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INVERSION OF DEGREE $n+2$

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Abstract. By the method of synthetic geometry, we define a seemingly new transformation of a three-dimensional projective space where the corresponding points lie on the rays of the first order, nth class congruence $\mathcal{C}_n^1$ and are conjugate with respect to a proper quadric Ψ. We prove that this transformation maps a straight line onto an $n+2$ order space curve and a plane onto an $n+2$ order surface which contains an n-ple (i.e. n-multiple) straight line. It is shown that in Euclidean space the pedal surfaces of the congruences $\mathcal{C}_n^1$ can be obtained by this transformation. The analytical approach enables new visualizations of the resulting curves and surfaces with the program *Mathematica*. They are shown in four examples.

Introduction

An inversion with respect to a quadric Ψ in a three-dimensional space (projective, affine or Euclidean) is a transformation which maps each point A onto a point A' which is conjugate to A with respect to Ψ, and the pair of points AA' satisfies additional requirements as follows:

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If the lines AA' form a bundle of lines $\{P\}$, the inversion is *quadratic inversion* with a pole P [8]. In the general case, this transformation maps a straight line onto a conic, and a plane onto a quadratic surface.

If the lines AA' form a linear congruence $\mathcal{C}_1^1$, the inversion is called *cubic inversion* [9]. It transforms a straight line and a plane, which are in general positions to $\mathcal{C}_1^1$ and Ψ, onto a twisted cubic and a cubic surface, respectively.

If the lines AA' form the first order and second class congruence $\mathcal{C}_2^1$, the inversion is called *quartic inversion* [4]. It transforms a straight line and a plane, which are in general positions to $\mathcal{C}_2^1$ and Ψ, onto a quartic space curve and a quartic surface, respectively.

In this paper, we shall define a new, more general inversion concept which includes the above-mentioned transformations as special cases.

1. $(1, n)$ congruences

A congruence $\mathcal{C}$ is a doubly infinite line system, i.e. it is a set of lines in three-dimensional space (projective, affine or Euclidean) depending on two parameters. The line $z \in \mathcal{C}$ is said to be the *ray* of the congruence. The *order* of a congruence is the number of its rays which pass through an arbitrary point; the *class* of a congruence is the number of its rays which lie in an arbitrary plane. $\mathcal{C}_n^m$ denotes an mth order nth class congruence. A point is called a *singular point* of a congruence if ∞^1 rays pass through it. A plane is called a *singular plane* of a congruence if it contains ∞^1 rays (1-parametrically infinite lines).

It has been proved that the rays of first order congruences are always the transversals of two curves, or they intersect the same space curve twice. Moreover, it has been proved that only the first order congruence, consisting of a system of lines meeting a proper curve twice, exists when the curve is a twisted cubic, ([10, p. 64], [16, pp. 1184-1185], [14, p. 32]).

There are only two types of first order congruences:

Type I: The 1st order nth class congruence $\mathcal{C}_n^1$ is the system of lines which intersect a space curve c^n of the order n and a straight line d, where c^n and d have $n-1$ common points.

Type II: The 1st order 3rd class congruence $\mathcal{B}_3^1$ is the system of lines which meet a twisted cubic twice.

They are elaborated in detail in [1]. We shall consider below only the congruences of Type I.

1.1. Directing lines. The directing lines of the congruence $\mathcal{C}_n^1$ are a space curve c^n of the order n and a straight line d which intersects c^n at $n-1$ points. Any proper nth order space curve can possess singular points with the highest multiplicity $n-2$. If the directing curve c^n has a multiple point, it must lie on the line d, otherwise it is a contradiction. Any k-multiple point of c^n which lies on d is denoted by D_i^k, where $1 \leqq i \leqq n-k$ (see Fig. 1).

Some of these points can coincide. There are the cases when d is the tangent line of c^n, the tangent at inflection, etc.

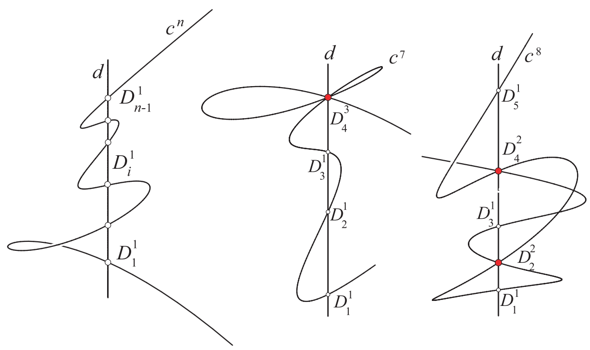


Fig. 1

1.2. Singular points and planes. All singular points of $\mathcal{C}_n^1$ (the points which contain ∞^1 rays of $\mathcal{C}_n^1$) lie on its directing lines c^n and d. If a point C lies on the curve c^n and $C \neq D_i^k$, then the rays of $\mathcal{C}_n^1$ which pass through C form a pencil of lines (C) in the plane δ which contains C and d, (see Fig. 2a). If a point D lies on d and $D \neq D_i^k$, then all the lines which join D with the points of the curve c^n are the rays of $\mathcal{C}_n^1$. They form an nth degree cone ζ_D^n with the apex D. Since c^n and d have $n-1$ common points, this cone intersects itself $n-1$ times through the line d, thus d is the $(n-1)$-ple generatrix of ζ_D^n, (see Fig. 2b). If a point D_i^k is the intersection point of c^n and d, and if it is the k-ple point of c^n, then the rays through D_i^k which cut c^n form an $(n-k)$th degree cone $\zeta_{D_i^k}^{n-k}$ with apex D_i^k. The line d is the $(n-k-1)$-ple generatrix of $\zeta_{D_i^k}^{n-k}$. Besides, the rays through the point D_i^k form k pencils of lines (D_i^k) in the planes determined by the line d and the tangent lines of c^n at D_i^k (see Fig. 2c).

