bar{A}_2)p(\bar{A}_3)p(\bar{A}_4)p(A_5); \quad F: 4/5$$

$$p(E_{22}^f) = p(\bar{A}_1)p(A_2)p(A_3)p(A_4)p(\bar{A}_5); \quad F: 2/5$$

$$p(E_{23}^f) = p(\bar{A}_1)p(A_2)p(A_3)p(\bar{A}_4)p(\bar{A}_5); \quad F: 3/5$$

$$p(E_{24}^f) = p(\bar{A}_1)p(\bar{A}_2)p(A_3)p(A_4)p(\bar{A}_5); \quad F: 3/5$$

$$p(E_{25}^f) = p(\bar{A}_1)p(\bar{A}_2)p(A_3)p(\bar{A}_4)p(\bar{A}_5); \quad F: 4/5$$

$$p(E_{26}^f) = p(A_1)p(\bar{A}_2)p(A_3)p(\bar{A}_4)p(A_5); \quad F: 2/5$$

$$p(E_{27}^f) = p(A_1)p(\bar{A}_2)p(A_3)p(\bar{A}_4)p(\bar{A}_5); \quad F: 3/5$$

$$p(E_{28}^f) = p(\bar{A}_1)p(\bar{A}_2)p(A_3)p(\bar{A}_4)p(A_5); \quad F: 3/5$$

$$p(E_{29}^f) = p(\bar{A}_1)p(\bar{A}_2)p(\bar{A}_3)p(A_4)p(A_5); \quad F: 3/5$$

$$p(E_{30}^f) = p(\bar{A}_1)p(A_2)p(\bar{A}_3)p(\bar{A}_4)p(A_5); \quad F: 3/5$$

$$p(E_{31}^f) = p(A_1)p(\bar{A}_2)p(\bar{A}_3)p(A_4)p(\bar{A}_5); \quad F: 3/5$$

$$p(E_{32}^f) = p(A_1)p(A_2)p(\bar{A}_3)p(\bar{A}_4)p(\bar{A}_5); \quad F: 3/5$$

Both the subsystems $\mathscr{O}$ and $\mathscr{F}$ for all three configurations are collected in a system of events $\mathscr{S}$. The system $\mathscr{S}$ is considered as a complete system of events in all configurations, since:

$$p(\mathscr{S}) = p(\mathscr{O}) + p(\mathscr{F}) = \sum_{i=1}^{N} p(E_i) = 1.0$$

Table 2
Uncertainties for a given target system reliability $p(\mathscr{O}) = 0.99$ for various reliabilities of corner and of central piles of a foundation supported by five piles

Pile reliabilities on corners	central	$H(\mathscr{O}+\mathscr{F})$ bits Eq (53)	$H(\mathscr{O},\mathscr{F})$ bits Eq (55)	$H(\mathscr{S}/\mathscr{O})$ bits Eq (49)	$H(\mathscr{S}/\mathscr{F})$ bits Eq (50)	$H(\mathscr{O}+\mathscr{F}) - H(\mathscr{O},\mathscr{F})$ bits Eq (57)	(Comments)
0.929	0.9^{12}	1.4756	0.0808	1.3916	1.7168	1.3948	(Non attainable)
0.930	0.9896	1.5651	0.0808	1.4794	1.9716	1.4843	
0.933	0.9330	1.7728	0.0808	1.6836	2.5234	1.6920	(Equal reliabilities)
0.935	0.8939	1.8760	0.0808	1.7854	2.7646	1.7952	
0.940	0.7789	2.0719	0.0808	1.9794	3.1454	1.9911	
0.945	0.6336	2.1769	0.0808	2.0839	3.2998	2.0951	
0.9465	0.5823	2.1846	0.0808	2.0917	3.3105	2.1038	(Maximal uncertainty)
0.950	0.4453	2.1369	0.0808	2.0438	3.2757	2.0561	
0.955	0.1942	1.7689	0.0808	1.6741	3.0664	1.6881	
0.958	0.0000	1.0055	0.0808	0.9062	2.7578	0.9247	(Zero reliability)

where

$$p(\mathscr{O}) = \sum_{i=1}^{N_o} p(E_i^o) \qquad \text{and} \qquad p(\mathscr{F}) = \sum_{i=N_o+1}^{N_o+N_f} p(E_i^f).$$

6.2. Event oriented system analysis of the vertically loaded foundation

The piles are first assumed to be of the same quality, also have identical component reliabilities. For a given target systems reliability of $p(\mathscr{O}) = 0.99$, different configurations of three, four and five piles are investigated (Fig. 1(a)–(c)). Such an assumption of unrealistically low target reliability provides presentable results of the event oriented system analysis, according to relations (49), (50), (53), (55) and (57) applied to various configurations of piles (Table 1).

A parametric study of a foundation supported by five piles by variation of pile's reliabilities is to be performed next. The corner piles are assumed to be of the same quality, also having identical reliabilities. The central pile is of another quality, with the reliability distinct from the corner piles. For a given target system reliability of $p(\mathscr{O}) = 0.99$, the effects of different combinations of reliabilities on the four corner piles and of the central pile on the system reliability and system uncertainties are investigated. The results of an event oriented system analysis are presented in Table 2 and Fig. 2.

