USAGE OF RELATIVE UNCERTAINTY MEASURES

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ABSTRACT This note considers an alternate probabilistic presentation of uncertainty of complex systems and subsystems of any level. The entropy defined in information theory is used for the uncertainty measure. The relative measure of uncertainty brings into the relation the actual entropy to the maximal attainable entropy of a considered system or subsystem. The relative uncertainty facilitates the interpretations of the uncertainty with respect to systems consisting always of same number of events. Such an approach is hopefully more user-friendly way for representation of uncertainties in engineering systems. Numerical examples are attached.

Key words: probability, engineering, system of events, reliability, uncertainty, entropy, information theory.

INTRODUCTION

The scope of this note is in the more readable probabilistic presentation of system uncertainties. It is particularly suited for engineering problems of complex systems of events. Engineering systems are usually consisting of more subsystems of events of any level, each of them pertaining to some characteristic of the system.

The uncertainty measure based on the entropy defined earlier in information theory [Wiener 1948, Shannon and Weaver 1949] is an adequate general uncertainty measure for complete systems of events. The Renyi's/Shannon's information of order one can be used to assess the uncertainty of incomplete systems [Renyi 1970]. The theorem about the information associated with the mixture of distributions can be used to assess the uncertainty of subsystems and system of subsystems of events [Žiha 1998]. The definition of a unit of uncertainty is not more and not less arbitrary then the choice of the unit of some physical quantity [Renyi 1970]. The unit of the Shannon's entropy conventionaly corresponds to a system of two equally probable events (simple alternatives). In this sense, it is general but not always easily interpretable measure for engineering purposes. The amount of the Shannon's entropy is not invariant on the applied bases of the logarithm. What is meant herein is not the relative information as it is defined earlier [e.g. Aczel and Daroczy 1975]. The idea underlined in the paper is to make use of the Shannon's information or of the Renyi's/Shannon's information to assess the system uncertainty by bringing it into the relation to the known maximal uncertainty of the reference system of events. Such a relative uncertainty measure provided for engineering systems built up of same number of events can be more readable for engineering purposes. It is also invariant on the applied bases of the logarithm. Moreover, the relative entropies of subsystems can be related to the uncertainty of subsystems themself, but also to the uncertainty of the whole system.

The presented relative uncertainty measures are applied to numerical examples in order to demonstrate its usefulness in the uncertainty analysis of systems and subsystems.

2. UNCERTAINTY MEASURES

Let us consider a system $\mathscr{S}$, constituted by the events E_i, $i=1,2,...,n$, and with the appropriate probabilities associated to each of the events $p_i=p(E_i)$, which can be presented as a finite scheme [Khinchin 1957], as follows:

$$\mathscr{S} = \begin{pmatrix} E_1 & E_2 & \cdots & E_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$$

The entropy of the complete system of events $\mathscr{S}$ [Shannon and Weaver 1949] is supposed to depend only of the probability distribution of considered events $\mathscr{P} = (p_1, p_2,..., p_n)$ and can be denoted in different ways, as:

$$H(\mathscr{S}) = H_n(\mathscr{S}) = H_n(p_1, p_2,..., p_n) = H_n(\mathscr{P}) = -\sum_{i=1}^{n} p_i \log p_i = \sum_{i=1}^{n} p_i \log \frac{1}{p_i} \qquad (1)$$

The quantity (1) is called entropy of the complete probability distribution or the entropy of the complete system of events, or the entropy of a finite scheme and it is usually denoted as the Shannon's entropy or the Shannon's information/uncertainty measure.

Another measure of uncertainty is the Renyi's entropy of order α, [e.g. Renyi 1970], which is defined for $\alpha \neq 1$:

$$H_n^{\alpha}(\mathcal{S}) = H^{\alpha}(\mathcal{S}) = \frac{1}{1-\alpha}\log_2\Big(\sum_{i=1}^{n} p_i^{\alpha} / \sum_{i=1}^{n} p_i\Big) \tag{2}$$

The quantity in (2) may also be viewed as a measure of the amount of uncertainty corresponding to either complete or incomplete distribution of probabilities pertaining to the system of events. If some outcomes of an "experiment" are not known or their probabilities can not be determined, e.g. only the observable outcomes are taken into account, an incomplete probability distribution p_k can be considered.