The other lines of the bundle $\{D_i^k\}$ are not regarded as the rays of the congruence $\mathcal{C}_n^1$.

All singular planes of $\mathcal{C}_n^1$ (the planes which contain ∞^1 rays of $\mathcal{C}_n^1$) are planes of the pencil $[d]$ through d. It is clear that in every plane $\delta \in [d]$ lies the pencil of rays (C) or (D_i^k) (see Fig. 3).

If d is the tangent line of c^n at D_i, then the rays of $\mathcal{C}_n^1$ form the pencil of lines (D_i) in the rectifying plane of c^n at D_i.

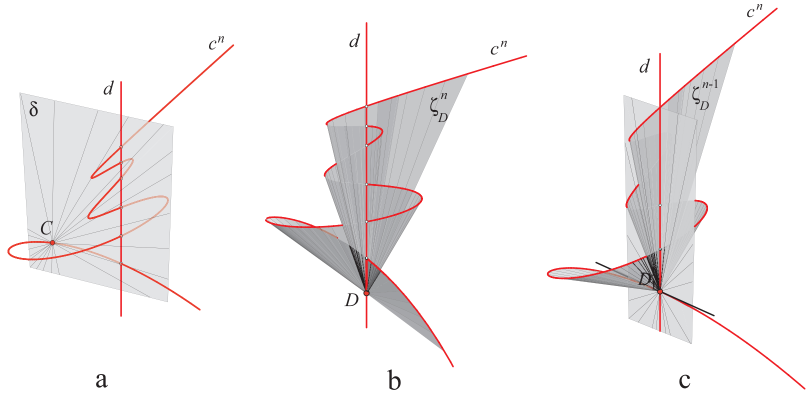


Fig. 2

It is possible that some tangent lines at the intersection points D_j^k lie in the same plane of the pencil $[d]$. In such a case, there is more than one pencil of lines in this plane.

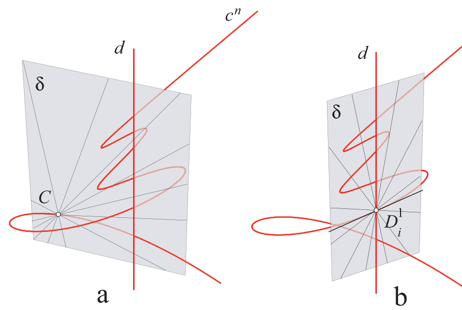


Fig. 3

1.3. Rays through an arbitrary point A **and plane** α Every point A which is not a singular point of $\mathcal{C}_n^1$, i.e. $A \notin c^n, A \notin d$, determines the unique plane $\delta_A \in [d]$ which cuts c^n at only one point C that does not lie on the line d, in general. The line AC, which cuts d at one point D, is the unique ray of $\mathcal{C}_n^1$ through A, denoted by z_A. If the plane δ_A contains one of the tangent lines of c^n at the intersection point D_i^k, then the points C and D coincide with D_i^k, and the line AD_i^k is the unique ray of $\mathcal{C}_n^1$ through A (see Fig. 4).

Every plane $\alpha \notin [d]$ contains n rays of the congruence $\mathcal{C}_n^1$. The plane α cuts the line d at one point D and intersects the nth order space curve c^n at n points $C_j, j = 1, ..., n$. The lines DC_j are n rays of the congruence $\mathcal{C}_n^1$ in the plane α. They are also the intersection of the plane α and the nth degree cone ζ_D^n, and

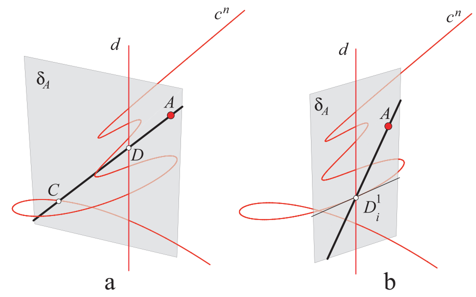


Fig. 4

can be real and different, coinciding or imaginary. If α cuts d at D_i^k, then the $n-k$ rays are the intersection of α and the cone $\zeta^{n-k}_{D_i^k}$, and the other k rays are the intersection of α and the planes through d determined by the tangent lines of c^n at D_i^k (see Fig. 5).

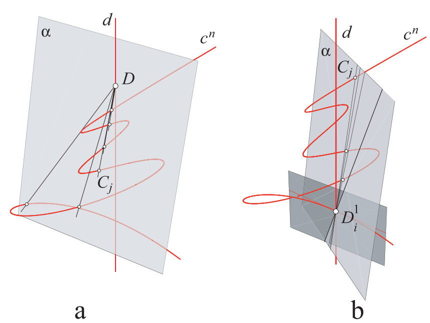


Fig. 5

2. Inversion of degree $n+2$

Two points $A, A' \in \mathbb{P}^3$ are said to be *conjugate points* with respect to a proper quadric Ψ if they are harmonic with respect to the points A_1, A_2 in which the line AA' meets Ψ. For every $A \in \mathbb{P}^3$, all the points A' which are conjugate to A with respect to Ψ lie in the plane π_A which is called the *polar plane* of A with respect to Ψ. $A \in \pi_A$ if and only if $A \in \Psi$, and π_A is the tangent plane of Ψ at the point $A \in \Psi$ [12, p. 266].

The rays of $\mathcal{C}_n^1$ which are the tangent lines of a quadric Ψ form the *ruled surface* Σ^{2n+2} of degree $2n+2$. It is the intersection of the congruence $\mathcal{C}_n^1$ and the 2nd degree complex of the tangent lines to Ψ [10, p. 38].

The ruled surface Σ^{2n+2} touches Ψ along the $(2n+2)$th order space curve e^{2n+2}. For every point E which lies on the curve e^{2n+2}, the ray z_E is the tangent line to Ψ with the contact point E (see Fig. 6).

