IS QUANTUM LOGIC A LOGIC?

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1 INTRODUCTION

Thirty five years ago, Richard Greechie and Stanley Gudder wrote a paper entitled *Is a Quantum Logic a Logic?* [Greechie and Gudder, 1971] in which they strengthen a previous negative result of Josef Jauch and Constantin Piron. [Jauch and Piron, 1970]

"Jauch and Piron have considered a possibility that a quantum propositional system is an infinite valued logic... and shown that standard propositional systems (that is, ones that are isomorphic to the lattice of all closed subspaces of a Hilbert space) are not conditional and thus cannot be logic in the usual sense." [Greechie and Gudder, 1971] A *conditional* lattice is defined as follows. We define a valuation $v[a]$ as a mapping from an element a of the lattice to the interval $[0, 1]$. We say that two elements a, b are conditional if there exists a unique c such that $v[c] = min\{1, 1 - v[a] + v[b]\}$. We call c the *conditional* of a and b and write $c = a \to b$. We say that the lattice is conditional if every pair a, b is conditional. Greechie and Gudder then proved that a lattice is conditional if and only if it contains only two elements 0 and 1.[1] This implies that [0,1] reduces to $\{0, 1\}$ and that the lattice reduces to a two-valued Boolean algebra. In effect, this result shows that one cannot apply the same kind of valuation to both quantum and classical logics.

It became obvious that if we wanted to arrive at a proper quantum logic, we should take an axiomatically defined set of propositions closed under substitutions and some rules of inference, and apply a model-theoretic approach to obtain valuations of every axiom and theorem of the logic. So, a valuation should not be a mapping to [0,1] or $\{0, 1\}$ but to the elements of a model. For classical logic, a model for logic was a complemented distributive lattice, i.e., a Boolean algebra. For quantum logics the most natural candidate for a model was the orthomodular lattice, while the logics themselves were still to be formulated. Here we come to the question of *what logic is.* We take that logic is about propositions and inferences between them, so as to form an axiomatic deductive system. The system always has some algebras as models, and we always define valuations that map its propositions to elements of the algebra—we say, the system always has its semantics—but our definition stops short of taking semantics to be a part of the

[1] We define 0 and 1 in a lattice in Section 2.

system itself. Our title refers to such a definition of logic, and we call quantum logic so defined *deductive quantum logic*.[2] Classical logic is deductive in the same sense.

In the early seventies, a number of results and a number of predecessors to deductive quantum logics were formulated. Jauch, Piron, Greechie, and Gudder above assumed the conditional—from now on we will call it *implication*—to be defined as $a \to_0 b = a' \cup b$ (see Section 2 for notation). However, it was already then known that in an orthomodular lattice[3] an implication so defined would not satisfy condition $a \to b = 1 \quad \Leftrightarrow \quad a \le b$, which holds in every Boolean algebra and which was considered plausible to hold in an orthomodular lattice too. In 1970, the following implication was found to satisfy this condition: $a \to_1 b = a' \cup (a \cap b)$ (the so-called *Sasaki hook*[4]) by Peter Mittelstaedt [Mittelstaedt, 1970] and Peter Finch [Finch, 1970]. The Sasaki hook becomes equal to $a' \cup b$ when the distributive law is added to an orthomodular lattice, i.e., when it becomes a Boolean algebra. The Sasaki implication first served several authors simply to reformulate the orthomodular lattice in a logic-like way and call it "quantum logic." [Finch, 1970; Clark, 1973; Piziak, 1974] In 1974 Gudrun Kalmbach proved that in addition to the Sasaki hook, there are four other "quantum implications" that satisfy the above plausible condition and that all reduce to $a' \cup b$ in a Boolean algebra.