For incomplete *system* ($p_k > 0$; $k=1, 2, \ldots, n$) there is $\sum_k p_k < 1$ and not necessarily $\sum_k p_k = 1$.

The entropy of a complete or of an incomplete system of events $\mathcal{S}$ can be determined as the limiting case of the term (2), for $\alpha \to 1$. It can be interpreted as the arithmetic mean (expected value) of the single enropies $-\log p_i$ with weights p_i. Renyi himself denoted this quantity as the Shannon's entropy of order one [Renyi 1970]:

$$H_n(\mathcal{S}) = H(\mathcal{S}) = H_n^1(\mathcal{S}) = H^1(\mathcal{S}) = \Big(-\sum_{i=1}^{n} p_i \log p_i\Big) / \sum_{i=1}^{n} p_i \tag{3}$$

Note that the entropy defined by (3) is often denoted as the Renyi's entropy of order one, [e.g. Aczel and Daroczi 1975]. The note will use the notation Renyi's/Shannon's entropy of order one.

The maximal entropy of a complete or incomplete system of events is obtained for $p_i = p(\mathcal{S})/n$, $p_i = 1,2,..,n$, is:

$$H_n(\mathcal{S})_{\max} = H_n^1(\mathcal{S})_{\max} = \frac{-\sum_{i=1}^{n} \frac{p(\mathcal{S})}{n}\log\frac{p(\mathcal{S})}{n}}{p(\mathcal{S})} = \log n - \log[p(\mathcal{S})] = \log\frac{n}{p(\mathcal{S})} \tag{4}$$

Let us consider a system $\mathcal{S}$ of events E_{ij}, with appropriate probabilities $p(E_{ij})$, as shown:

$$\mathcal{S} = \begin{pmatrix} E_{11} & \ldots & E_{1m_1} & \ldots & E_{i1} & \ldots & E_{im_i} & \ldots & E_{n1} & \ldots & E_{nm_n} \\ p(E_{11}) & \ldots & p(E_{1m_1}) & \ldots & p(E_{i1}) & \ldots & p(E_{im_i}) & \ldots & p(E_{n1}) & \ldots & p(E_{nm_n}) \end{pmatrix}$$

The events of system $\mathcal{S}$ can be grouped regardless of the ordering of events within the system, into subsystems of events $\mathcal{S}_i$, $i=1,2,\ldots,n$, each containing E_{ij}, $j=1,2,\ldots,m_i$ elements, as presented next:

$$\mathcal{S}_i = \begin{pmatrix} E_{i1} & \ldots & E_{ij} & \ldots & E_{im_i} \\ p(E_{i1}) & \ldots & p(E_{ij}) & \ldots & p(E_{im_i}) \end{pmatrix}, \qquad p(\mathcal{S}_i) = \sum_{j=1}^{m_i} p(E_{ij})$$

(5)

Note that neither the probability nor the entropy depends on the sequence of events in subsystem $\mathcal{S}_i$.

The system $\mathcal{S}$ can be now presented as a summa of disjoint subsystems of events, as shown:

$$\mathcal{S} = (\mathcal{S}_1 + \ldots + \mathcal{S}_i + \ldots + \mathcal{S}_n)$$

The system $\mathcal{S}$ can be also viewed as a compound of subsystems of events $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$, denoted as $\mathcal{S}'$, as shown:

$$\mathcal{S}' = (\mathcal{S}_1, \ldots, \mathcal{S}_i, \ldots, \mathcal{S}_n) = \begin{pmatrix} \mathcal{S}_1 & \ldots & \mathcal{S}_i & \ldots & \mathcal{S}_n \\ p(\mathcal{S}_1) & \ldots & p(\mathcal{S}_i) & \ldots & p(\mathcal{S}_n) \end{pmatrix}$$

The probability associated to systems $\mathcal{S}$ and $\mathcal{S}'$ is defined as follows:

$$p(\mathcal{S}) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} p(E_{ij}) = p(\mathcal{S}') = \sum_{i=1}^{n} p(\mathcal{S}_i) \tag{6}$$

Subsystems of events are considered to be associated with a mixture of partial distributions, [Žiha 1998].