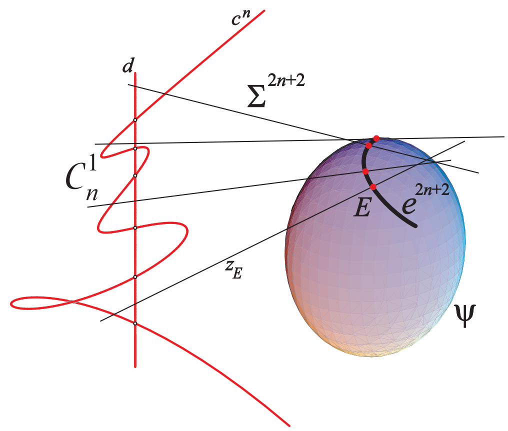


Fig. 6

Definition 1. Let Ψ and $\mathcal{C}_n^1$ be a proper quadric and a congruence in the projective space $\mathbb{P}^3$. Let the directing lines of the congruence $\mathcal{C}_n^1$ (the curve c^n and the straight line d) be in general position to Ψ, i.e. $\Psi \cap d = \{P_1, P_2\}$, $\Psi \cap c^n = \{K_1, ..., K_{2n}\}$. For any point $A \in \mathbb{P}^3$, let $i_\Psi(A)$ be the intersection of π_A (the polar plane of A with respect to Ψ) and the rays of the congruence $\mathcal{C}_n^1$ which pass through A.

(i) If $A \notin d \cup c^n \cup e^{2n+2}$, then $i_\Psi(A) = \pi_A \cap z_A$ is a unique point. Considering the fundamental properties of the polarity with respect to Ψ, A and $i_\Psi(A)$ correspond involutively, i.e. it holds in general that $i_\Psi(i_\Psi(A)) = A$ (see Fig. 7.), especially for $A \in \Psi$, $i_\Psi(A) = A$.

(ii) If $A = E \in e^{2n+2}$, then $i_\Psi(E) = z_E \in \mathcal{C}_n^1$. $i_\Psi(E)$ is a straight line.

(iii) If $A = C \in c^n$ and $C \neq D_i^k$, then $i_\Psi(C) = \pi_C \cap \delta_C$. $i_\Psi(C)$ is a straight line.

(iv) If $A = D \in d$ and $D \neq D_i^k$, then $i_\Psi(D) = \pi_D \cap \zeta_D^n$. $i_\Psi(D)$ is an nth order plane curve.

(v) If $A = D_i^k$, then $i_\Psi(D_i^k) = \pi_{D_i^k} \cap (\zeta_{D_i^k}^{n-k} \cup \delta_{i,k})$. $i_\Psi(D_i^j)$ is the degenerated nth order plane curve. It breaks up into the $(n-k)$th order plane curve $\pi_{D_i^k} \cap \zeta_{D_i^k}^{n-k}$ and the straight lines $\pi_{D_i^k} \cap \delta_{i,k}$.

According to (i)$-$(v), $i_\Psi : \mathbb{P}^3 \to \mathbb{P}^3$ is a (birational) *Cremona transformation* [12, p. 230] with the *singular points* on d, c^n and e^{2n+2}.

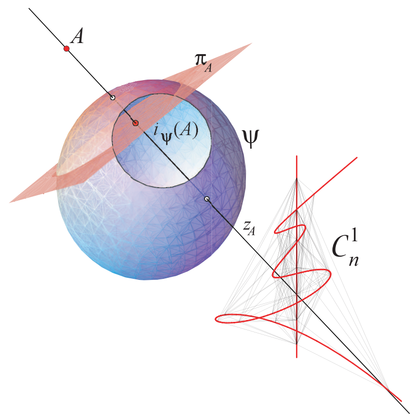


Fig. 7

THEOREM 1. *If l is a straight line and $l \neq d$, then $i_\Psi(l)$ is an $(n+2)$ order space curve.*

PROOF. (i) Assume that a line l does not contain any singular point of the transformation i_Ψ, i.e. $l \cap (d \cup c^n \cup e^{2n+2}) = \{\phi\}$. For any $A \in l$, $i_\Psi(A) = z_A \cap \pi_A$, where z_A is the ruling on an $(n+1)$ degree ruled surface $\Theta^{n+1} = \{z_A \in \mathcal{C}_n^1 : A \in l\}$ and $\pi_A \in [l']$, where l and l' are the conjugate lines with respect to Ψ.

According to the *Chasles formula* [7, p. 48], the result of the (1,1) correspondence between the $(n+1)$ order set of Θ^{n+1} and the pencil of planes $[l']$ is the space curve of the order $(n+1)\cdot 1 + 1 \cdot 1 = n+2$.

(ii) If a line l contains the singular points of the transformation i_Ψ, i.e. if l is in a special position with respect to d, c^n or e^{2n+2}, then the result of the $(1,1)$ correspondence $\Theta^{n+1} \leftrightarrow [l']$ degenerates to the space curve of the order $n+2$. The following cases occur:

(ii$_1$) If a line l contains the point $D \in d$, then the ruled surface Θ^{n+1} splits into the cone ζ_D^n and the plane $\delta_l \in [d]$ determined by l and d. The image $i_\Psi(l)$ splits into the nth order plane curve $\zeta_D \cap \pi_D$ and the conic c_{δ_l} which is the image of l with respect to the plane quadratic inversion determined by the conic $\delta_l \cap \Psi$ and the pole $C = c^n \cap \delta_l$ in δ_l.

If $D = D_i^k$, then the cone of rays through D_i^k splits into the $\zeta_{D_i^k}^{n-k}$ and k planes through d. Therefore, $i_\Psi(l)$ splits into the plane curve of the order $n-k$, k lines in the plane $\pi_{D_i^k}$ and the inverse conic of l in the plane $\delta_l \in [d]$.

If D is the intersection point of d and Ψ (P_1 or P_2), then the intersection curve $\zeta_D^n \cap \pi_D$ splits into n lines through D.