In the very same year, four genuine (i.e. propositional) deductive quantum logics—using three different implications and none at all, respectively—were formulated by Gudrun Kalmbach [Kalmbach, 1974] (a standard propositional logic based on the *Kalmbach implication*,[5]) Hermann Dishkant [Dishkant, 1974] (a first-order predicate logic based on the *Dishkant implication*[6]), Peter Mittelstaedt [Mittelstaedt, 1974] (a dialog logic based on the Sasaki hook), and Robert Goldblatt [Goldblatt, 1974] (a binary logic with no implication—the binary inference '$\vdash$' represented the lattice '$\le$'). Several other quantum logics were later formulated by Maria Luisa Dalla Chiara [Dalla Chiara, 1977] (first-order quantum logic), Jay Zeman [Zeman, 1978] (*normal logic*), Hirokazu Nishimura [Nishimura, 1980] (Gentzen sequent logic), George Georgacarakos [Georgacarakos, 1980] (*orthomodular logics* based on *relevance*,[7] Sasaki, and Dishkant implications), Michael Dunn [Dunn, 1981] (predicate binary logic), Ernst-Walter Stachow [Stachow, 1976] (*tableaux calculus*, a Gentzen-like *calculus of sequents*, and a Brouwer-like logic), Gary Hard-

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[2]Note that many authors understand quantum logic as simply a lattice [Jauch, 1968] or a poset [Varadarajan, 1968; Pták and Pulmannová, 1991]. Quantum logics so defined do not have the aforementioned valuation and are not deductive quantum logics. Such a definition stems from an operationalist approach, which started with the idea that quantum logic might be empirical. It was argued that propositions might be measured and that properties such as orthomodularity for quantum systems or distributivity for classical ones can be experimentally verified. [Jauch, 1968]

[3]The lattice of all closed subspaces of a Hilbert space is an orthomodular lattice. See Section 2.

[4]The Sasaki hook is an orthocomplement to the *Sasaki projection* [Sasaki, 1964].

[5]Kalmbach implication is defined as $a \to_3 b = (a' \cap b) \cup (a' \cap b') \cup (a \cap (a' \cup b))$.

[6]Dishkant implication is defined as $a \to_2 b = b' \to_1 a'$.

[7]Relevance implication is defined as $a \to_5 b = (a \cap b) \cup (a' \cap b) \cup (a' \cap b')$.

egree [Hardegree, 1981] (*orthomodular calculus*), John Bell [Bell, 1986] (quantum "attribute" logic), Mladen Pavičić [Pavičić, 1987] (binary quantum logics with *merged implications*[8]), Mladen Pavičić [Pavičić, 1989] (unary quantum logic with *merged implications*),[9] Mladen Pavičić and Norman Megill [Pavičić and Megill, 1999] (unary quantum logics with *merged equivalences*[10]), etc. Logics with the $v(a) = 1$ lattice valuation corresponding to $\vdash a$ we call *unary* logics and logics with the $v(a) \le v(b)$ lattice valuation corresponding to $a \vdash b$ we call *binary* logics.

Still, the parallels with classical logic were a major concern of the researchers at the time. "I would argue that a 'logic' without an implication ... is radically incomplete, and indeed, hardly qualifies as a theory of deduction" (Jay Zeman, 1978). [Zeman, 1978] So, an extensive search was undertaken in the seventies and eighties to single out a "proper quantum implication" from the five possible ones on purely logical grounds,[11] but none of the attempts proved successful.

In 1987 Mladen Pavičić [Pavičić, 1987; Pavičić, 1989] proved that there is no "proper quantum implication" since any one of the conditions $a \to_i b = 1 \;\Leftrightarrow\; a \le b$, $i = 1, \ldots, 5$[12] *is* the very orthomodularity which, when added to an orthocomplemented lattice (the so-called *ortholattice*), makes it orthomodular. In terms of a logic, the corresponding logical rules of inference turn any *orthologic* or *minimal quantum logic* into a quantum logic. He also proved that when we add the condition $a \to_0 b = 1 \;\Leftrightarrow\; a \le b$ to an ortholattice, it turns the lattice into a complemented distributive one, that is, into a Boolean algebra.[13] A corresponding logical rule of inference turns any orthologic into a classical logic.

