Logical Foundations of Metaphysics

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Hume's Principle and Sortal Concepts

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1. Introduction.

In this talk I try to evaluate the neo-Fregeans' program by concentrating on the "Caesar Problem" (the problem Frege himself raised for what was, in effect, his anticipation of neo-logicism) and neo-fregeans' solution to it, as well as on some issues concerning Hume's Principle in particular (and abstraction principles in general).

Contemporary neo-Fregean logicism attempts to salvage the insights of Frege's logicism from the havoc wrought by the contradictoriness of Basic Law V[1].

Firstly, it attempts to vindicate the spirit, if not the letter, of the basic doctrines of Frege's logicism, by developing a systematic treatment of arithmetic and analysis which *approaches* the requirements of Frege's doctrine while avoiding inconsistency.

Secondly, it attempts to consolidate this doctrine's approach to the ontological and epistemological difficulties which have always dogged platonism.

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[1] As Vol. 2 of the *Grundgesetze* was about to go to press in 1903, Bertrand Russell wrote to Frege, showing how to derive Russell's paradox from Basic Law V. (This letter and Frege's reply are translated in Jean van Heijenoort (1967). Hence the system of the *Grundgesetze* was inconsistent. Frege wrote a hasty last-minute appendix to vol. 2, deriving the contradiction and proposing to eliminate it by modifying Basic Law V. Frege's proposed remedy was subsequently shown to imply that there is but one object in the universe of discourse, and hence is worthless (indeed this is a contradiction in Frege's system because he takes True and False to be distinct objects; see e.g. Michael Dummett (1973). From Wikipedia

2. The Caesar Problem

The Caesar problem is the problem Frege himself raised for what was, in effect, his anticipation of neo-logicism

Let me just remind you briefly what the Cesar problem amounts to.

Having defined equinumerosity, Frege is in a position to assert the criterion of identity for numbers formally.

This criterion has been called "Hume's Principle" and it states necessary and sufficient conditions for the identity of “the number of F’s” and “the number of G’s” – namely, one-one correspondence between the F’s and the G’s.

Hume’s Principle:

The number of Fs is identical to the number of Gs
if and only if
the concept F is equinumerous with the concept G[2].

In symbols:

$$\forall F \, \forall G \, (n(F) = n(G) \; \Leftrightarrow \; F \approx G)$$

we take n(...) to abbreviate 'the number of ...s', and $F \approx G$ to abbreviate 'F is equinumerous with G'.

Is Hume’s principle the criterion of identity for numbers that Frege seeks? No. His reason for saying so is that while it serves perfectly well to give meaning to *some* numerical identities, it does not suffice to give meaning to them *all*.

Namely, although it determines the truth value of identities of the form:

'the number of F = the number of G', for any two concepts F and G,

it does not determine the truth value of sentences of the form:

'the number of F = x', for arbitrary x.

The consequence Frege draws from this limitation is that:

> ... we can never - to take a crude example - decide by means of our definitions whether any concept has the number Julius Caesar

[2] *Grundlagen*, § 73.

> belonging to it, or whether that conqueror of Gaul is a number or is not.[3]

The problem Frege raises here, whereby Hume's principle seems to leave the truth value – and hence the meaning - of so-called “mixed” identities undetermined is now called the 'Caesar problem.’

Frege himself takes this problem to be insoluble, and resortes to an explicit definition of natural number from which Hume’s Principle can be derived. This treatment by-passes the Caesar Problem.

According to Frege’s definition, natural numbers are the extensions of higher-order concepts of the form “equinumerous with the concept F.” Since these extensions comprise lower-order concepts, and Caesar is not a concept, the sense, and truth value of a statement “the number of F’s is Julius Caesar” is determined.

In the explicit definition of an object, its criterion of identity is given as comprehensively as possible.

The explicit definition of “the number of F’s” and, of course, of numerical singular terms leads him into the theory of extensions (his definition of e.g. the former being ‘the extension of the second-order concept: "being equinumerous with F"’), and, hence, to the disastrous Basic Law V.

Once the inconsistency of Basic Law V becames known to Frege, he quickly abandones his logicism programme as a whole.[4]

Neo-Fregeans, however, advocate a more measured response.

They observe that the explicit definition of numbers, and hence the theory of extensions, and therefore the inconsistent Basic Law V, serves no purpose, in effect, other than the derivation of Hume's Principle.

