SEFA 5

5th Conference of The Spanish Society for Analytic Philosophy

University of Barcelona -Spain

7 - 9 September 2007

Are There Any Sets Out There?

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1. Introduction

In the philosophy of mathematics, one of main concern is whether mathematical statements have objective truth values. One possible answer to the question is the realistic one: mathematics concerns itself with certain objects - numbers, sets, functions, groups etc. – and the claims it makes about these objects are determinately and objectively true or false. At this point a question naturally arises, namely:

Where are these objects, and what sorts of things are they?

In the paper I first concentrate on one of the most tempting contemporary formulations of the so called “faint of heart” realistic answer (if not the only one): Maddy's “set-theoretic” realism that I characterise, and then try to criticize. I aim to show that, quite generally, faint of heart realism is intrinsically flawed.

2. Faint of heart realism

Generally, 'faint of heart' realism denies that mathematical objects are abstract: it holds that at least some mathematical objects are *concrete* objects, so that at least some mathematical objects are spatio-temporally located.

3. Maddy’s ‘set-theoretic realism’

Penelope Maddy's 'set-theoretic realism' is the theory according to which some sets are concrete objects located in space and time, and we grasp them in pretty much the same way in which we see physical objects.

Let us have a closer look at such a theory:

To illustrate what she has in mind, we'll consider her own example.

Steve who needs two eggs for a certain recipe, and opens the fridge door. He finds there an egg carton and sees three eggs in it. According to Maddy, he does not just see three eggs. He sees something more:

> My claim is that Steve has perceived a *set* of three eggs. By the account of perception just canvassed, this requires that there be a set of three eggs in the carton, that Steve acquire perceptual beliefs about it, and that the set of eggs participate in the generation of these perceptual beliefs in the same way that my hand participates in the generation of my belief that there is a hand before me when I look at it in good light.[1] (my emphasis)

On Maddy's account, Steve can see the *set* of three eggs, and not just the three eggs, because like the eggs themselves that set is spatiotemporally located. It is located in the same place in which its elements, the three eggs, are located. We can generalise from this example, Maddy embraces an extreme version of set-theoretic realism.

Maddy holds that *all* sets which contain only physical objects, and whose members contain only physical objects, and whose members of whose members contain only physical objects ... are located in space and time!

More precisely, sets in which physical objects, and only physical objects, are implicated in this way, are located where those physical objects are located.

For example, a set of higher order, like the set whose two members are the set of the three eggs and the set of Steve's two hands, is likewise located where its members are located. So, since the set of three eggs and the set of Steve's hands are located where the three eggs and Steve's hands are, this higher order set is located there too.

The same holds true even in the case of extremely complicated sets in which all and only physical objects are implicated. Indeed, she generalises further on the basis of the physicalist doctrine that everything is physical. For she conjoins physicalism with the considerations just adduced to derive the conclusion that *all* sets, without exception, are located where the physical objects that are implicated in them are located.

But what, one might ask, about "pure" sets? (i.e. the sets built up from null set).

Unlike the strictly impure sets built up from physical objects, neither the empty set, nor, hence, the pure sets built up from it in standard set theory, can be located

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[1]Maddy, *Realism in Mathematics*, p. 58.

anywhere in space and time. For Maddy, this is just to say that the empty set and the pure sets generated from it do not exist. She says:

> the pure sets aren't really needed. The set theoretic realist who would simultaneously embrace physicalism can take the subject matter of set theoretic science to be the radically impure hierarchy generated from the set of physical individuals by the usual power set operation, except that the empty set is omitted at each stage.... So the set theoretic realist can locate all the sets she needs in space and time.[2]

In addition to construing the strictly impure sets as physical objects, and locating them in physical space, Maddy goes even further in the case of the singleton sets whose single members are physical objects.

Where x is a physical object, Maddy does not merely locate singleton x (i.e. whose only member is x) where x is located. She actually *identifies* singleton x with x itself:

> we take it that the physical objects, x, the individuals from which the generation of the iterative hierarchy begins, are such that x={x}.[3]

In sum, then, Maddy's set theoretic ontology is characterised by two modifications:

> I've suggested two minor alterations in the set theoretic realist's ontology: the identification of physical objects with their singletons and the elimination of pure sets.[4]

Accordingly, Maddy maintains that pure sets are not necessary to get the Zermelo (or von Newmann) ordinals, or to accept a set theory strong enough to perform the mathematical tasks standard set theory performs.

In particular, the existence of two physical objects x and y enables the ordinals to be generated in the following way: x, {x, y}, {x, y, {x, y}} and so on.

On the other hand,

On purely pragmatic grounds Maddy is prepared to relax her restriction of set theory to the strictly impure sets built up from physical objects.

Maddy thinks that for practical reasons only, it is best to keep the empty set, which she treats as a notational convenience.[5]

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[2]Maddy, *Realism in Mathematics*, pp. 156-157.
[3]Maddy, *Realism in Mathematics*, p. 153.
[4]Maddy, *Realism in Mathematics*, p. 157.
[5]Maddy, *Realism in Mathematics*, p. 157, footnote 10.

