USAGE OF AVERAGE UNCERTAINTY MEASURES

Kalman Žiha
University of Zagreb, Faculty for Mechanical Engineering and Naval Architecture
I. Lućića 5, Zagreb, Croatia
kziha@fsb.hr

Abstract: *This note considers alternative probabilistic presentations of uncertainties of complex systems and subsystems of events of any level, other than the entropy defined in information theory. The average measures of uncertainty in the note are expressed by the average probability and by the average number of events of a considered system or subsystem of events and are related to the entropy of the system. The average uncertainty measures are expected to facilitate the interpretations of the system uncertainty. Such an approach is hopefully an additional, user-friendly way for the observation of system uncertainties. Numerical examples are attached to illustrate the use of the proposed average uncertainty measures.*
Key words: entropy, information theory, average uncertainty measure, relative uncertainty measure, engineering

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1. Introduction

The scope of this note is the more readable probabilistic presentation of system uncertainties in addition to the commonly adopted entropy of systems of events. It is particularly suited for engineering problems of complex systems of events, consisting of more subsystems of events of any level, each of them pertaining to the specific function of the system.
The uncertainty measure based on the entropy defined earlier in information theory (Wiener, 1948, Shannon and Weaver, 1949) is an adequate general uncertainty measure for systems of events. The Renyi's/Shannon's information of order one can be used to assess the uncertainty of incomplete systems (Renyi, 1970). The theorem about the information associated with the mixture of distributions can be used to assess the uncertainty of subsystems and system of subsystems of events (Žiha, 1998). The definition of a unit of uncertainty is not more and not less arbitrary then the choice of the unit of some physical quantity (Renyi, 1970). The unit of the Shannon's entropy conventionally corresponds to a system of two equally probable events (simple alternatives). In this sense, it is a general but not always easily interpretable measure for engineering purposes. The relative measure of uncertainty brings into the relation the actual entropy to the maximal attainable entropy of a considered system or subsystem (Žiha, 1999). The aim of the paper is to define some other measures like average number of events and average probabilities assigned to systems and subsystems of events in order to assess the system uncertainty (e.g. Aczel and Daroczy, 1975). Such average uncertainty measures provided for engineering systems can be more adequate for some engineering purposes. Moreover, the average uncertainty measures of subsystems can be related to the uncertainty of the whole system. The presented average uncertainty measures are applied to numerical examples in order to demonstrate their usefulness in system uncertainty analysis.

2. Average uncertainty measures for systems of events

Let us consider a complete system $\mathcal{S}$ constituted of N events E_i, $i=1,2,...,N$., as a finite scheme (Khinchin, 1957):

$$\mathcal{S} = \begin{pmatrix} E_1 & E_2 & \cdots & E_n \\ p(E_1) & p(E_2) & \cdots & p(E_n) \end{pmatrix}$$

2.1. Average probability of a system of events

In analogy to the entropy of a single event E with probability $p(E)$ defined as $H(E)=-\log p(E)$, (Wiener, 1948), the Shannon's entropy (Shannon and Weaver, 1949) $H_N(\mathcal{S})$ of a considered system of N events denoted as $\mathcal{S}$, can be defined as follows:

$$H_N(\mathcal{S}) = -\log G_N(\mathcal{S}) = \log F_N(\mathcal{S}) = -\sum_{i=1}^{N} p(E_i)\log p(E_i) = \sum_{i=1}^{N} p(E_i)\log\frac{1}{p(E_i)} \quad (1)$$

In (1), the $G_N(\mathcal{S})$ is a kind of average probability, namely, the geometric mean of the probabilities in system $\mathcal{S}$ weighted with the same probabilities as weights (Aczel and Daroczy, 1970):

$$G_N(\mathcal{S}) = G_N\left[p(E_1), p(E_2),..., p(E_N)\right] = 2^{-H_N(\mathcal{S})} = \prod_{i=1}^{N} p(E_i)^{p(E_i)} \quad (2)$$

More generally, for either complete or incomplete systems of events with known system probability $p(\mathcal{S}) = \sum_{i=1}^{N} p(E_i)$, the average probability can be defined as follows:

$$G_N(\mathcal{S}) = 2^{-H_N^1(\mathcal{S})} = \prod_{i=1}^{N} p(E_i)^{\frac{p(E_i)}{p(\mathcal{S})}} \quad (3)$$