(ii$_2$) If l contains $C \in c^n$, $C \neq D_i^k$, the surface Θ^{n+1} splits into the ruled surface Θ^n (with directing lines c^n, d and l) and the plane $\delta_C \in [d]$. Therefore, $i_\Psi(l)$ splits into the nth order space curve, the result of the (1,1) correspondence between Θ^n and $[l']$, and the line $\delta_C \cap \pi_C$.

If l contains s points $C_j \in c^n$, $s < n-1$, $C_j \neq D_i^k$, then the surface Θ^{n+1} splits into the ruled surface Θ^{n-s+1} (with directing lines c^n, d and l) and s planes $\delta_{C_j} \in [d]$. Therefore $i_\Psi(l)$ splits into the $(n-s+1)$th order space curve, the result of the (1,1) correspondence between Θ^{n-s+1} and $[l']$, the s lines $\delta_{C_j} \cap \pi_{C_j}$ and the line $\delta_C \cap \pi_C$.

(ii$_3$) If l contains $E \in e^{2n+2}$, then the (1,1) correspondence $\Theta^{n+1} \leftrightarrow [l']$ has one pair of incident elements ($z_E \subset \pi_E \in [l']$). Thus the result of this correspondence is an $(n+1)$th order space curve and the line z_E.

If l contains two points of e^{2n+2}, then $i_\Psi(l)$ splits into an nth order space curve and the lines z_{E_1}, z_{E_2}.

(ii$_4$) If l contains $D \in d$ and $C \in c^n$, it is the ray of $\mathcal{C}_n^1$ and then $i_\Psi(l)$ splits into $\zeta_D^n \cap \pi_D$, $\delta_C \cap \pi_C$ and l.

It is easy to see, by combining (ii$_1$), (ii$_2$) and (ii$_3$), that in all other cases when a line l contains the singular points of i_Ψ, the image $i_\Psi(l)$ is the degenerated space curve of the order $n+2$. □

Tehorem 2. $i_\Psi(e^{2n+2}) = \Sigma^{2n+2}$

Proof. For every $E \in e^{2n+2}$, $i_\Psi(E) = z_E$ which is the ruling on Σ^{2n+2}.
□

Theorem 3. *$i_\Psi(c^n)$ is an $(n+1)$ degree ruled surface with the n-ple line d.*

Proof. For the points $C \in c^n$, the polar planes π_C, with respect to Ψ, form the nth class torse Π^n. Since for any $C \in c^n$, $i_\Psi(C) = \pi_C \cap \delta_C$, then the image $i_\Psi(c^n)$ is the result of the (1,1) correspondence between the planes of Π^n and $[d]$. The result of this (1,1) correspondence is the ruled surface of the order $n+1$ [7, p. 48] which contains the line d (the directing line). Since every plane $\delta_C \in [d]$ cuts the ruled surface $i_\Psi(c^n)$ into the line $i_\Psi(C)$ and the line d, we can conclude that d is the n-ple line of $i_\Psi(c^n)$. □

Theorem 4. *$i_\Psi(d)$ is an $(n+1)$ order surface which contains d', c^n, e^{2n+2} and the $(n-1)$-ple line d.*

Proof.For every point $D \in d$, the image $i_\Psi(D)$ is $\zeta_D^n \cap \pi_D$. The cones ζ_D^n, $D \in d$, form the pencil of nth order developables $(\mathbb{Z}^n)$ by the directing lines d and c^n. According to the Chasles formula, the result of the (1,1) correspondence $\mathbb{Z}^n \leftrightarrow [d']$ $(\zeta_D^n \leftrightarrow \pi_D)$ is a surface of the order $n \cdot 1 + 1 \cdot 1 = n+1$.

It is clear that the resulting surface $i_\Psi(d)$ contains the directing lines of $[d']$ and $\mathbb{Z}^n$ (d', c^n and the $(n-1)$-ple line d). It also contains e^{2n+2} because for any $E \in e^{2n+2}$, $E \in \pi_E$ and E is conjugate with respect to Ψ to each point on z_E.

Since z_E cuts the line d at the point D, then for any $E \in e^{2n+2}$, there exists $D \in d$, $E \in i_\Psi(D)$. $\square$

THEOREM 5. *If α is a plane, $i_\Psi(\alpha)$ is an $(n+2)$ order surface which contains the n-ple line d, the curves c^n, e^{2n+2} and the conic $f = \alpha \cap \Psi$.*

PROOF. (i) $\alpha \neq \delta \in [d]$. From Definition 1 and the fact that $\mathcal{C}_n^1$ is a doubly infinite line system, it follows that the locus of the points $i_\Psi(P)$, $P \in \alpha$, is a surface.

Let t be a given line. According to the property $i_\Psi(i_\Psi(A)) = A$, it could be considered that $i_\Psi(i_\Psi(t)) = t$ (excluding the residual images which are the images of the singular points of the transformation i_Ψ). According to Theorem 1, $i_\Psi(t)$ cuts the plane α at $n+2$ points. Thus, $i_\Psi(\alpha)$ and the given line t have also $n+2$ common points.

Every nth order plane curve $i_\Psi(D)$, $D \in d$, cuts α at n points. These points, transformed by i_Ψ, give the point $D \in d$, $D \in i_\Psi(\alpha)$. Therefore, every point of d is the image of the n points of α, i.e., d is the n-ple line of $i_\Psi(\alpha)$.

Every line $i_\Psi(C)$, $C \in c^n$, cuts α at one point. This point, transformed by i_Ψ, gives the point $C \in c^n$, $C \in i_\Psi(\alpha)$. Therefore, every point of c^n is the image of one point of α, i.e., c^n is the curve on $i_\Psi(\alpha)$.

Every line $i_\Psi(E) = z_E$, $E \in e^{2n+2}$, cuts α in one point. This point, transformed by i_Ψ, gives the point $E \in e^{2n+2}$, $E \in i_\Psi(\alpha)$. Therefore, every point of e^{2n+2} is the image of one point of α, i.e., e^{2n+2} is the curve on $i_\Psi(\alpha)$.