Since Frege himself showes that Hume’s Principle alone suffices for the derivation of the Peano Axioms in second order logic (this is “Frege’s theorem”), the *formal* requirements of arithmetic at least can therefore be met, without inconsistency, by simply adding Hume’s Principle to the second-order logic as an supplementary axiom.[5]

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[3]*Grundlagen*, § 56.

[4]Frege did, in his latest years, tried to base arithmetic on geometry but I have been concentrated in the paper exclusively on his logicist programme as a platonistic one.

[5]Hume's Principle, in the language of second-order logic, is:

$$\forall F\ \forall G\ (\ n(F) = n(G) \Leftrightarrow \exists R\ (\ \forall x\ (\ (Fx \rightarrow \exists! y\ (Gy \wedge xRy))\ \wedge (\ Gx \rightarrow \exists!\ y\ (\ Fy \wedge yRx)\ )\ )\ )$$

According to Frege's neo-logicist followers, then, his logicism is correct in all fundamental respects except for two related points at which his judgement goes awry. Firstly, Frege overestimates the significance of the Caesar Problem. And secondly, he underestimates the significance of his derivation of the Peano Axioms from Hume's Principle in second order logic, and of the possibility of *grounding* the claim that arithmetic is analytic in Hume's Principle.

But what about the Caesar problem? What solution does neo-Fregean logicism have to offer?

Neo-Fregeans resist Frege's treatment of this problem.

They disagree with Frege that introducing the natural numbers by means of Hume's Principle leaves the criteria of identity undetermined in a way made explicit in the Caesar Problem. In effect, they do so on the grounds that he misconceives the nature of so-called "sortal concepts".

The key to their proposal is the notion of a "sortal concept", and in particular, the idea that sortal concepts themselves fall into different categories.[6]

What is a sortal concept? A concept F is sortal iff it always makes sense to ask how many F's are there that satisfy a certain condition (given that the stated condition itself makes sense).

So, clearly, the concepts cat, person, number, book and so on are sortal. By contrast, to give Frege's own example, the concept 'red' appears not to be.

No doubt in certain contexts one might sensibly ask "How many red things are there?", but one cannot do this in general. This is because the answer cannot be determinate unless context supplements the question with some further concept – 'book', 'pen' etc. – which is itself sortal. In the absence of such supplementation, the question appears not to have a determinate answer. To be sure, if, after an accident, one's faculties were being tested, and, confronted with a table on which seven figurines were placed, three of them red, one was asked "How many red things are there?", one might immediately answer "Three." But that is just to say that one understood the question to be "How many figurines on the table are red?" Had one not provided a supplementary sortal concept in this way, one wouldn't have known what to answer. Here is one red figurine. But it has two arms which are red, and two legs too? So that makes four red things? No. For the arms have hands, and they are red ...

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[6]See, for example, Hale and Wright, *The Reasons's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*, pp. 336-396.

The concept of natural number is paradigmatically sortal: numbers are used to count, but they can also be counted. As Hale and Wright say, when Frege says that numbers are objects, 'he is best taken to mean that Number is a (non-empty) sortal concept in this technical sense.'[7]

It is clear from their definition that sortal concepts involve a criterion of individuation: Are A and B one book or two books? And a criterion of identity over time: Is A the same book I bought yesterday? These criteria are essential to the concepts. For example, unless books are determinately individuated, the question "How many books are on the table?" would have no answer. Accordingly, the concept of natural number must likewise embody criterion of individuation. Unless whether two numbers were identical were always a determinate matter, it would not be the case that the question "How many numbers are there?" always makes sense.

This much does little more than to sharpen our understanding of the Caesar Problem. In effect, it merely allows a restatement of Frege's claim that the concept of natural number embodies a criterion of individuation which Hume's Principle alone is unable to provide. However, we are now in a position to understand the neo-Fregeans' rebuttal of Frege's attitude. But, before that, we just have to introduce

As I said, this turns on the notion of "categories" of sortal concepts. A category 'may usefully and naturally be identified with a *maximally extensive sortal*'[8]. A maximally extensive sortal F is a sortal concept such that all the sub-sortals have the same criterion of identity and no other concept can be added as sharing the same criterion of individuation.

An example might be the category, that is, the maximally extensive sortal concept, of "being a spatio-temporal located object". The criterion of individuation that works for all spatio-temporal located objects *might* be:

Given two spatio-temporally located objects S_1, S_2:

$S_1 = S_2$

if and only if

at all times, S_1 occupies the same place as S_2.[9]

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[7]Hale and Wright, *The Reasons's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*, p. 367. All the other quotations in this talk are from the same book!