6.3. Points for the discussion of the results of the event oriented system analysis

1. For the target system reliability of $p(\mathscr{O}) = 0.99$, the entropies of the vertically loaded foundation are increasing functions of a number of piles (three, four and five) (Fig. 1(a–c)) for different considered configurations with an increasing number of modes (Table 1).
2. For the reliability of corner piles equal to 0.958 of the foundation with five piles (Table 2 and Fig. 2), the central pile is entirely ineffective with respect to the foundation target reliability.
3. The foundation configuration with five piles for the reliability of the corner piles equal to 0.958 is, from the

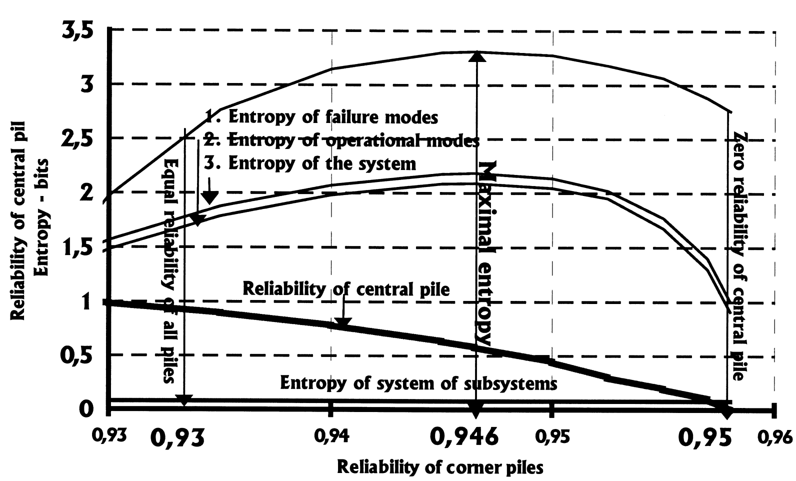


Fig. 2. Uncertainties for a given target reliability $p(Q) = 0.99$ with respect to the reliability variation of corner and central piles of a vertically loaded foundation.

reliability point of view, equivalent to the configuration with four piles (Table 1), where the reliability of all four piles is 0.958002. The entropies for four and five piles are identical due to the property of entropy that events with zero reliability do not effect uncertainty.

4. The maximum entropy of the foundation is encountered when corner pile reliability equals 0.9465 and central pile reliability equals 0.5823. Then both entropies of operational modes and of failure modes reach their maximal values, indicating that the probabilities of alternative operational and failure modes are maximally uniformly distributed.
5. For the reliability of corner piles under 0.929, the central pile reliability amounting to $0.9^{(12)}$ is practically unattainable with respect to the foundation target reliability of 0.99.
6. The target system reliability of $p(\mathcal{O}) = 0.99$ is also accomplished when all five piles are of identical reliability of 0.933.
7. The entropy of the system of operational and failure subsystems is constant, $H(\mathcal{O}, \mathcal{F}) = 0.0808$ due to the imposed constant system target reliability of $p(\mathcal{O}) = 0.99$.
8. The highest entropy of the operational modes indicates the optimal foundation operational abundance.
9. The highest entropy of the failure modes indicates the optimal foundation endurance under distress.

7. Conclusion

This article suggests that the traditional probabilistic engineering system analysis based on physical and/or technical components of a system, may be extended by an event oriented system analysis. Such an analysis should take into account different random events in the system's lifetime service. The presented procedure can be consistently applied to problems of exclusive or inclusive events by adequate partitioning of the event space. The uncertainties in system's operation originate from the unpredictability of possible events. A practical uncertainty measure, in addition to other complex system performance measures, convey knowledge about the number of operational and failure modes and their probabilities. The relation of the uncertainties of the system and of subsystems to the overall system performance, as it is defined in event oriented system analysis, may be helpful in different fields of engineering in the refinement of system performance.

Shannon's entropy can be used for uncertainty assessment of complete systems and Renyi's entropy for incomplete systems. The theorems about the mixture of distributions and dependent systems can be applied for bringing into the relation, the probabilities and uncertainties of the systems and those of the subsystems. The entropy, as the only rational measure of system uncertainty, does not depend on anything else other than possible events and in this sense is entirely objective.

The assessment of the uncertainty of systems by representing them by systems of events and the application of the entropy as defined in the information theory has been well known in engineering. The reason that the system uncertainty analysis is not widely adopted in engineering practice could be the fact that the entropy of a system itself in general is not particularly helpful in the assessment of system performance. However, the uncertainties of important subsystems of events, such as the operational and failure modes, as well as their relations to the uncertainty and reliability of the entire system, can provide a better insight into the system performance. In many engineering problems, the difficulty is to consider all relevant circumstances. An event may be random with respect to some circumstances, and at the same time it may be completely determined with respect to some other circumstances. The randomness or determinedness of an event depends on whether the circumstance do or do not determine the occurrence or non-occurrence of the event. The choice of circumstances depends on the observer and there is certain freedom of choice within the limits of possibilities. Within each subsystem, other groups or subgroups of modes can be of interest to the designers and to the users, like modes of equal operational capacity or modes with equal failure rates etc. The event oriented analysis can also be applied at any level of subsystem partitioning.

The article tackles the problem of distinction among complex system, including also possible redundancy and robustness, performing identical function, with the same level of reliability but with various probability distributions or with different number of operational and failure modes. The system uncertainty can be thought of as a design decision attribute, which takes into account the number of events and the dispersion of their probabilities, over all possible events and important subsystems of events, which is not included in design considerations about safety and economy. Such an approach based on event oriented system analysis, could provide an improved alternative to strengthening lifeline networks, updating or inverse analysis with observations made on system behaviour and in general better system designs.

At present, the event oriented system analysis faces possible numerical problems in dealing with larger systems. For a complete event oriented system analysis, an enumeration of all the possible events is needed. Most of the quantitative methods are economical in the use of only the most influential events in order to reduce the computational efforts. The methods presented in the paper also allows the uncertainty assessments of incomplete systems, consisting of only observable or only of important events, being then numerically more efficient and perhaps more practically applicable. An enormous increase in numerical capacities of recent computer systems could further encourage the development of even oriented system analysis.

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