By taking the logarithm of (3) the Renyi's/Shannon's entropy of order one denoted $H^1_N(\mathcal{S})$ is:

$$H_N^1(\mathcal{S}) = -\log G_N(\mathcal{S}) = \log F_N(\mathcal{S}) = -\frac{\sum_{i=1}^{N} p(E_i)\log p(E_i)}{p(\mathcal{S})} = \frac{1}{p(\mathcal{S})}\sum_{i=1}^{N} p(E_i)\log\frac{1}{p(E_i)} \quad (4)$$

Since $\log p(\mathcal{S}) \le H_N^1(\mathcal{S}) \le \log\frac{N}{p(\mathcal{S})}$, it is clear that the bounds of the average probabilities in general are as follows:

$$\frac{p(\mathcal{S})}{N} \le G_N(\mathcal{S}) \le p(\mathcal{S}) \quad (5)$$

$G_n(\mathcal{S})$ in above expressions represents an average probability of occurrence of any of the events within the system of events $\mathcal{S}$. Moreover, the events can be grouped regardless of the ordering of events within the system $\mathcal{S}$, as shown:

$$\mathcal{S} = \begin{pmatrix} E_{11} & ... & E_{1m_1} & ... & E_{i1} & ... & E_{im_i} & ... & E_{n1} & ... & E_{nm_n} \\ p(E_{11}) & ... & p(E_{1m_1}) & ... & p(E_{i1}) & ... & p(E_{im_i}) & ... & p(E_{n1}) & ... & p(E_{nm_n}) \end{pmatrix}$$

Events are distributed into subsystems $\mathcal{S}_i$, $i=1,2,...,n$, each containing E_{ij}, $j=1,2,...,m_i$ events:

$$\mathcal{S}_i = \begin{pmatrix} E_{i1} & ... & E_{ij} & ... & E_{im_i} \\ p(E_{i1}) & ... & p(E_{ij}) & ... & p(E_{im_i}) \end{pmatrix}$$

The probability associated to subsystems $\mathcal{S}_i$, $i=1,2,...,n$, is defined as $p(\mathcal{S}_i) = \sum_{j=1}^{m_i} p(E_{ij})$.

The system $\mathcal{S}$ can be presented as a summa of disjoint subsystems of events, as $\mathcal{S} = (\mathcal{S}_1 + \cdots + \mathcal{S}_i + \cdots + \mathcal{S}_n)$. More generally, if there are some events in common with more subsystems, the system is presented as a union of subsystems $\mathcal{S} = (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_i \cup \cdots \cup \mathcal{S}_n)$.
The system $\mathcal{S}$ can be also viewed as a compound of subsystems of events $\mathcal{S}_1$, $\mathcal{S}_2$, ..., $\mathcal{S}_n$, denoted $\mathcal{S}'$, and presented like as $\mathcal{S}' = (\mathcal{S}_1,...,\mathcal{S}_i,...,\mathcal{S}_n) = \begin{pmatrix} \mathcal{S}_1 & ... & \mathcal{S}_i & ... & \mathcal{S}_n \\ p(\mathcal{S}_1) & ... & p(\mathcal{S}_i) & ... & p(\mathcal{S}_n) \end{pmatrix}$

The probability of systems $\mathcal{S}$ and $\mathcal{S}'$ is defined as $p(\mathcal{S}) = \sum_{i=1}^{n}\sum_{j=1}^{m_i} p(E_{ij}) = p(\mathcal{S}') = \sum_{i=1}^{n} p(\mathcal{S}_i)$.

Subsystems of events are considered to be associated with a mixture of partial distributions, (Žiha, 1998). The average probability of the system of subsystems $\mathcal{S}'$ is defined as follows:

$$G_n(\mathcal{S}') = 2^{-H_n^1(\mathcal{S}')} = \prod_{i=1}^{n} p(\mathcal{S}_i)^{\frac{p(\mathcal{S}_i)}{p(\mathcal{S})}} \tag{6}$$

and (6) can be related to the entropy of order one of the system of subsystems as follows:

$$H_n^1(\mathcal{S}') = -\log G_n(\mathcal{S}') = \log F_n(\mathcal{S}') = -\frac{\sum_{i=1}^{n} p(\mathcal{S}_i)\log p(\mathcal{S}_i)}{p(\mathcal{S})} = \frac{1}{p(\mathcal{S})}\sum_{i=1}^{n} p(\mathcal{S}_i)\log\frac{1}{p(\mathcal{S}_i)} \tag{7}$$