Since it follows from $A \in \Psi$ and $A \notin e^{2n+2}$ that $i_\Psi(A) = A$, it is clear that $i_\Psi(\alpha)$ contains $f = \alpha \cap \Psi$.

(ii) If $\alpha = \delta \in [d]$, $i_\Psi(\alpha)$ degenerates to an $(n+1)$ order surface $i_\Psi(d)$ with the $(n-1)$-ple line d and the plane δ. According to Theorem 4, it contains c^n and e^{2n+2}. $\square$

$i_\Psi : \mathbb{P}^3 \to \mathbb{P}^3$, given above by Definition 1, is called the $(n+2)$ *degree inversion* with respect to the congruence $\mathcal{C}_n^1$ and the quadric Ψ. It will be denoted below by i_Ψ^{n+2}.

For $n = 0$ we could consider that the congruence $\mathcal{C}_0^1$ is a bundle of lines and then the inversion i_Ψ^2 is the *quadratic inversion* [8]. If $n = 1$, it is the *cubic inversion* [9]. If $n = 2$, it is the *quartic inversion* [4].

3. Some properties of surfaces given by inversion i_Ψ^{n+2}

The class of the mth order surfaces with the $(m-2)$-ple straight line d was elaborated in detail by Sturm [13, pp. 315-328]. Some properties of such surfaces, denoted by $\mathbb{F}_{m-2}^m$, are the following:

(1) Every plane in the pencil of planes $[d]$ cuts $\mathbb{F}^m_{m-2}$ in the $(m-2)$-ple line d and one conic.

(2) There are $3m-4$ planes of the pencil $[d]$ where the intersection conics split up in two lines.

(3) $2(3m-4)$ straight lines exist on the surface $\mathbb{F}^m_{m-2}$. They lie in the planes of the pencil $[d]$.

According to Theorem 5, for every $\alpha \notin [d]$ the surface $i^{n+2}_\Psi(\alpha)$ belongs to the above-mentioned class of surfaces (for $m = n+2$).

Ad (1) Every plane $\delta_C \in [d]$ which contains the point $C \in c^n$ cuts $i^{n+2}_\Psi(\alpha)$ into the n-ple line d and the conic which is the inverse image of the intersection line $\alpha \cap \delta_C$ given by the plane quadratic inversion with respect to the point C and the intersection conic $\delta_C \cap \Psi$.

Ad (2,3) The plane $\alpha \notin [d]$ cuts the quadric Ψ in the conic f^α, the line d at the point D^α, the curve c^n at the points $C^\alpha_1, ..., C^\alpha_n$ and the curve e^{2n+2} at the points $E^\alpha_1, ..., E^\alpha_{2n+2}$ which lie on the conic f^α. According to Theorem 1, the images of the lines $D^\alpha C^\alpha_i$ $(i = 1, ..., n)$ and $D^\alpha E^\alpha_j$ $(j = 1, ..., 2n+2)$ degenerate to the nth order plane curve $\zeta^n_{D^\alpha} \cap \pi_{D^\alpha}$ and two lines in the planes determined by d and $D^\alpha C^\alpha_i$ or $D^\alpha E^\alpha_j$. Thus, there are $2(3n+2)$ lines on the surface $i_\Psi(\alpha)$ which lie in the $3n+2$ planes of the pencil $[d]$.

In the Euclidean space $\mathbb{E}^3$, the *pedal surface* of a congruence $\mathcal{C}^m_n$ with respect to a pole P is the locus of the foot points of the perpendiculars from the point P to the rays of the congruence $\mathcal{C}^m_n$ [6].

THEOREM 6. *The pedal surface of a congruence $\mathcal{C}^1_n$ (given by the directing lines d and c^n) with respect to a pole P is an $(n+2)$ order surface with the n-ple line d which contains the curve c^n and the absolute conic of $\mathbb{E}^3$.*

PROOF. For every sphere Ψ with the center P and the point at infinity A^∞, the polar plane π_{A^∞} is perpendicular to any line through A^∞ and passes through P. Thus, according to Theorem 5, the pedal surface of $\mathcal{C}^1_n$ for the pole P is the image of the plane at infinity given by the inversion of degree $n+2$ with respect to any sphere with the center P and the congruence $\mathcal{C}^1_n$. According to the same theorem, this $i^{n+2}_\Psi(\alpha^\infty)$ is an $n+2$ order surface with the n-ple line d which contains c^n and the absolute conic (the intersection of the sphere Ψ and the plane at infinity). □

4. Analytical approach and *Mathematica* visualizations

We use the homogeneous Cartesian point coordinates $(x_1 : x_2 : x_3 : x_4)$ in the usual notation where $(0,0,0,0) \neq (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, for any $k \in \mathbb{R} \setminus \{0\}$, $(x_1 : x_2 : x_3 : x_4) = k\,(x_1 : x_2 : x_3 : x_4)$. The relations between the homogeneous and affine Cartesian coordinates (x,y,z) are: $x=\frac{x_1}{x_4},\ y=\frac{x_2}{x_4},\ z=\frac{x_3}{x_4}$.