[8]Hale and Wright, *The Reasons's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*, p. 389.

[9]This criterion is nevertheless contentious. Wiggins for example would not agree with it since he thinks it is dodge to have examples like this: "This lump of clay is identical to this statue" even though clay and statue occupy the same place at all times. Wiggins holds that the lump of clay is not identical to the

How does the notion of a maximally extensive sortal solve the Caesar Problem?
The crucial consequence of such a distinction is that no concept can belong to two different categories.
So, inside of a category, objects can be differentiated by referring to the criterion of identity while objects that do not belong to the same category are differentiated precisely by the fact that they belong to different categories.
So, if we take an object to which the criterion of individuation associated with the concepts of a category C could not apply, we have a guarantee that it does not fall under any of those concepts.
It follows that Hume's Principle alone guarantees that Julius Caesar is not a number!
Since, Julius Caesar falls under the concept "person" (or Person), the criterion of individuation that applies to him is the criterion embodied in the concept person and, all the other concepts in the category to which the concept person belongs (whatever that is).
In contrast, Hume's Principle tells us that the concept "natural number" belongs to a different category: for its criterion of individuation, as provided by that principle – in terms of one-one correspondence between the entities which fall under a concept – is not a criterion which applies to people.
Since Caesar is not subject to the criterion of individuation to which natural numbers are subjected (by Hume's Principle), and vice-versa, it follows that Caesar cannot fall under the concept natural number.
Hence, contrary to Frege's fears Hume's Principle does tell us, after all, that Caesar is not a number.
Perhaps it might be objected that such a solution to the Caesar problem requires a developed underlying theory that characterises or defines the concepts or the objects in question *prior* to applying Hume's Principle, and that the need for such a theory is precisely what Frege is complaining about. However, this criticism misses the point.
According to neo-Fregeans, Hume's Principle doesn't merely offer a criterion of identity for the (sortal) concept of number, or merely *partially* explains the concept:

> The neo-Fregean, however, makes a stronger claim - that by stipulating that the number of *F*s is the same as the number of *G*s

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statue; it just constitutes the statue. But I appealed to this criterion of individuation only to give an example. It is not my concern to defend it.

> just in the case the *F*s are one-one correlated with the *G*s, we can set up *number* as a sortal concept, i.e. that Hume's Principle *suffices* to explain the concept of *number* as a sortal concept.[10]

It is of the essence of neo-Fregeanism that Hume's Principle alone suffices to give a *complete* explanation of the sortal concept "natural number", and, eventually, once suitable truths involving numerical singular terms have been proven, of the objects which fall under it.

Now, what is my comment on that?

I agree with neo-Fregean logicism in its dispute with Frege regarding the Caesar Problem. The criterion of identity invoked in Hume's Principle does settle the question as to whether Caesar is a natural number in the negative.

3. Neo-Fregeanism' Platonism

To say agree that Hume's Principle establishes that Caesar is not a natural number (given that Caesar is a person), is to agree that it establishes that no identity statement of the form "Julius Caesar is identical to the number of F's" is true.

And, hence, on the further assumption that every number is such that for some F, it is the number of F's, it is to agree that no identity statement of the form "Julius Caesar is n", where n is any numerical singular term, is true.

However, this concession falls far short of what neo-Fregeans require of Hume's Principle. For they take it, in combination with uncontentious facts, to generate *true* identity statements of the form "the number of F's is identical to the number of G's." And they take it to guarantee that the singular terms these identities involve refer, and, hence, that existential generalisation applies to them.

In particular, one can infer from the truth of an identity of the kind just quoted that there is some x, such that x is identical to the number of G's. But to concede that the Caesar Problem can be overcome is not to concede this much. Let me explain why.

Following Frege himself, neo-Fregeans[11] maintain that it is possible to define abstract sortal concepts - that is concepts whose instances are abstract objects of a certain kind - by stipulation. What has to be stipulated is the truth of an abstraction principle.

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[10]Hale and Wright, *The Reasons's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*, p. 15.

[11]See, for example Hale, Bob (1999) 'Intuition and Reflection in Arithmetic II', Proceedings of the Aristotelian Society, 73 (Suppl.), pp. 75-98.