This finding was soon complemented by a proof given by Jacek Malinowski in 1990 that "no logic determined by any class of orthomodular lattices admits the deduction theorem," [Malinowski, 1990] where the *deduction theorem* says that if we can derive b from $S \bigcup \{a\}$ then we can derive $a \to b$ from S.[14] He also proved that no extension of quantum logic, i.e., no logic between the quantum and the classical one, satisfies the deduction theorem. [Mortensen, 1991] The conclusion was: "Since orthomodular logic is algebraically well behaved, this perhaps shows that implication is not such a desirable operation to have." [Mortensen, 1991]

The conjecture was confirmed by Mladen Pavičić in 1993 [Pavičić, 1993]. The

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[8] Under *merged implications* all six implications are meant; $a \to_i b$, $i = 0, 1, 2, 3, 5$ are defined above; $a \to_4 b = b' \to_3 a'$ is called *non-tollens implication*. In these logics of Pavičić, axioms of identical form hold for each of the implications yielding five quantum logics and one classical (for $i = 0$).

[9] Again, axioms of identical form hold for all implications.

[10] Merged equivalences, $a \equiv_i b$, $i = 0, \ldots, 5$, are explicit expressions (by means of $\cup, \cap, '$) of $(a \to_i b) \cap (b \to_j a)$, $i = 0, \ldots, 5$, $j = 0, \ldots, 5$, in any orthomodular lattice as given by Table 1 of Ref. [Pavičić and Megill, 1999]. In these logics, axioms of identical form hold for all equivalences.

[11] An excellent contemporary review of the state of the art was written in 1979 by Gary Hardegree [Hardegree, 1979].

[12] $a \to_i b$, $i = 1, \ldots, 5$ are defined above. See footnotes Nos. 8 and 9.

[13] In any Boolean algebra all six implications merge.

[14] It should be stressed here that the deduction theorem is not essential for classical logic either. It was first proved by Jaques Herbrand in 1930. [Herbrand, 1931] All classical logic systems before 1930, e.g., the ones by Whitehead and Russell, Hilbert, Ackermann, Post, Skolem, Łukasiewicz, Tarski, etc., were formulated without it.

above orthomodularity condition does not require implications. One can also have it with an essentially weaker equivalence operation: $a \equiv b = 1 \;\Leftrightarrow\; a = b$, where $a \equiv b = (a \cap b) \cup (a' \cap b')$; we say a and b are *equivalent.* [Pavičić, 1993; Pavičić and Megill, 1999] As above, when this condition is added to an ortholattice it makes it orthomodular.[15] Moreover in any orthomodular lattice $a \equiv b = (a \to_i b) \cap (b \to_i a)$, $i = 1, \dots, 5$. The analogous classical condition $a \equiv_0 b = 1 \;\Leftrightarrow\; a = b$, where $a \equiv_0 b = (a' \cup b) \cap (a \cup b')$, amounts to distributivity: when added to an ortholattice, it makes it a Boolean algebra. [Pavičić, 1998; Pavičić and Megill, 1999]

On the other hand, it turned out that everything in orthomodular lattices is sixfold defined: binary operations, unary operation, variables and even unities and zeros. They all collapse to standard Boolean operations, variables and 0,1 when we add distributivity. For example, as proved by Norman Megill and Mladen Pavičić [Megill and Pavičić, 2001] $0_{1(a,b)} = a \cap (a' \cup b) \cap (a \cup b')$,...,$0_{5(a,b)} = (a \cup b) \cap (a \cup b') \cap (a' \cup b) \cap (a' \cup b')$; $a \equiv_3 b = (a' \cup b) \cap (a \cup (a' \cap b'))$; etc. [Megill and Pavičić, 2002] Moreover, we can express any of such expressions by means of every appropriate other in a huge although definite number of equivalence classes. [Megill and Pavičić, 2002] For example, the shortest expression for $\cup$ expressed by means of quantum implications is $a \cup b = (a \to_i b) \to_i (((a \to_i b) \to_i (b \to_i a)) \to_i a)$, $i = 1, \dots, 5$. [Megill and Pavičić, 2001; Megill and Pavičić, 2002; Pavičić and Megill, 1998a; Megill and Pavičić, 2003]

For such a "weird" model the question emerged as to whether it is possible to formulate a proper deductive quantum logic as a general theory of inference and how independent of its model this logic can be. In other words, can such a logic be more general than its orthomodular model?