Without loss of generality we can assume that the line d, the directing straight line of $\mathcal{C}_n^1$, is the axis z, i.e. it is given by the following parametric equations:

$$x_1=0,\quad x_2=0,\quad x_3=u,\quad x_4=1,\quad u\in\mathbb{R}. \tag{1}$$

The nth order space curve c^n, the second directing line of $\mathcal{C}_n^1$, is given by the following parametric equations:

$$x_1=x_1^{c^n}(v),\quad x_2=x_2^{c^n}(v),\quad x_3=x_3^{c^n}(v),\quad x_4=x_4^{c^n}(v),\quad v\in I\subseteq\mathbb{R}, \tag{2}$$

where $\exists\ v_i\in I, i\in\{1,...,n-1\}$ such that $x_1^{c^n}(v_i)=x_2^{c^n}(v_i)=0$. It is clear that the $n-1$ intersection points of d and c^n are

$$D_i(0:0:x_3^{c^n}(v_i):x_4^{c^n}(v_i)). \tag{3}$$

If $A=(a_1:a_2:a_3:a_4)$ is an arbitrary point in general position to the directing lines of $\mathcal{C}_n^1$, then the plane $\delta_A\in[d]$ which contains A and d, given by the equation

$$a_2x_1-a_1x_2=0,\quad a_1,a_2\in\mathbb{R},\ a_1\neq 0\vee a_2\neq 0, \tag{4}$$

intersects the curve c^n at the $n-1$ points D_i, which are given by the formula (3), and the point

$$C_A=(x_1^{c^n}(v_C):x_2^{c^n}(v_C):x_3^{c^n}(v_C):x_4^{c^n}(v_C)) \tag{5}$$

where $C_A\notin d$, i.e. $x_1^{c^n}(v_C)\neq 0\vee x_2^{c^n}(v_C)\neq 0$.

Now, the line $AC_A=z_A$, the unique ray of $\mathcal{C}_n^1$ through A, is given by the following parametric equations:

$$\left\{\begin{array}{ll} x_1^z(s)=\big(x_1^{c^n}(v_C)-a_1\big)s+a_1, & x_2^z(s)=\big(x_2^{c^n}(v_C)-a_2\big)s+a_2\\ x_3^z(s)=\big(x_3^{c^n}(v_C)-a_3\big)s+a_3, & x_4^z(s)=\big(x_4^{c^n}(v_C)-a_4\big)s+a_4, \end{array}\right.\quad s\in\mathbb{R}. \tag{6}$$

Let a quadric Ψ be given by the following equation

$$\sum_{i,j=1}^{4}\alpha_{ij}x_ix_j=0,\quad \alpha_{ij}=\alpha_{ji}. \tag{7}$$

The symmetric bilinear form

$$\Phi(X,\, \tilde{X}) = \sum_{i,j=1}^{4} \alpha_{ij} x_i \tilde{x}_j = 0, \quad \alpha_{ij} = \alpha_{ji}, \tag{8}$$

is called the *polar form* to Ψ. Two points X', X'' with $\Phi(X', X'') = 0$ are conjugate points with respect to Ψ [2]. Hence, for an arbitrary point $A = (a_1 : a_2 : a_3 : a_4)$ the polar plane π_A with respect to the quadric Ψ is given by the equation

$$\Phi(A,\, X) = \sum_{i,j=1}^{4} \alpha_{ij} a_i x_j = 0, \quad \alpha_{ij} = \alpha_{ji}. \tag{9}$$

The point $i_\Psi^{n+2}(A)$ (see Definition 1) is the intersection point of z_A and π_A. Thus, if the substitution $x_j = x_j^z(s),\ j = 1, ..., 4$ is made in the formula (9), the following value for the parameter s is obtained

$$s = \frac{\Phi(A,\, A)}{\Phi(A,\, A) - \Phi(A,\, C_A)} = s_A, \tag{10}$$

and the coordinates of the point $i_\Psi^{n+2}(A)$ are $(x_1^z(s_A) : x_2^z(s_A) : x_3^z(s_A) : x_4^z(s_A))$, where

(11)

$$\begin{cases} x_1^z(s_A) = \left(x_1^{c^n}(v_C) - a_1\right) s_A + a_1, \quad x_2^z(s_A) = \left(x_2^{c^n}(v_C) - a_2\right) s_A + a_2, \\ x_3^z(s_A) = \left(x_3^{c^n}(v_C) - a_3\right) s_A + a_3, \quad x_4^z(s_A) = \left(x_4^{c^n}(v_C) - a_4\right) s_A + a_4. \end{cases}$$

The formulas (6) and (11) enable *Mathematica* visualizations of the rays of $\mathcal{C}_n^1$ and the points, curves and surfaces which are the images given by the inversion i_Ψ^{n+2}. Some examples are shown below.

4.1. Example 1. In the Euclidean space $\mathbb{E}^3$, the directing lines of $\mathcal{C}_2^1$ are a straight line d and a circle c^2 which lies in the plane perpendicular to d and cuts d at the point D (see Fig. 8).

Let the directing lines of $\mathcal{C}_2^1$ (d and c^2) and the quadric Ψ (see Fig. 9a) be given by the following equations

$$\begin{cases} d \ldots\ x = 0, \quad y = 0, \\ c^2 \ldots\ (x-1)^2 + y^2 - 1 = 0, \quad z = 0, \\ \Psi \ldots\ (x-2)^2 + (y-2)^2 + (z-2)^2 - 1 = 0. \end{cases}$$

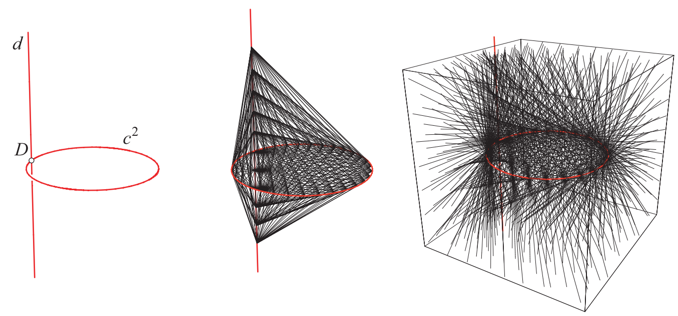
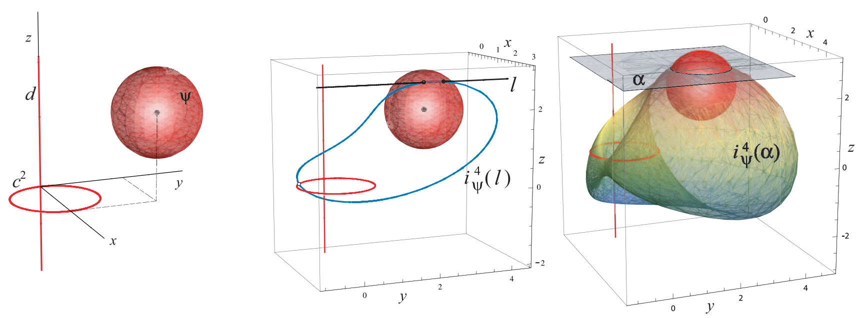