The bounds of the average probabilities of system of subsystems in general are as follows:

$$\frac{p(\mathcal{S})}{n} \le G_n(\mathcal{S}') \le p(\mathcal{S}) \tag{8}$$

The average probability of a subsystem within a system is related to the probability of the event under the condition that the subsystem itself occurs, and can be written as follows:

$$G_{m_i}(\mathcal{S}/\mathcal{S}_i) = \prod_{j=1}^{m_i}\left[\frac{p(E_{ij})}{p(\mathcal{S}_i)}\right]^{\frac{p(E_{ij})}{p(\mathcal{S}_i)}} = 2^{-H_{m_i}(\mathcal{S}/\mathcal{S}_i)} \tag{9}$$

It can be noted that (9) is also related to the entropy of subsystema as shown:

$$H_{m_i}(\mathcal{S}/\mathcal{S}_i') = -\log G_{m_i}(\mathcal{S}/\mathcal{S}') = \log F_{m_i}(\mathcal{S}/\mathcal{S}') = -\sum_{j=1}^{m_i}\frac{p(E_{ij})}{p(\mathcal{S}_i)}\log\frac{p(E_{ij})}{p(\mathcal{S}_i)} \tag{10}$$

The range of the average probability of a subsystem is as follows

$$\frac{1}{m_i} \le G_{m_i}(\mathcal{S}/\mathcal{S}_i) \le 1 \tag{11}$$

The relation of the average conditional probabilities of events of subsystems in (9) to the average probability of events of the system defined by (3) and (6), can be expressed for both the incomplete and complete systems of events, as the weighted geometric mean of terms in (9), as follows:

$$\prod_{i=1}^{n} G_{m_i}(\mathcal{S}/\mathcal{S}_i)^{p(\mathcal{S}_i)} = \left[\frac{G_N(\mathcal{S})}{G_n(\mathcal{S}')}\right]^{p(\mathcal{S})} = \left[\frac{F_n(\mathcal{S}')}{F_N(\mathcal{S})}\right]^{p(\mathcal{S})} \tag{12}$$

The average probability of a complete system of events is maximal and amounts to unity, when one of the events is a “sure” event, i.e. has the probability of one, and all the other probabilities equal zero. The average probability of a complete system is minimal and amounts to *1/N* when all the probabilities are equal.

2.2. Average number of events of a system of events

Let us consider also the reciprocal value of the average probability, denoted as follows:

$$F_N(\mathcal{S}) = \frac{1}{G_N(\mathcal{S})} = \prod_{i=1}^{N}\left[\frac{1}{p(E_i)}\right]^{\frac{p(E_i)}{p(\mathcal{S})}} = 2^{H_N^1(\mathcal{S})} \tag{13}$$

$$F_n(\mathcal{S}') = \frac{1}{G_n(\mathcal{S}')} = \prod_{i=1}^{n}\left[\frac{1}{p(\mathcal{S}_i)}\right]^{-\frac{p(\mathcal{S}_i)}{p(\mathcal{S})}} = 2^{H_n^1(\mathcal{S}')} \tag{14}$$

In analogy to (3), the value of $F_N(\mathcal{S})$ can be interpreted as a kind of average number of events. The bounds of the average number of events are as follow:

$$\frac{1}{p(\mathcal{S})} \le F_N(\mathcal{S}) \le \frac{N}{p(\mathcal{S})} \tag{15}$$

$$\frac{1}{p(\mathcal{S})} \le F_n(\mathcal{S}') \le \frac{n}{p(\mathcal{S})} \tag{16}$$

The average number of events of a subsystem is related to the average probability and to the conditional entropy under the condition that the subsystem itself occurs, as follows:

$$F_{m_i}(\mathcal{S}/\mathcal{S}_i) = \frac{1}{G_{m_i}(\mathcal{S}/\mathcal{S}_i)} = \prod_{j=1}^{m_i} \left[\frac{p(E_{ij})}{p(\mathcal{S}_i)}\right]^{-\frac{p(E_{ij})}{p(\mathcal{S}_i)}} = 2^{H_{m_i}(\mathcal{S}/\mathcal{S}_i)} \tag{17}$$