The uncertainty of the subsystem $\mathcal{S}_i$ can be expressed as the Shannon's entropy applied only to the partial distribution of the system $\mathcal{S}$ considered under the condition that the subsystem $\mathcal{S}_i$ occurs. Such a conditional entropy does not depend on the system probability $p(\mathcal{S})$. It is independent on whether the system $\mathcal{S}$ is complete or incomplete. According to the definition of the entropy, it follows:

$$H_{m_i}(\mathcal{S}/\mathcal{S}_i) = -\sum_{j=1}^{m_i} \frac{p(E_{ij})}{p(\mathcal{S}_i)}\log\frac{p(E_{ij})}{p(\mathcal{S}_i)} \tag{7}$$

Note also that the entropies of the subsystems in (7) depend only on the states of the subsystem $\mathcal{S}_i$ and not on the any other states of the system $\mathcal{S}$. The following relation expresses the loss of the entropy of a subsystem $\mathcal{S}_i$ viewed as a partial distribution, with respect to the entropy $H^1(\mathcal{S}_i)$ of a subsystem $\mathcal{S}_i$ viewed as an incomplete one:

$$H_{m_i}(\mathcal{S}\ /\mathcal{S}_i) = H^1_{m_i}(\mathcal{S}_i) + \log p(\mathcal{S}_i) \tag{8}$$

The Renyi's/Shannon's entropy of order one $H^1(\mathcal{S}_i)$ in (8) is related to the entropy of an incomplete system as:

$$H^1_{m_i}(\mathcal{S}_i) = H_{m_i}(\mathcal{S}_i)/\,p(\mathcal{S}_i) \quad \text{where the partial summa is} \quad H_{m_i}(\mathcal{S}_i) = -\sum_{j=1}^{m_i} p(E_{ij})\log p(E_{ij}) \tag{9}$$

The maximal attainable conditional entropy of subsystem $\mathcal{S}_i$ in (7) is obtained for m_i equally probable events is:

$$H_{m_i}(\mathcal{S}/\mathcal{S}_i)_{\max} = \log m_i \tag{10}$$

By taking the weighted summa of all the subsystem's entropies, the following relation can be derived:

$$\sum_{i=1}^{n} p(\mathcal{S}_i)\cdot H_{m_i}(\mathcal{S}/\mathcal{S}_i) = p(\mathcal{S})\cdot\left[H^1_N(\mathcal{S}) - H^1_n(\mathcal{S}')\right] = p(\mathcal{S})\cdot\left[H_N(\mathcal{S}/\mathcal{S}) - H_n(\mathcal{S}'/\mathcal{S}')\right] = H_N(\mathcal{S}) - H_n(\mathcal{S}') \tag{11}$$

The expressions $H_N(\mathcal{S}\ /\mathcal{S}')$ and $H_n(\mathcal{S}\ /\mathcal{S}')$ in (11) may be considered as conditional entropies of the system $\mathcal{S}$ and $\mathcal{S}'$ with respect to the system of subsystems $\mathcal{S}'$. The average of the entropy of the subsystems $\mathcal{S}_i$, with weights equal to the associated probabilities $p(\mathcal{S}_i)$, is equal to the entropy of the system of events $\mathcal{S}$, reduced for the entropy of the system of subsystems $\mathcal{S}'$. The reduction in the entropy of system $\mathcal{S}$ is a consequence of the knowledge about its partitioning into subsystems. The relation (11) does not depend on whether the systems $\mathcal{S}$ and $\mathcal{S}'$ are complete systems, i.e. $p(\mathcal{S}) = p(\mathcal{S}') = 1$, $H_N(\mathcal{S})=H^1_N(\mathcal{S})$ and $H_n(\mathcal{S}')=H^1_n(\mathcal{S}')$, or incomplete systems due to $p(\mathcal{S}) = p(\mathcal{S}') < 1$, $H_N(\mathcal{S})=H^1_N(\mathcal{S})+\log p(\mathcal{S})$ and $H_n(\mathcal{S}')= H^1(\mathcal{S}')+\log p(\mathcal{S}')$.