Abstraction principles have the general form:

$$\forall f \forall g \, (\psi(f) = \psi(g) \leftrightarrow f \approx g)$$

Here, f and g are variables referring to entities of a certain kind (objects or concepts usually), ψ is an operator which forms singular terms when applied to f and g – so that $\psi(f)$ and $\psi(g)$ are singular terms referring to objects, and $\approx$ is an equivalence relation on entities denoted by f and g.

Frege himself invokes three abstraction principles:

The direction principle:

The direction of the line *a* is identical to the direction of the line *b*
if and only if
the line *a* is parallel to the line *b*.

Hume's principle:

The number belonging to the concept F is identical
to the number belonging to the concept F
if and only if
the concept F is equinumerous with the concept G.

Basic Law V:

The extension of the concept F is identical with the extension of the concept G
if and only if
every object that falls under F falls under G and conversely.[12]

The third one notoriously turns out to be inconsistent.

Abstraction principles are of major importance for neo-Fregean logicism. It is abstraction principles which bear the main burden of the task of reconciling logicist or neo-Fregean logicist thesis that arithmetic and analysis are pure logic. In so far as they are *stipulations* they can aspire to explain in one stroke both how *logic* can be committed to abstract objects, and how it is possible to have knowledge of these objects.

[12]Frege formulates the first two in *Grundlagen*, while the third one in *Grundgesetze*.

For, in typical cases – such as the direction principle – our knowledge of instances of the right-hand side of the equivalences is relatively unproblematic.

We know on occasion that one line is parallel to another. If the corresponding abstraction principle is to be believed, then, a mere stipulation affords us knowledge on this basis of something else – namely, that one abstract object – the direction of the one line – is identical to another – the direction of the other.

On the other hand, the burden neo-Fregeans place on abstraction principles, following Frege, might seem unbearable.

The problem is however the following one:

How can mere stipulation ensure the existence of the objects apparently referred to on the left hand side of an abstraction principles' equivalence sign, when nothing on the right hand side makes even apparent reference to such objects? How can objects, the existence of which is independent of us and our practices, be *stipulated* into existence?

The neo-Fregeans' answer is the following one:

The stipulation guarantees – or shows, even – that the relevant abstract objects' existence *consists* in the truth instances of the other (right-hand) sides of such principles. As Hale points out in the case of the direction principle:

> directions simply are (on this explanation) objects for whose identity (and therefore for whose existence) it is necessary and sufficient that the corresponding statement of line-parallelism be true.[13]

Fundamental to this conception of the (possible) role of abstraction principles is the idea that the objects on the left hand side of an abstraction principles' equivalence sign introduce no further *content* than is already presupposed on the right hand side. So to speak, for example, if it is given that line b is parallel to line c, then the idea that an object – the direction of b – is identical to another – the direction of c – is given too. This idea gives *the same fact*. In Hale's terminology, the right and the left-hand sides of abstraction principles are simply "carving up" one and the same content in a different way.

The neo-Fregean perspective on abstraction principles promises to be, then, something of a holy grail of platonism. Our knowledge that one line is parallel to another is not in doubt. But now we are told that, in effect, this knowledge *is* knowledge of abstract objects. Our knowledge that there is a one-one correspondence

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[13]Hale, B. 'Intuition and Reflection in Arithmetic II', p. 94.

between the F's and the G's is not in doubt. But now we know that, in effect, this knowledge *is* knowledge of numbers – and, in particular, of the number of F's – namely, that this number is identical to the number of G's.

Lets' concentrate now specifically on Hume's principle.

It is very easy to be seduced into thinking that the neo-Fregeans' claims for Hume's Principle are legitimate. We all unthinkingly believe that (a suitably restricted version of) Hume' Principle is true! Hence,

once we learn that the knives are one-one correlated with the forks, we automatically infer that the number of knives is identical to the number of forks. But of course neo-Fregeans construe the proposition we infer in this way differently from what we might suppose, and once their suppositions are made explicit, our readiness to make the inference might diminish.

Neo-Fregeans read into our unthinking "the number of knives is identical to the number of forks" the further proposition that there *exists* an object – an abstract object no less – such that this object is identical to the number of knives. Of course, they argue that this further existential proposition is, upon reflection, a commitment which can be teased out of the numerical identity we unthinkingly endorse.

But the fact remains that once it is made explicit, and accept it, we are liable to be less sure than we were that the one-one correlation between knives and forks itself suffices for the truth of that identity.

My present reluctance to accept the neo-Fregean attitude to Hume's principle, and to abstraction principles in general, it's not to mean my refusal of Platonism in general though.