The answer turned out to be affirmative. In 1998 Mladen Pavičić and Norman Megill showed that the deductive quantum logic is not only more general but also very different from their models. [Pavičić and Megill, 1998b; Pavičić and Megill, 1999] They proved that

- Deductive quantum logic is not orthomodular.
- Deductive quantum logic has models that are ortholattices that are not orthomodular.
- Deductive quantum logic is sound and complete under these models.

This shows that quantum logic is not much different from the classical one since they also proved that [Pavičić and Megill, 1999]

- Classical logic is not distributive.[16]

[15]The same holds for $a \equiv_i b$, $i = 1, \dots, 5$ from footnote No. 10, as well. [Pavičić and Megill, 1999]

[16]Don't be alarmed. This is *not* in contradiction with anything in the literature. The classical logic still stands intact, and the fact that it is not distributive is just a feature of classical logic that—due to Boole's heritage—simply has not occurred to anyone as possible and which therefore has not been discovered before. See the proof of Theorem 30, Theorem 45, Lemma 50, and the discussion in Section 10.

- Classical logic has models that are ortholattices that are not orthomodular and therefore also not distributive.
- Classical logic is sound and complete under these models.

These remarkably similar results suggest that quantum logic is a logic in the very same way in which classical logic is a logic. In the present chapter, we show these results in some detail.

The chapter is organized as follows. In Section 2, we define the ortholattice, orthomodular lattice, complemented distributive lattice (Boolean algebra), weakly orthomodular lattice WOML (which is not orthomodular), weakly distributive lattice WDOL (which is neither distributive nor orthomodular), and some results that connect the lattices. In Section 3, we define quantum and classical logics. In Sections 4 and 5, we prove the soundness of quantum logic for WOML and of classical logic for WDOL, respectively. In Sections 6 and 7, we prove the completeness of the logics for WOML and WDOL, respectively. In Sections 8 and 9, we prove the completeness of the logics for OML and Boolean algebra, respectively, and show that the latter proofs of completeness introduce hidden axioms of orthomodularity and distributivity in the respective Lindenbaum algebras of the logics. In Section 10, we discuss the obtained results.

2 LATTICES

In this section, we introduce two models for deductive quantum logic, orthomodular lattice and WOML, and two models for classical logic, Boolean algebra and WDOL. They are gradually defined as follows.

There are two equivalent ways to define a lattice: as a partially ordered set (poset)[17] [Maeda and Maeda, 1970] or as an algebra [Birkhoff, 1948, II.3. *Lattices as Abstract Algebras*]. We shall adopt the latter approach.

DEFINITION 1. An *ortholattice*, OL, is an algebra $\langle \mathcal{OL}_0, ', \cup, \cap \rangle$ such that the following conditions are satisfied for any $a, b, c \in \mathcal{OL}_0$ [Megill and Pavičić, 2002]:

$$a \cup b = b \cup a \tag{1}$$
$$(a \cup b) \cup c = a \cup (b \cup a) \tag{2}$$
$$a'' = a \tag{3}$$
$$a \cup (b \cup b') = b \cup b' \tag{4}$$
$$a \cup (a \cap b) = a \tag{5}$$
$$a \cap b = (a' \cup b')' \tag{6}$$

In addition, since $a \cup a' = b \cup b'$ for any $a, b \in \mathcal{OL}_0$, we define:

$$1 \stackrel{\text{def}}{=} a \cup a', \qquad 0 \stackrel{\text{def}}{=} a \cap a' \tag{7}$$

[17] Any two elements a and b of the poset have a least upper bound $a \cup b$—called *join*—and a greatest lower bound $a \cap b$—called *meet*.

and

$$a \le b \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad a \cap b = a \quad \Longleftrightarrow \quad a \cup b = b \tag{8}$$

Connectives $\to_1$ (*quantum implication*, *Sasaki hook*), $\to_0$ (*classical implication*), $\equiv$ (*quantum equivalence*), and $\equiv_0$ (*classical equivalence*) are defined as follows:

DEFINITION 2. $a \to_1 b \stackrel{\text{def}}{=} a' \cup (a \cap b)$, $\qquad a \to_0 b \stackrel{\text{def}}{=} a' \cup b$.