Fig. 8

The 4th order inversion $i^4_\Psi : \mathbb{E}^3 \to \mathbb{E}^3$, with respect to the congruence $\mathcal{C}^1_2$ and the sphere Ψ, transforms the straight line l ($x=2.6$, $z=2.75$) into the 4th order space curve $i^4_\Psi(l)$ (see Fig. 9b), and the plane α ($z=2.5$) into the 4th order surface $i^4_\Psi(\alpha)$ (see Fig. 9c). The surface $i^4_\Psi(\alpha)$ contains the double line d, the circle c^2 and the intersection circle $\alpha \cap \Psi$. Besides the circle $\alpha \cap \Psi$, the plane α cuts the quartic $i^4_\Psi(\alpha)$ into the pair of isotropic lines through the point $d \cap \alpha$ which are the rays of $\mathcal{C}^1_2$ in α.



a b c

Fig. 9

Three surfaces in Fig. 10 are the images of the plane $z=2$ obtained by the inversions i^4_Ψ with respect to $\mathcal{C}^1_2$ and three spheres with the same center $(2, 2, 2)$ but different radii (1, 2 and 4).

4.2. Example 2. As it is shown in the proof of Theorem 6, the pedal surface of a congruence $\mathcal{C}^1_n$ for a pole P is $i^{n+2}_\Psi(\alpha^\infty)$ where Ψ is any sphere with the center P and α^∞ is the plane at infinity. Since the formulas (6) and (11) are valid for the points at infinity, they enable the construction of the pedal surfaces of $\mathcal{C}^1_n$.

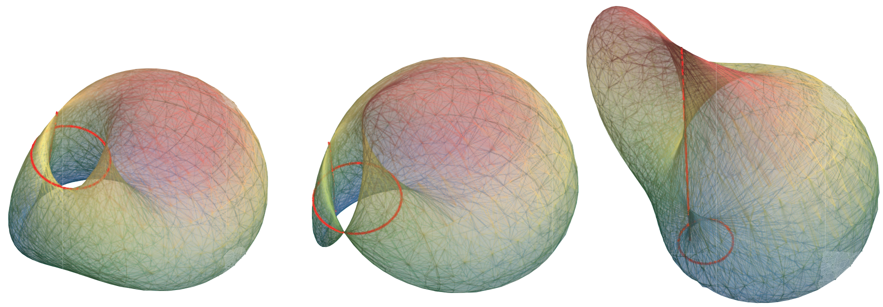
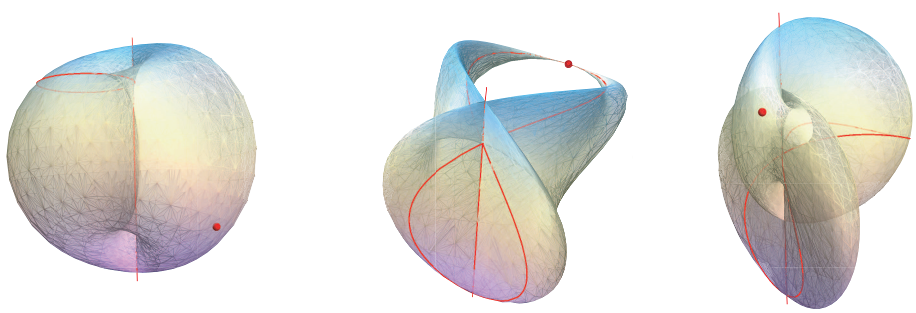


Fig. 10

When the spherical coordinates (r, ϕ, θ) are used, the points at infinity can be presented in the following way

$$A^{\infty} = (\cos\phi\sin\theta : \sin\phi\sin\theta : \cos\theta : 0), \qquad (\phi, \theta) \in [0, \pi)^2.$$

If the directing line d is given by the formulas $x_1 = 0$ and $x_2 = 0$, the point $C_{A^\infty} = c^n \cap \delta_{A^\infty}$ is the intersection of the curve c^n and the plane $x_1 \cdot \sin\phi - x_2 \cdot \cos\phi = 0$. The points A^∞, C_{A^∞}, formulas (10) and (11) give the parametric equations of the pedal surface of the congruence $\mathcal{C}_n^1$.



a b c

Fig. 11

Fig. 11a shows the pedal surface of the same $\mathcal{C}_2^1$ as in Example 1 with the pole $(-1, 2, -4)$. It is a quartic surface with the double line d. (The pedal surfaces of $\mathcal{C}_2^1$ are classified in [3]). In figures 11b and 11c the pedal surfaces of one special $\mathcal{C}_4^1$ are shown. The directing lines of $\mathcal{C}_4^1$ are the Viviani's curve c^4 and the line d which cuts it at two points, where one of them is the double point of c^4. The pedals are sextic surfaces with the quadruple line d. The coordinates of the pole P are $(-2\sqrt{2}, 0, 0)$ in the case (b) and $(1, 1, 1)$ in the case (c).

4.3. Example 3. The following polynomial parametric equations for the directing lines of $\mathcal{C}_n^1$ were used for illustrations in the sections 2 and 3.