The range of the average number of events of a subsystem is as follows

$$1 \le F_{m_1}(\mathcal{S}/\mathcal{S}_i) \le m_i \tag{18}$$

The relation of the average conditional number of events (17) of considered subsystems to the average probabilities in (3) and (6), as well as to the average number of events of the system defined by (13) and (14), can be expressed for both the incomplete and complete systems of events, as the weighted geometric mean of terms in (17), as follows:

$$\prod_{i=1}^{n} F_{m_i}(\mathcal{S}/\mathcal{S}_i)^{p(\mathcal{S}_i)} = \left[\frac{G_n(\mathcal{S}')}{G_N(\mathcal{S})}\right]^{p(\mathcal{S})} = \left[\frac{F_N(\mathcal{S})}{F_n(\mathcal{S}')}\right]^{p(\mathcal{S})} \tag{19}$$

The average number of events of a complete system is maximal when all the events are of the same probability and it amounts exactly to N, i.e. the number of events of the basic system. The minimal average number of events is equal to 1, if there is only one “sure” event.
The entropy $H_2(\mathcal{S}) = -x\log x - (1-x)\log(1-x)$, the average probability $G_2(\mathcal{S}) = 2^{-H_2(\mathcal{S})}$ and the average number of events $F_2(\mathcal{S}) = 1/G_2(\mathcal{S}) = 2^{H_2(\mathcal{S})}$ of a system of event of two events, sometimes denoted as simple alternatives (the non-occurrence of an event is itself an event), are presented on Fig. 1.

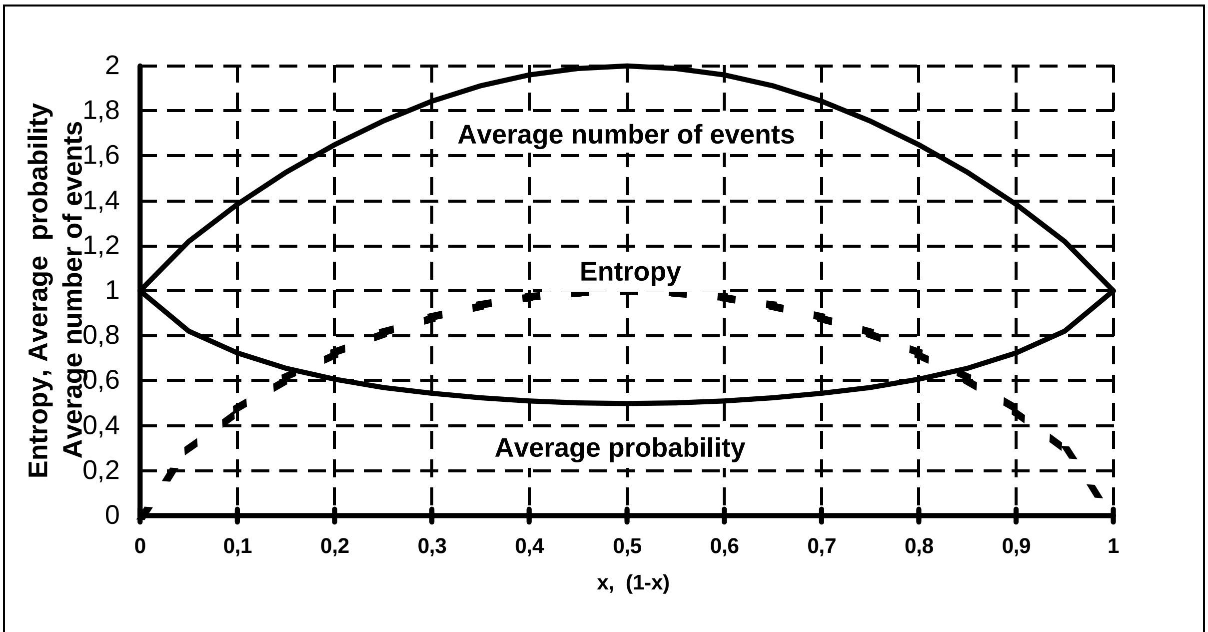

Fig. 1. Entropy, average probability and average number of events of a system of two events