Finally, what are numbers in this theory? According to Maddy, numbers are not sets. If they were sets, it would be possible to identify what sets they are, which it is not.[6] Numbers cannot be any other objects either.

If, Maddy argues, the number 5 were an object, this object would have (outside the natural number sequence) certain properties that are not relevant for the numerical functioning of such object. Since there are no arguments according to which such properties could be identified, the number 5 cannot be an object.

The conclusion is that numbers are not objects. What are numbers then? They are properties of sets:

> for the set theoretic realist, sets have number properties in the same sense that physical objects have length.[7]

As we compare different lengths we can also compare sets according to their "size"; number are sets's properties, analogously to physical properties.[8] We grasp the "measure" of a set - that is the number of its element when grasping the set itself - in the same way in which we grasp physical properties of a physical object when grasping the object itself. Numbers are not included in the set-theoretic ontology since, as Maddy holds, there is nothing of mathematical relevance in number-theory that cannot be expressed without explicit reference to numbers. All we can say about numbers can be said by using the von Neumann's (or Zermelo's or some other) ordinals; for example: '2 is prime' says 'if x is equinumerous with $\{\{\}, \{\{\}\}\}$, then there are not two sets of of cardinality less than $\{\{\}, \{\{\}\}\}$ but greater than $\{\{\}\}$ whose cross product is equinumerous with x'.[9]

4. Critique of Maddy's 'set-theoretic' realism

Is Maddy's 'set-theoretic' realism acceptable? I will try to give reasons why I think not.

Maddy maintains that it is possible to construe the empty set as a mere device which, though convenient in practice, is in principle dispensible, while at the same time

[6]There are several possible reductions of numbers to set theory and no mathematical result can sort that out. See more about this problem in the next chapter.

[7]Maddy, *Realism in Mathematics*, p. 98.

[8]The only disanalogy consists just in the fact that it is possible to “measure” sets with different scales which is possible when measuring the length or mass or density or suchlike.

[9]Maddy, *Realism in Mathematics*, p. 97.

identifying each physical object x with its singleton set, and maintaining the physicalist doctrine that all things are physical.

But is this really possible, is it acceptable?

Trivially, it is impossible if there is only one physical object.

For in that case the set-theoretic hierarchy collapses immediately.

The starting point, given a physical object x, is that x = {x}. But if the physical object x is identical to singleton x, then the set whose only member is singleton x – viz. {{x}}- must be identical to singleton x. Let us take the set {{x}} in which x is an -apple. Its only element is the set {x} which is identical with x, which means that {{x}}={x}, and that means (since {x}=x) that {{x}}=x and so on for all the others ordinals. So, the identity does preclude von Neumann's (or Zermelo's) reduction of the ordinals.

This point is not lost on Maddy.

Indeed, she herself observes that at least two physical objects are needed if her set-theoretic realism is to be viable.

With two individuals x and y, she says 'a version of the ordinals can be constructed without pure sets - x, y, {x, y}, {x, y, {x, y}}, and so on'[10].

Of course, Maddy is right in this. However, that she is right about it merely serves to demonstrate the strangeness of her theory.

Since a set-theoretic hierarachy is generated with two objects, but not with one, in her theory, she thinks of the set-operator as forming a new object – {x,y} – out of two objects, but not out of one (since {x}=x).

But why should this be?

Let me remind you that

her motive for identifying {x} with x, when x is a physical object, is simply the fact that there is no perceptible difference between {x} and x.

Maddy’s motive for identifying {x} with x is equally pressing in the case of doubleton {x,y}.

Supposedly, this object is located where x and y are located. But the difference between this object and the objects x and y is no more perceptible than is the difference between {x} and x. One cannot *see* that the object {x,y} is different from the objects x and y.

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[10]Maddy, *Realism in Mathematics*, p. 157, footnote 10.

On the other hand, there would seem to be a logical reason precluding the identification of a doubleton set with its members. For whereas the doubleton set {x,y} is *one*, the objects x and y are *two*.

But we might solve this problem by using what is called "plural quantification", and use plural identity as a form of relative identity; by then again, many would disagree with it.

But how significant is this?

We are familiar nowadays with plural quantification: we know that there are sentences in which the subject is ineliminably plural, in that the predicate of the sentence does not attach to the each of the subjects included in the plurality individually (as in e.g. "The men surrounded the city.")

Why then shouldn't there be a similarly *plural* identity

"x and y are identical to the one set {x,y}" in which the predicate "is identical to the set {x,y}" applies to a plural subject?

Admittedly, some have argued that a plural identity in this sense is incoherent, in that it violates the identity of indiscernibles. Suppose Bob and Alice are plurally identical to the one object Xynt. Then Bob and Alice appear to have a property – being two – which Xynt lacks.[11]

But perhaps this argument is too swift: Bob and Alice – and hence Xynt – are two people, but one F. So the doctrine of plural identity is simply a version of relative identity in the sense advocated by Peter Geach.[12]

On the other hand, relative identity has had a bad press.