DEFINITION 3.[18] $a \equiv b \stackrel{\text{def}}{=} (a \cap b) \cup (a' \cap b')$.

DEFINITION 4. $a \equiv_0 b \stackrel{\text{def}}{=} (a \to_0 b) \cap (b \to_0 a)$.

Connectives bind from weakest to strongest in the order $\to_1$ ($\to_0$), $\equiv$ ($\equiv_0$), $\cup$, $\cap$, and $'$.

DEFINITION 5. (Pavičić and Megill [Pavičić and Megill, 1999]) An ortholattice to which the following condition is added:

$$a \equiv b = 1 \qquad \Rightarrow \qquad (a \cup c) \equiv (b \cup c) = 1 \tag{9}$$

is called a *weakly orthomodular ortholattice*, WOML.

DEFINITION 6. (Pavičić [Pavičić, 1993]) An ortholattice to which the following condition is added:

$$a \equiv b = 1 \qquad \Rightarrow \qquad a = b, \tag{10}$$

is called an *orthomodular lattice*, OML.

Equivalently:

DEFINITION 7. (Foulis [Foulis, 1962], Kalmbach [Kalmbach, 1974]) An ortholattice to which either of the following two conditions is added:

$$a \cup (a' \cap (a \cup b)) = a \cup b \tag{11}$$

$$a\, \mathcal{C}\, b \quad \& \quad a\, \mathcal{C}\, c \quad \Rightarrow \quad a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{12}$$

where $a\, \mathcal{C}\, b \stackrel{\text{def}}{\Longleftrightarrow} a = (a \cap b) \cup (a \cap b')$ (a *commutes* with b), is called an *orthomodular lattice*, OML.

DEFINITION 8. (Pavičić and Megill [Pavičić and Megill, 1999]) An ortholattice to which the following condition is added:[19]

$$(a \equiv b) \cup (a \equiv b') = (a \cap b) \cup (a \cap b') \cup (a' \cap b) \cup (a' \cap b') = 1 \tag{13}$$

is called a *weakly distributive ortholattice*, WDOL.

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[18] In every orthomodular lattice $a \equiv b = (a \to_1 b) \cap (b \to_1 a)$, but not in every ortholattice.

[19] This condition is known as *commensurability.* [Mittelstaedt, 1970, Definition (2.13), p. 32] Commensurability is a weaker form of the commutativity from Definition 7. Actually, a metaimplication from commensurability to commutativity is yet another way to express orthomodularity. They coincide in any OML.

DEFINITION 9. (Pavičić [Pavičić, 1998]) An ortholattice to which the following condition is added:

$$a \equiv_0 b = 1 \qquad \Rightarrow \qquad a = b \tag{14}$$

is called a *Boolean algebra*.

Equivalently:

DEFINITION 10. (Schröder [Schröder, 1890]) An ortholattice to which the following condition is added:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{15}$$

is called a *Boolean algebra*.

The opposite directions in Eqs. (10) and (14) hold in any OL.

Any finite lattice can be represented by a Hasse diagram that consists of points (*vertices*) and lines (*edges*). Each point represents an element of the lattice, and positioning element a above element b and connecting them with a line means $a \leq b$. For example in Figure 1 we have $0 \leq x \leq y \leq 1$. We also see that in this lattice, e.g., x does not have a relation with either x' or y'.

Definition 11 and Theorems 12 and 14 will turn out to be crucial for the completeness proofs of both quantum and classical logics in Sections 6 and 7.

DEFINITION 11. We define O6 as the lattice shown in Figure 1, with the meaning $0 < x < y < 1$ and $0 < y' < x' < 1$,

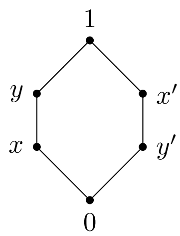

Figure 1. Ortholattice O6, also called *benzene ring* and *hexagon*.