3. Numerical examples for usage of average uncertainty measures

3.1. Incomplete system of events

Let us first consider an incomplete system of events $\mathcal{S}$ consisting of six events and subdivided into two subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, as follows:

$$\mathcal{S}=\begin{pmatrix} E_{11} & E_{12} & E_{21} & E_{22} & E_{23} & E_{24} \\ \frac{6}{24} & \frac{5}{24} & \frac{4}{24} & \frac{3}{24} & \frac{2}{24} & \frac{1}{24} \end{pmatrix},\ \mathcal{S}_1=\begin{pmatrix} E_{11} & E_{12} \\ \frac{6}{24} & \frac{5}{24} \end{pmatrix}\quad \mathcal{S}_2=\begin{pmatrix} E_{21} & E_{22} & E_{23} & E_{24} \\ \frac{4}{24} & \frac{3}{24} & \frac{2}{24} & \frac{1}{24} \end{pmatrix}$$

The probabilities associated with subsystems are $p(\mathcal{S}_1)=\frac{11}{24}$ and $p(\mathcal{S}_2)=\frac{10}{24}$. Systems $\mathcal{S}$ and $\mathcal{S}$' are incomplete since $p(\mathcal{S})=21/24<1$.

The entropy, and the relative measures of uncertainty are considered as first (Žiha, 1999). The average measures of uncertainty applied to the subsystems are calculated according to expressions given herein, and presented relative to their minimal and maximal values. The results for the incomplete system are presented in the sequence.

$H_2(\mathcal{S}/\mathcal{S}_1)=0.9940$	$H_4(\mathcal{S}/\mathcal{S}_2)=1.8464$	$H_6^1(\mathcal{S})=2.5909$	$H_2^1(\mathcal{S}')=1.1910$
$\log 2=1.0000$	$\log 4=2.0000$	$\log 6-\log(7/8)=2.77$	$\log 2-\log(7/8)=1.19$
$h_{2,2}(\mathcal{S}/\mathcal{S}_1)=0.9940$	$h_{4,4}(\mathcal{S}/\mathcal{S}_2)=0.9232$	$h^1_{6,6}(\mathcal{S})=0.9328$	$h^1_{2,2}(\mathcal{S}')=0.9984$
$h^1_{2,6}(\mathcal{S}/\mathcal{S}_1)=0.3578$	$h^1_{4,6}(\mathcal{S}/\mathcal{S}_2)=0.6647$	$h^1_{6,6}(\mathcal{S})=0.9328$	$h^1_{2,6}(\mathcal{S}')=0.4288$
$h_{2,6}(\mathcal{S}/\mathcal{S}_1)=0.3845$	$h_{4,6}(\mathcal{S}/\mathcal{S}_2)=0.7143$	$h_{6,6}(\mathcal{S})=1.0023$	$h_{2,6}(\mathcal{S}')=0.4607$
$G_{min}=0.5$	0.25	0.1458	0.4375
$G_2(\mathcal{S}/\mathcal{S}_1)=0.5021$	$G_4(\mathcal{S}/\mathcal{S}_2)=0.2781$	$G_6(\mathcal{S})=0.1659$	$G_2(\mathcal{S}')=0.4380$
$G_{max}=1.0$	1.0	0.8750	0.8750
$G/G_{min}=1.0042$	1.1124	1.1377	1.0011
$G_{max}/G=1.9916$	3.5958	5.2742	1.9977
$F_{min}=1.0$	1.0	1.1428	1.1428
$F_2(\mathcal{S}/\mathcal{S}_1)=1.9916$	$F_4(\mathcal{S}/\mathcal{S}_2)=3.5960$	$F_6(\mathcal{S})=6.0277$	$F_2(\mathcal{S}')=2.2831$
$F_{max}=2.0$	4.0	6.8571	2.2857
$F/F_{min}=1.9916$	3.5958	5.2745	1.9977
$F_{max}/F=1.0042$	1.1124	1.1377	1.0011
$F/2=0.9985$	1.7980	3.0123	1.1415

3.2. Complete system of events

Let us consider next another system $\mathcal{S}$, now as a complete system of events:

$$\mathcal{S}=\begin{pmatrix} E_{11} & E_{12} & E_{21} & E_{22} & E_{23} & E_{24} \\ \frac{8}{24} & \frac{6}{24} & \frac{4}{24} & \frac{3}{24} & \frac{2}{24} & \frac{1}{24} \end{pmatrix},\quad \mathcal{S}_1=\begin{pmatrix} E_{11} & E_{12} \\ \frac{8}{24} & \frac{6}{24} \end{pmatrix} \mathcal{S}_2=\begin{pmatrix} E_{21} & E_{22} & E_{23} & E_{24} \\ \frac{4}{24} & \frac{3}{24} & \frac{2}{24} & \frac{1}{24} \end{pmatrix}$$

The probabilities associated with subsystems are $p(\mathcal{S}_1)=\frac{14}{24}$ and $p(\mathcal{S}_2)=\frac{10}{24}$, see (7).