3. Relative measures of uncertainty

What is meant herein is not the relative information as it is defined earlier [e.g. Renyi 1970, Aczel and Daroczy 1975]. For engineering purposes it might be more readable to express the uncertainty of a system either relative to the maximal uncertainty of the system itself or relative to some other appropriate system or subsystem, say target or reference system, instead in terms of "standard" system of two equally probable events. The relative measure of uncertainty hopefully more suitable for engineering purposes will be denoted with small leters $h_{n,N}(\mathcal{S})$ instead of capitals for entropy by definition $H_N(\mathcal{S})$. The index n emphasises the number of events in a considered system or subsystem. The index N is the number of events in a reference system or subsystem relative to which the uncertainty is to be expressed. For complete and incomplete systems of events the relative measure of uncertainty can be expressed in dimensionless form with respect to any system, let us say of N events, and it reads:

$$h^1_{n,N}(\mathcal{S}) = \frac{H^1_n(\mathcal{S})}{H^1_N(\mathcal{S})_{\max}} = \frac{H^1_n(\mathcal{S})}{\log N - \log(p(\mathcal{S}))} = \frac{H^1_n(\mathcal{S})}{\log(N/p(\mathcal{S}))} \tag{12}$$

The relative uncertainty measure with respect to the considered system of N events itself, is as shown:

$$h^1_{N,N}(\mathcal{S}) = \frac{H^1_N(\mathcal{S})}{H^1_N(\mathcal{S})_{\max}} = \frac{H^1_N(\mathcal{S})}{\log N - \log(p(\mathcal{S}))} = \frac{H^1_N(\mathcal{S})}{\log(N/p(\mathcal{S}))} \tag{13}$$

Superscripts "1" in (12) and (13) emphasise that the entropy is related to an incomplete system of events. The terms (12) and (13) are valid for complete systems too. The relations (12) and (13) can be viewed as the application of logarithm of base B instead of base 2, where $B=N/p(\mathcal{S})$ is the base of the applied logarithm. In those cases, entropy is exactly equal to unity for any system of N equally probable events with probability of observable events equal to $p(\mathcal{S})$.

However, for incomplete systems some can find sometimes as more appropriate to express the uncertainty with respect to a hypothetically complete system of N events (no superscript "1" used in (14) and (15)), as follows:

$$h_{n,N}(\mathcal{S}) = \frac{H^1_n(\mathcal{S})}{H_N(\mathcal{S})_{\max}} = \frac{H^1_n(\mathcal{S})}{\log N} \tag{14}$$

$$h_{N,N}(\mathcal{S}) = \frac{H^1_N(\mathcal{S})}{H_N(\mathcal{S})_{\max}} = \frac{H^1_N(\mathcal{S})}{\log N} \tag{15}$$

To the basic relations for relative entropy (12, 13, 14 and 15), the relation $\log_N x = \dfrac{\log_b x}{\log_b N}$ can be applied.

The value of $h_{n,N}(\mathcal{S})$ represents the fraction of the maximal attainable entropy, equal to the entropy of the system of N equally probable events. It expresses how many times is the entropy of the considered system less of the maximal attainable entropy of the target system. The relative measure defined in such a way is possibly a closer expression of intuitive perception of uncertainty in engineering problems.

The main drawback of the relative measures of uncertainty in (12, 13, 14 and 15), compared to the absolute measure expressed by the Shannon's entropy (1) or Renyi's/Shannon's entropy (3), is that it is expressed by two quantities: the entropy and by the number of events constituting the considered reference system. The advantage is that the relative measure of uncertainty does not depend on the bases of the logarithm applied in the definition of the entropy, being in this sense dimensionless.