So the correct attitude in the current state of knowledge is probably scepticism about the logical coherence of the identification of {x,y} with the objects x, y.

Accordingly, the operation of set formation *has* to generate a hierarchy from two physical objects, even if it is powerless to generate one from a single physical object.

But emphasising in this way that doubleton {x,y} is *one* object, not two as x and y are, and hence distinct from them, makes Maddy's view more puzzling, not less so.

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[11] For a variant of this argument that plural identity in this sense is contradictory, see Byeong-Yi (1999), "Is mereology ontologically innocent," Philosophical Studies 93, 141-60.

[12] Cf. P. Geach (1967 "Identity," Review of Metaphysics 21, 3-12.

For if {x,y} is distinct from x and y, how can it be located where they are? If it is located *in its entirety* where *each* of them is, {x,y} is no longer an object at all in the traditional sense: it is a universal.

But if it is located at x only in part, and at y only in part, it would appear that x and y must be parts of it. But that seems wrong too.

The doubleton set {x,y} cannot be the object whose only parts are x and y. That would make it indistinguishable from the mereological sum of x and y.[13]

And now that we have mentioned it, how is this latter object distinguished from singleton set {{x,y}}?

Both are one, and both are located where x and y are. And finally, how is {x,y} to be distinguished from a mixed set like {x,{x,y}}?

Both sets are located where the implicated physical objects are located. Hence, both are located where x and y are.

But this is to say that infinitely many sets are located there: for {{{x,y}}} is no less distinct from {{x,y}} than is the latter from {x,y}, and so on up through the hierarchy past {{{{x,y}}}} and beyond. That's a lot of imperceptible differences.

Maddy's asymmetrical treatment of the singleton sets {x} and {y} on the one hand, and the doubleton set {x,y} on the other, is puzzling.

And it may be that on reflection she should relinquish it by distinguishing {x} from x after all, even where x is a physical object.

But restoring symmetry in this regard simply exarcerbates the problem arising from her insistence that the set-theoretic hierarchy be located in space-time.

Whether or not she admits an infinite hierarchy of distinct objects x, {x}, {{x}}, {{{x}}}, … for each physical object x, she is committed to one of the form {x,y}, {{x,y}}, {{{x,y}}} for each pair of physical objects x,y.

We have already noted the incongruity between her contention that sets are perceptible and the obvious fact that at least after the first pair in the hierarchy, the differences between these objects are imperceptible.

But the supposition that each of these objects has the same location in space-time – namely, the place where x and y are – is no less incongruous.

[13] The mereological sum of certain entities is the object whose parts are all those entities, together with all of their parts.

The conviction that it is impossible for two physical objects to be in the same place at the same time has a long history, and it remains as appealing today as it ever was.[14]

5. Are the other versions of 'faint of heart' realism untenable too?

Even were **the incongruity of infinitely many imperceptibly different objects being located at the same location** put to one side,

Maddy’s set-theoretic realism encounters a general difficulty which any version of faint of heart realism will encounter.

We might ask if there are enough concrete objects, located in space and time, for classical mathematics in the first place. Are there really *infinite* objects out there?

At best, there is an issue about it. As Hilbert points out, even though Euclidean geometry does imply an infinite space, the elliptical geometry offers a model of a finite space and all the physical, that is astronomical, results are compatible with the latter.[15]

Einstein's results show that the Euclidean geometry has to be ruled out after all and that a finite universe is possible.

This gives more reasons for abandoning the view that mathematical objects, which classical mathematics is dealing with, can be identified with certain concrete, physical objects.

Of course, to suppose that space-time is finite in the large is not to deny that it is infinite in the small: the supposition that space-time is e.g. the surface of a(n n-dimensional) sphere, leaves open the possibility of its comprising non-denumerably many points.

But infinity of this ilk is nothing to the set-theoretic hierarchy, and hence to mathematics!

The cardinality of the continuum barely touches upon the cardinalities in the set-theoretic hierarchy: it is so small as to be practically nothing.

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[14] It has to be admitted that there has been some movement away from this conviction in recent times, prompted by a desire to distinction between a functional object – such as a statue or ship – and the physical matter of which it is composed. (One of the many examples which prompts this distinction is the age-old one of Theseus’s ship.) But several thinkers have been concerned to restore the conviction by formal devices which accommodate the examples. See especially D. Lewis (1971), “Counterparts of Persons and their Bodies,” Journal of Philosophy 68, 203-11.

[15] Hilbert’s, I would say wrong, conclusion is that infinity exists just in our thinking.
See Hilber, David ‘On the infinite’, in Benacerraf and Putnam (1983) *Philosophy of Mathematics*, p. 186.

Are we really to suppose that the cardinality of the physical world reaches up into the remoter regions of the set-theoretic hierarchy? No, surely not. The matter is impossible.

The cardinalities of even ZFC are too big.

The efforts of theorists such as Maddy notwithstanding, faint of heart realism is untenable. It tries to pack sets in to a space – the physical world – which is simply too small to accommodate them.