I'm simply expressing doubts about the neo-Fregeans' conception of the *ground* of platonism.

I'd like to resist here the view that that numbers exist, and are abstract objects, is stipulatively implicit, as a matter of sheer logic, in facts like: the knives are 1-1 correlated with the forks!

On the contrary, I emphasise the indispensability of mathematics to physical science (in addition to the "obviousness" of certain mathematical truths) as the ground of realism about truth value (in the strong sense), and, hence, for realism about ontology (and hence platonism). There was no commitment in any of that to the neo-Fregeans' conception of the conceptual and epistemological connection between number theory and Hume's Principle; quite the opposite.

Ultimately, I think, the neo-Fregeans' employment of Hume's Principle boils down to Frege's thesis that syntax is prior to, and the arbiter of, ontology. It is hard – though perhaps not impossible – to maintain that a 1-1 correlation between the knives and the forks does not *itself* suffice for the *truth* of the identity: the number of knives is identical to the number of forks.

What more could suffice for it?! What more, so to speak, does the world have to do in order to make it the case that the number of knives is identical to the number of forks?

Nevertheless, it is an extra step from the truth of this identity to the *objective* truth of this identity *in the strong sense (of chapter 2)*, and, hence to the existence of numbers.

Following Frege's priority of syntax thesis, neo-Fregeans see this as no step at all. According to them, the required condition for singular terms to refer is they

> occur in true statements free of all epistemic, modal, quotational, and other forms of vocabulary standardly recognized to compromise straightforward referential function. For if certain expressions function as singular terms in various true extensional contexts, there can be no further question but that those expressions have reference, and, since they are singular terms, refer to objects.[14]

A proposition that contains singular terms actually cannot be true unless those singular terms do refer. And when they refer, they refer to objects.

In the case of mathematical expressions containing numerical singular terms, that is numerals, such expressions cannot be true unless there exist objects – numbers – to which the included numerals refer to.

But in my terms this is question-begging.

It is true enough for strong truth, for truth in the objective sense. But it is incorrect for truth in the weaker sense. Consequently, even if we agreed that Hume's Principle is stipulatively true, the crucial question would arise as to whether stipulated truth is objective truth in the strong sense.

Neo-Fregeans are simply wrong to assert that this question does not arise.

To settle that the statement "the number of knives is identical to the number of forks" is true, by stipulative reference to a 1-1 correlation between the knives and the forks, is *not* to establish that the numerical singular terms it involves refer. The truth of the identity "Hamlet is identical to Hamlet" tells us that much.

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[14]Hale and Wright, *The Reasons's Proper Study: Essays towards a Neo-Fregean Philosophy of Mathematics*, p. 8.

Failure to recognise this fact lands neo-Fregeans with a problem of circularity.

Neo-Fregeans offer criteria for determining when an expression functions as a singular term. But by so doing they undermine the criteria we normally employ for ascertaining that the criterion is met. For example, the effect of the supposition that a 1-1 correlation between the knives and the forks *suffices* for the truth of the identity “the number of knives is identical to the number of forks” to be true, and, hence, for the existence of an abstract object to which the singular term “the number of knives” refers, is to make me wary of my earlier belief that the knives *are* 1-1 correlated with the forks.

After all, it looks now as if in order to determine whether that much is true I have to ascertain whether the term “the number of knives” refers. Nothing has been said to prevent that from being a real issue, since nothing has been said to preclude the possibility that, contrary to my pre-reflective belief, it is not the case that the knives are 1-1 correlated with the forks.

Moreover, I am left in an inescapable quandary.

Besides, I can no longer be confident as to whether the condition for “the number of knives” having a referent is met, until I first ascertain whether “the number of knives” has a referent.

The problem can be illustrated too by identity statements in general. According to Frege, an object exists if statements about it are objectively true; '*par excellence* statements of identity'.

But how can an identity statements like '0=0' be distinguished from identity statements like 'Pegasus=Pegasus'? It seems that, no matter what we insert for 't', the identity statement 't=t' is always true.

One answer might be that 'Pegasus=Pegasus' is not objectively true because 'Pegasus' is not referential. But the point is that we are supposed to know that 'Pegasus=Pegasus' is not objectively true prior to our knowing that 'Pegasus' is not referential. We are supposed to distinguished identity 't=t' in which 't' is referential from those in which 't' is not referential *before* we acknowledge if the term 't' is referential or not. This problem seems insoluble.