Systems $\mathcal{S}$ and $\mathcal{S}$' are complete since $p(\mathcal{S})=24/24=1$.

The entropy, and the relative measures of uncertainty are resumed as first (Žiha, 1999). The average measures of uncertainty applied to the subsystems are calculated according to expressions given herein, and presented as relative to their minimal and maximal values.

The results for the complete system are summarized next.

$H_2(\mathcal{S}/\mathcal{S}_1)=0.9852$	$H_4(\mathcal{S}/\mathcal{S}_2)=1.8464$	$H_6(\mathcal{S})=2.3239$	$H_2(\mathcal{S}')=0.9798$
$\log 2=1.0000$	$\log 4=2.0000$	$\log 6=2.5849$	$\log 2=1.0000$
$h_{2,2}(\mathcal{S}/\mathcal{S}_1)=0.9852$	$h_{4,4}(\mathcal{S}/\mathcal{S}_2)=0.9232$	$h_{6,6}(\mathcal{S})=0.8990$	$h_{2,2}(\mathcal{S}')=0.9798$
$h_{2,6}(\mathcal{S}/\mathcal{S}_1)=0.3811$	$h_{4,6}(\mathcal{S}/\mathcal{S}_2)=0.7143$	$h_{6,6}(\mathcal{S})=0.8990$	$h_{2,6}(\mathcal{S}')=0.3790$
$G_{min}=0.5$	0.25	0.1666	0.5
$G_2(\mathcal{S}/\mathcal{S}_1)=0.5051$	$G_4(\mathcal{S}/\mathcal{S}_2)=0.2781$	$G_6(\mathcal{S})=0.1997$	$G_2(\mathcal{S}')=0.5070$
$G_{max}=1.0$	1.0	1.0	1.0
$G/G_{min}=1.0102$	1.1124	1.1982	1.0140
$G_{max}/G=1.9798$	3.5960	5.0075	1.9724
$F_{min}=1.0$	1.0	1.0	1.0
$F_2(\mathcal{S}/\mathcal{S}_1)=1.9798$	$F_4(\mathcal{S}/\mathcal{S}_2)=3.5960$	$F_6(\mathcal{S})=5.0075$	$F_2(\mathcal{S}')=1.9724$
$F_{max}=2.0$	4.0	6.0	2.0
$F_{max}/F=1.0102$	1.1124	1.1982	1.0140
$F/2=0.9899$	1.7980	2.5037	0.9862

4. Conclusion

The recently introduced event oriented system analysis (Žiha, 2000) requires appropriate presentation and interpretation of system uncertainties. The reason that the system uncertainty analysis is not widely adopted in engineering practice could also be in the fact that the engineers are not familiar with interpretation of uncertainties in ‘bits’ or ‘nits’. The important feature of the uncertainty is not the scale of units in which it is measured but rather it is the meaning. The note tackles problem of the system uncertainty representation in an average manner instead of standard units like “bits” or “nits” and relative measures. It is argued in the note that the uncertainty can be perceived with respect to average probability and average number of events associated to systems and subsystems of events. Such an approach can help the decision process in system evaluation and system design with respect to uncertainties. The average uncertainty measures can be interpreted in a simple manner. The average probability represents such a probability, which, if considered as equal for all events, gives the same entropy for the average number of events, as it is the entropy of the basic system. The last statement can easily be proven both for complete and incomplete systems as follows:

$$-F_N(\mathcal{S})\cdot G_N(\mathcal{S})\log G_N(\mathcal{S}) = -2^{H_N^1(\mathcal{S})}\cdot 2^{-H_N^1(\mathcal{S})}\log 2^{-H_N^1(\mathcal{S})} = H_N^1(\mathcal{S})$$

Each of the presented ways for representation of the system uncertainties can be useful. A system designer can apply the adequate and rational methods of uncertainty representations to meet his or her needs as well the needs of system users. Implementation of average uncertainty measures to system analysis may facilitate engineering presentation of complex system uncertainties and a more intensive application of event oriented system analysis.

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