THEOREM 12. *An ortholattice is orthomodular if only if it does not include a subalgebra isomorphic to the lattice* O6*.*

Proof. Samuel Holland [Holland, 1970]. See also Gudrun Kalmbach [Kalmbach, 1983, p. 22]. ■

COROLLARY 13. O6 *violates the distributive law.*

Proof. Distributivity implies orthomodularity. We can also easily verify on the diagram: $y \cap (x \cup x') = y \cap 1 = y$, but $(y \cap x) \cup (y \cap x') = x \cup 0 = x$. ■

THEOREM 14. *All conditions of* WOML *and* WDOL *hold in* O6.

Proof. As given by Mladen Pavičić and Norman Megill. [Pavičić and Megill, 1998b; Pavičić and Megill, 1999] It boils down to the fact that O6 violates none of the conditions given by Eqs. (1-6), (9), and (13) ■

THEOREM 15. *There exist* WDOL *lattices that are not orthomodular and therefore not distributive,* WOML *lattices that are not orthomodular, ortholattices that are neither* WOML *nor* WDOL*, and there are* WOML *lattices that are not* WDOL.

Proof. As given by Mladen Pavičić and Norman Megill. [Pavičić and Megill, 1998b; Pavičić and Megill, 1999]. ■

On the one hand, the equations that hold in OML and Boolean algebra properly include those that hold in WOML and WDOL, since WOML and WDOL are strictly more general classes of algebras. But on the other hand, there is also a sense in which the equations of WOML and WDOL can be considered to properly include those of OML and Boolean algebra, via mappings that the next theorems describe.

THEOREM 16. *The equational theory of* OML*s can be simulated by a proper subset of the equational theory of* WOML*s.*

Proof. The equational theory of OML consists of equality conditions, Eqs. (1)–(6) together with the orthomodularity condition Eq. (11) (or Eq. (10) or Eq. (12)). We construct a mapping from these conditions into WOML as follows. We map each of the OML conditions, which is an equation in the form $t = s$ (where t and s are terms), to the equation $t \equiv s = 1$, which holds in WOML. Any equational proof in OML can then be simulated in WOML by replacing each axiom reference in the OML proof with its corresponding WOML mapping. [Megill and Pavičić, 2006] Such a mapped proof will use only a proper subset of the equations that hold in WOML: any equation whose right-hand side does not equal 1, such as $a = a$, will never be used. ■

COROLLARY 17. *No set of equations of the form $t \equiv s = 1$, where t and s are terms in* OML *and where $t = s$ holds in* OML*, determines an* OML *when added to an ortholattice.*

Proof. Theorem 16 shows that all equations of this form hold in a WOML and none of WOML conditions given by Eqs. (1-6,9) is violated by O6. Hence, Theorem 12 completes the proof. ■

THEOREM 18. *The equational theory of Boolean algebras can be simulated by a proper subset of the equational theory of* WDOL*s.*

Proof. The equational theory of Boolean algebras consists of equality conditions Eqs. (1)–(6) together with the distributivity condition Eq. (15). We construct a mapping from these conditions into WDOL as follows. We map each of the Boolean algebra conditions, which is an equation in the form $t = s$ (where t and s are terms), to the equation $t \equiv_0 s = 1$, which holds in WDOL. Any equational proof in a Boolean algebra can then be simulated in WDOL by replacing each condition reference in the Boolean algebra proof with its corresponding WDOL mapping. [Megill and Pavičić, 2006] Such a mapped proof will use only a proper subset of the equations that hold in WDOL: any equation whose right-hand side does not equal 1, such as $a = a$, will never be used. ■

COROLLARY 19. *No set of equations of the form $t \equiv_0 s = 1$, where t and s are terms in any Boolean algebra and where $t = s$ holds in the algebra, determines a Boolean algebra when added to an ortholattice.*

Proof. Theorem 18 shows that all equations of this form hold in a WDOL and none of WDOL conditions given by Eqs. (1-6,8) is violated by O6. Hence, Corollary 13 completes the proof. ■

3 LOGICS

Logic, $\mathcal{L}$, is a language consisting of propositions and a set of conditions and rules imposed on them called axioms and rules of inference.