The relative entropy of a subsystem of m_i events with respect to an incomplete system of N events, is defined as:

$$h^1_{m_i,N}(\mathcal{S}\ /\mathcal{S}_i)=\frac{H_{m_i}(\mathcal{S}\ /\mathcal{S}_i)}{H^1_N(\mathcal{S})_{\max}}=\frac{H_{m_i}(\mathcal{S}\ /\mathcal{S}_i)}{\log N-\log p(\mathcal{S})}=\frac{H^1_{m_i}(\mathcal{S}_i)+\log p(\mathcal{S}_i)}{\log(N/p(\mathcal{S}))} \tag{16}$$

However, for incomplete systems the uncertainty can also be expressed with respect to a hypothetically complete system of N events, as follows:

$$h_{m_i,N}(\mathcal{S}\ /\mathcal{S}_i)=\frac{H_{m_i}(\mathcal{S}\ /\mathcal{S}_i)}{H_N(\mathcal{S})_{\max}}=\frac{H_{m_i}(\mathcal{S}\ /\mathcal{S}_i)}{\log N}=\frac{H^1_{m_i}(\mathcal{S}_i)+\log p(\mathcal{S}_i)}{\log N} \tag{17}$$

The relative entropy of a subsystem of m_i events with respect to the subsystem itself, is defined as follows:

$$h_{m_i,m_i}(\mathcal{S}\ /\mathcal{S}_i)=\frac{H_{m_i}(\mathcal{S}\ /\mathcal{S}_i)}{H_{m_1}(\mathcal{S}/\mathcal{S}_i)_{\max}}=\frac{H_{m_i}(\mathcal{S}\ /\mathcal{S}_i)}{\log m_i}=\frac{H^1_{m_i}(\mathcal{S}_i)+\log p(\mathcal{S}_i)}{\log m_i}=\frac{H^1_{m_i}(\mathcal{S}_i)+H_1\left[p(\mathcal{S}_i)\right]}{\log m_i} \tag{18}$$

The three useful relations are derived next.

By substitution of definition (12), (13) and (18) into the relation (11), the first expression is obtained:

$$\sum_{i=1}^{n} p(\mathcal{S}_i)\cdot\log m_i\cdot h_{m_i,m_i}(\mathcal{S}/\mathcal{S}_i)=p(\mathcal{S})\cdot\left[\log(N/p(\mathcal{S}))\cdot h^1_{N,N}(\mathcal{S})-\log(n/p(\mathcal{S}))\cdot h^1_{n,n}(\mathcal{S}')\right] \tag{19}$$

Division of the equation (11) with *log N/p(*$\mathcal{S}$*)* gives the second expression:

$$\sum_{i=1}^{n} p(\mathcal{S}_i)\cdot h_{m_i,N/p(\mathcal{S})}(\mathcal{S}/\mathcal{S}_i)=p(\mathcal{S})\cdot\left[h^1_{N,N/p(\mathcal{S})}(\mathcal{S})-h^1_{n,N/p(\mathcal{S})}(\mathcal{S}')\right] \tag{20}$$

Division of the equation (11) with *log N* gives the third expression:

$$\sum_{i=1}^{n} p(\mathcal{S}_i)\cdot h_{m_i,N}(\mathcal{S}/\mathcal{S}_i)=p(\mathcal{S})\cdot\left[h^1_{N,N}(\mathcal{S})-h^1_{n,N}(\mathcal{S}')\right] \tag{21}$$

4. NUMERICAL EXAMPLES

4.1. Incomplete system

Let us consider first an incomplete system of events $\mathcal{S}$ consisting of six events as shown:

$$\mathcal{S}=\begin{pmatrix} E_{11} & E_{12} & E_{21} & E_{22} & E_{23} & E_{24}\\ \frac{6}{24} & \frac{5}{24} & \frac{4}{24} & \frac{3}{24} & \frac{2}{24} & \frac{1}{24}\end{pmatrix}$$