(12)

$$\begin{cases} x_1^d(u)=0, \quad x_2^d(u)=0, \quad x_3^d(u)=u, \quad x_4^d=1, \quad u\in\mathbb{R}, \\ x_1^{c^n}(v)=(v-v_1)\cdots(v-v_{n-1}), \qquad v,v_1,\dots,v_{n-1}\in\mathbb{R}. \\ x_2^{c^n}(v)=v\,x_1^{c^n}(v), \quad x_3^{c^n}(v)=v, \quad x_4^{c^n}(v)=1, \end{cases}$$

It is clear that the line d is the axis z and c^n is the nth order space curve which cuts the axis z at the points $D_i(0{:}0{:}v_i{:}1)$, $i\in\{1,...,n-1\}$.

If the polynomial $x_1^{c^n}(v)$ from (12) contains the factor $(v-v_i)^s$, then $i\leqq n-s$ and d and c^n have an $s-$ple contact at the point $D_i(0{:}0{:}v_i{:}1)$.

If the polynomial $x_3^{c^n}(v)$ from (12) takes the form

$$\text{(13)} \qquad \begin{cases} x_3^{c^n}(v)=v(v-v_{i_1})\cdots(v-v_{i_k}),\ v_{i_j}\neq 0, \\ i_1,\dots,i_k\in\{1,\dots,n-1\}, \quad k\leqq n-2, \end{cases}$$

then $(0:0:0:1)$ is the $k-$ple singular point of c^n, and the coordinates of the intersection points of c^n and d are $(0{:}0{:}x_3^{c^n}(v_i){:}1)$.

Fig. 12 shows some rays of the two congruences with the directing lines given by equations (12). There occurs $\mathcal{C}_3^1$ in the case (a) and $\mathcal{C}_7^1$ in the case (b). Since the congruences $\mathcal{C}_n^1$ of this kind have real rays at infinity, the surfaces given by the inversion with respect to them have the sheets with the real intersections at infinity and are not easily displayed. Fig. 12c shows one of these examples.

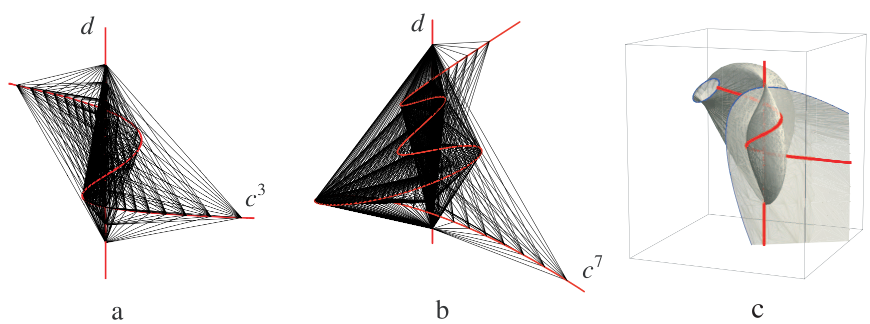


Fig. 12

4.4. Example 4. A special class of $\mathcal{C}_n^1$ arises if the directing lines are an nth order plane curve c^n with the $(n-1)$-ple point M and a straight line d which cuts c^n in M.

Without loss of generality, we assume that the plane of the curve c^n is perpendicular to the line d and that the point M is the center of the Cartesian coordinate system $O(x, y, z)$. According to [11, p. 27], the directing lines of these congruences can be given as follows:

$$d \ldots \; x = 0, \quad y = 0, \qquad c^n \ldots \; p^n + p^{n-1} = 0, \quad z = 0,$$

where p^n, p^{n-1} are homogeneous polynomial in x, y (of degree n and $n-1$, respectively) and the $n-1$ tangent lines of c^n at $(0,0,0)$ are represented by equation $p^{n-1} = 0$. Fig. 13 shows three examples of such congruences.

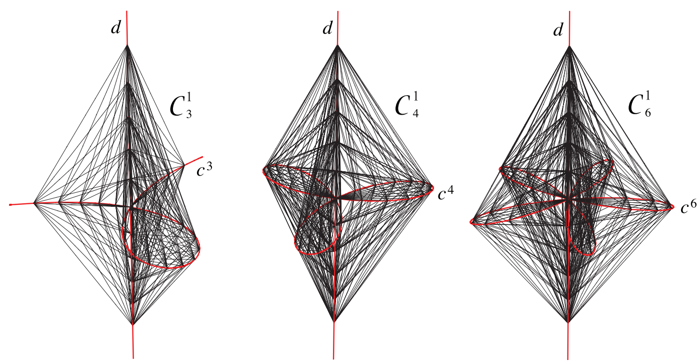


Fig. 13

Fig. 14 shows the three examples of the surfaces $i_\Psi^{n+2}(\alpha)$, where α are the planes perpendicular to d, Ψ is a sphere, and congruences are $\mathcal{C}_n^1$ from Fig. 13. If n is an odd number, surface $i_\Psi^{n+2}(\alpha)$ has real intersection at infinity (the first case in Fig. 14), and it cannot be displayed completely.

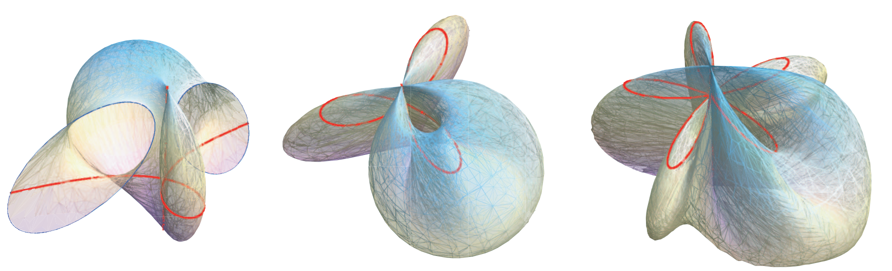


Fig. 14

Conclusion. Examples given in this paper show only a small number of the above-mentioned surfaces $\mathbb{F}_n^{n+2}$ which could be obtained by the inversion i_Ψ^{n+2}. Furthermore, for future research it would be worth exploring if the whole class of surfaces $\mathbb{F}_n^{n+2}$ could be obtained by that inversion and visualized with the program *Mathematica*.

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