The system $\mathcal{S}$ is subdivided into two subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows:

$$\mathcal{S}_1=\begin{pmatrix} E_{11} & E_{12}\\ \frac{6}{24} & \frac{5}{24}\end{pmatrix} \qquad \mathcal{S}_2=\begin{pmatrix} E_{21} & E_{22} & E_{23} & E_{24}\\ \frac{4}{24} & \frac{3}{24} & \frac{2}{24} & \frac{1}{24}\end{pmatrix}$$

The probabilities associated with systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are $p(\mathcal{S}_1)=\frac{11}{24}$ and $p(\mathcal{S}_2)=\frac{10}{24}$, see equation. (5).

Systems $\mathcal{S}$ and two-element system $\mathcal{S}'$ are incomplete since p($\mathcal{S}$)<1. The relative measures of uncertainty for incomplete system with respect to maximal entropy of the incomplete system, are according to (12, 13) as follows:

$$h^1_{6,6}(\mathcal{S})=\frac{H^1_6(\mathcal{S})}{H^1_6(\mathcal{S})_{\max}}=\frac{H^1_6(\mathcal{S})}{\log 6-\log\frac{21}{24}}, \qquad h^1_{2,2}(\mathcal{S}')=\frac{H^1_2(\mathcal{S}')}{H^1_2(\mathcal{S}')_{\max}}=\frac{H^1_2(\mathcal{S}')}{\log 2-\log\frac{21}{24}} \quad \text{and}$$

$$h^1_{2,6}(\mathcal{S}') = \frac{H^1_2(\mathcal{S}')}{H^1_6(\mathcal{S})_{\max}} = \frac{H^1_2(\mathcal{S}')}{\log 6 - \log\frac{21}{24}}.$$

The relative measures of uncertainty for incomplete system with respect to maximal entropy of the system of $N=6$ events, are calculated according to (14) and (15) as follows:

$$h_{6,6}(\mathcal{S}) = \frac{H^1_6(\mathcal{S})}{H_6(\mathcal{S})_{\max}} = \frac{H^1_6(\mathcal{S})}{\log 6}, \qquad h_{2,2}(\mathcal{S}') = \frac{H^1_2(\mathcal{S}')}{H_2(\mathcal{S}')_{\max}} = \frac{H^1_2(\mathcal{S}')}{\log 2} \quad \text{and}$$

$$h_{2,6}(\mathcal{S}') = \frac{H^1_2(\mathcal{S}')}{H_6(\mathcal{S})_{\max}} = \frac{H^1_2(\mathcal{S}')}{\log 6}.$$

The relative measure of uncertainty applied to the subsystems are calculated according (16), (17) and (18) as follows:

$$h^1_{2,6}(\mathcal{S}/\mathcal{S}_1) = \frac{H_2(\mathcal{S}/\mathcal{S}_1)}{H^1_6(\mathcal{S})_{\max}} = \frac{H_2(\mathcal{S}/\mathcal{S}_1)}{\log 6 - \log\frac{21}{24}} \qquad h_{4,6}(\mathcal{S}/\mathcal{S}_2) = \frac{H_4(\mathcal{S}/\mathcal{S}_2)}{H^1_6(\mathcal{S})_{\max}} = \frac{H_4(\mathcal{S}/\mathcal{S}_2)}{\log 6 - \log\frac{21}{24}}$$

$$h_{2,6}(\mathcal{S}/\mathcal{S}_1) = \frac{H_2(\mathcal{S}/\mathcal{S}_1)}{H_6(\mathcal{S})_{\max}} = \frac{H_2(\mathcal{S}/\mathcal{S}_1)}{\log 6} \qquad h_{4,6}(\mathcal{S}/\mathcal{S}_2) = \frac{H_4(\mathcal{S}/\mathcal{S}_2)}{H_{6\max}(\mathcal{S})} = \frac{H_4(\mathcal{S}/\mathcal{S}_2)}{\log 6 - \log\frac{21}{24}}$$

$$h_{2,2}(\mathcal{S}/\mathcal{S}_1) = \frac{H_2(\mathcal{S}/\mathcal{S}_1)}{H_2(\mathcal{S}/\mathcal{S}_1)_{\max}} = \frac{H_2(\mathcal{S}/\mathcal{S}_1)}{\log 2} \qquad h_{4,4}(\mathcal{S}/\mathcal{S}_2) = \frac{H_4(\mathcal{S}/\mathcal{S}_2)}{H_4(\mathcal{S}/\mathcal{S}_2)_{mx}} = \frac{H_4(\mathcal{S}/\mathcal{S}_2)}{\log 4}$$

The results for the incomplete system are resumed in the sequel.

$H_2(\mathcal{S}_1)$ =0.9940,	$H_4(\mathcal{S}_2)$ =1.8464,	$H_6^1(\mathcal{S})$ =2.5909,	$H_2^1(\mathcal{S}')$ =1.1910
log 2 =1.0000,	log 4 =2.0000,	log 6-log(21/24) =2.7776,	log 2-log(21/24) =1.1926
$h_{2,2}(\mathcal{S}_1)$=0.9940,	$h_{4,4}(\mathcal{S}_2)$ =0.9232,	$h^1_{6,6}(\mathcal{S})$ =0.9328,	$h^1_{2,2}(\mathcal{S}')$ =0.9984
$h^1_{2,6}(\mathcal{S}_1)$=0.3578,	$h^1_{4,6}(\mathcal{S}_2)$=0.6647,	$h^1_{6,6}(\mathcal{S})$ =0.9328,	$h^1_{2,6}(\mathcal{S}')$ =0.4288
$h_{2,6}(\mathcal{S}_1)$ =0.3845,	$h_{4,6}(\mathcal{S}_2)$ =0.7143,	$h_{6,6}(\mathcal{S})$ =1.0023,	$h_{2,6}(\mathcal{S}')$ =0.4607

4.2. Complete system

Let us consider next an another system $\mathcal{S}$, now as a complete system of events:

$$\mathcal{S} = \begin{pmatrix} E_{11} & E_{12} & E_{21} & E_{22} & E_{23} & E_{24} \\ \frac{8}{24} & \frac{6}{24} & \frac{4}{24} & \frac{3}{24} & \frac{2}{24} & \frac{1}{24} \end{pmatrix}$$

The system $\mathcal{S}$ is subdivided into two subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows

$$\mathcal{S}_1 = \begin{pmatrix} E_{11} & E_{12} \\ \frac{8}{24} & \frac{6}{24} \end{pmatrix} \qquad \mathcal{S}_2 = \begin{pmatrix} E_{21} & E_{22} & E_{23} & E_{24} \\ \frac{4}{24} & \frac{3}{24} & \frac{2}{24} & \frac{1}{24} \end{pmatrix}$$

The probabilities associated with systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are $p(\mathcal{S}_1) = \frac{14}{24}$ and $p(\mathcal{S}_2) = \frac{10}{24}$, see equation. (7).

Systems $\mathcal{S}$ and two-element system $\mathcal{S}'$ are complete since p($\mathcal{S}$)=1. The relative measure of uncertainty applied to the complete system and system of subsystems are calculated according to (14) and (15) as follows:

$$h_{6,6}(\mathcal{S})=\frac{H_6(\mathcal{S})}{H_6(\mathcal{S})_{\max}}=\frac{H_6(\mathcal{S})}{\log 6}, \qquad h_{2,2}(\mathcal{S}')=\frac{H_2(\mathcal{S}')}{H_2(\mathcal{S}')_{\max}}=\frac{H_2(\mathcal{S}')}{\log 2} \quad \text{and}$$

$$h_{2,6}(\mathcal{S}')=\frac{H_2(\mathcal{S}')}{H_6(\mathcal{S})_{\max}}=\frac{H_2(\mathcal{S}')}{\log 6}.$$

The relative measure of uncertainty applied to the subsystems are calculated according to (17) and (18) as follows:

$$h_{2,6}(\mathcal{S}/\mathcal{S}_1)=\frac{H_2(\mathcal{S}/\mathcal{S}_1)}{H_6(\mathcal{S})_{\max}}=\frac{H_2(\mathcal{S}/\mathcal{S}_1)}{\log 6} \qquad h_{4,6}(\mathcal{S}/\mathcal{S}_2)=\frac{H_4(\mathcal{S}/\mathcal{S}_2)}{H_6(\mathcal{S})_{\max}}=\frac{H_4(\mathcal{S}/\mathcal{S}_2)}{\log 6}$$

$$h_{2,2}(\mathcal{S}/\mathcal{S}_1)=\frac{H_2(\mathcal{S}/\mathcal{S}_1)}{H_{2\max}(\mathcal{S}/\mathcal{S}_1)}=\frac{H_2(\mathcal{S}/\mathcal{S}_1)}{\log 2} \qquad h_{4,4}(\mathcal{S}/\mathcal{S}_2)=\frac{H_4(\mathcal{S}/\mathcal{S}_2)}{H_4(\mathcal{S}/\mathcal{S}_2)_{\max}}=\frac{H_4(\mathcal{S}/\mathcal{S}_2)}{\log 4}$$

The results for the complete system are resumed in the sequel.

$H_2(\mathcal{S}_1)$ =0.9852,	$H_4(\mathcal{S}_2)$ =1.8464,	$H_6(\mathcal{S})$ =2.3239,	$H_2(\mathcal{S}')$ =0.9798
log 2 =1.0000,	log 4 =2.0000,	log 6 =2.5849,	log 2 =1.0000
$h_{2,2}(\mathcal{S}_1)$ =0.9852,	$h_{4,4}(\mathcal{S}_2)$ =0.9232,	$h_{6,6}(\mathcal{S})$ =0.8990,	$h_{2,2}(\mathcal{S}')$ =0.9798
$h_{2,6}(\mathcal{S}_1)$ =0.3811,	$h_{4,6}(\mathcal{S}_2)$ =0.7143,	$h_{6,6}(\mathcal{S})$ =0.8990,	$h_{2,6}(\mathcal{S}')$ =0.3790

5. CONCLUSION

The note tackles the problem of the system uncertainty representation in a relative manner instead of standards units like "bits" or "nits". Such an approach could possibly help the decision process in system evaluation and design. Each of the presented ways of representation of the system and subsystems uncertainties can be sometimes useful under some circumstances. It is up to the system designer to apply the adequate and rational methods of uncertainty representations, which will utmost meet his needs as well the needs of the system users.

The tables of results of the illustrative examples contains the following methods for presentations of the system uncertainties, given in the article by rows:

- The first row represents the system and subsystems entropy values by definition in bits,
- The second row represents the maximal attainable entropies of considered system and subsystems in bits,
- The third row represents the relative uncertainties, with respect to each system and subsystems themselfs,
- The fourth row represents the relative uncertainties with respect to the maximal attainable uncertainties of the system and of the subsystems,
- The fifth row represents the relative uncertainties only for incomplete system, with respect to the maximal attainable uncertainty of the system considered as a complete one.

The assessment of the uncertainty of systems by representing them by systems of events and application of the entropy for uncertainty measure as defined in information theory, is known in engineering from earlier. The reason that the system uncertainty analysis is not widely adopted in engineering practice could be also in the fact that the engineers are not too much familiar with interpretation of uncertainties in bits or nits. The presented application of the relative entropy for assessment of the uncertainty of complete or incomplete systems of events as well as the reinterpretation of the theorem about the mixture of distribution for assessment of the uncertainty of subsystems, offers a more comprehensive insight in the system features. The uncertainty can be perceived as a fraction or percentage of some known and recognisable reference system uncertainty. Implementation of a relative uncertainty measure to system analysis can facilitate engineering presentation of complex system uncertainties and hopefully a more intensive application of system uncertainty analysis.

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