The propositions we use are well-formed formulas (wffs), defined as follows. We denote elementary, or primitive, propositions by $p_0, p_1, p_2, \ldots$, and have the following primitive connectives: $\neg$ (negation) and $\vee$ (disjunction). The set of wffs is defined recursively as follows:

p_j is a wff for $j = 0, 1, 2, \ldots$

$\neg A$ is a wff if A is a wff.

$A \vee B$ is a wff if A and B are wffs.

We introduce conjunction with the following definition:

DEFINITION 20. $A \wedge B \stackrel{\text{def}}{=} \neg(\neg A \vee \neg B)$.

The statement calculus of our metalanguage consists of axioms and rules from the object language as elementary metapropositions and of compound metapropositions built up by means of the following metaconnectives: $\sim$ (*not*), & (*and*), $\veebar$ (*or*), $\Rightarrow$ (*if…, then*), and $\Leftrightarrow$ (*iff*), with the usual *classical* meaning. Our metalanguage statement calculus is actually the very same classical logic we deal with in this chapter, only with the {0,1} valuation. We extend the statement calculus of the metalanguage with first-order predicate calculus—with quantifiers $\forall$ (*for all*) and $\exists$ (*exists*)—and informal set theory in the usual way.

The operations of implication are the following ones (classical, Sasaki, and Kalmbach) [Pavičić, 1987]:

DEFINITION 21. $A \to_0 B \overset{\text{def}}{=} \neg A \vee B.$

DEFINITION 22. $A \to_1 B \overset{\text{def}}{=} \neg A \vee (A \wedge B).$

DEFINITION 23. $A \to_3 B \overset{\text{def}}{=} (\neg A \wedge B) \vee (\neg A \wedge \neg B) \vee (A \wedge (\neg A \vee B)).$

We also define the *equivalence* operations as follows:

DEFINITION 24. $A \equiv B \overset{\text{def}}{=} (A \wedge B) \vee (\neg A \wedge \neg B).$

DEFINITION 25. $A \equiv_0 B \overset{\text{def}}{=} (A \to_0 B) \wedge (B \to_0 A).$

Connectives bind from weakest to strongest in the order $\to$, $\equiv$, $\vee$, $\wedge$, $\neg$.

Let $\mathcal{F}^\circ$ be the set of all propositions, i.e., of all wffs. Of the above connectives, $\vee$ and $\neg$ are primitive ones. Wffs containing $\vee$ and $\neg$ within logic $\mathcal{L}$ are used to build an algebra $\mathcal{F} = \langle \mathcal{F}^\circ, \neg, \vee \rangle$. In $\mathcal{L}$, a set of axioms and rules of inference are imposed on $\mathcal{F}$. From a set of axioms by means of rules of inference, we get other expressions which we call theorems. Axioms themselves are also theorems. A special symbol $\vdash$ is used to denote the set of theorems. Hence $A \in \vdash$ iff A is a theorem. The statement $A \in \vdash$ is usually written as $\vdash A$. We read this: "A is provable" since if A is a theorem, then there is a proof for it. We present the axiom systems of our propositional logics in schemata form (so that we dispense with the rule of substitution).

3.1 Quantum Logic

All unary quantum logics we mentioned in the Introduction are equivalent. Here we present Kalmbach's quantum logic because it is the system which has been investigated in the greatest detail in her book [Kalmbach, 1983] and elsewhere [Kalmbach, 1974; Pavičić and Megill, 1998b]. Quantum logic, $\mathcal{QL}$, is defined as a language consisting of propositions and connectives (operations) as introduced above, and the following axioms and a rule of inference. We will use $\vdash_{\mathcal{QL}}$ to denote provability from the axioms and rule of $\mathcal{QL}$ and omit the subscript when it is clear from context (such as in the list of axioms that follow).

Axioms

A1	$\vdash A \equiv A$	(16)
A2	$\vdash A \equiv B \to_0 (B \equiv C \to_0 A \equiv C)$	(17)
A3	$\vdash A \equiv B \to_0 \neg A \equiv \neg B$	(18)
A4	$\vdash A \equiv B \to_0 A \wedge C \equiv B \wedge C$	(19)
A5	$\vdash A \wedge B \equiv B \wedge A$	(20)
A6	$\vdash A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$	(21)
A7	$\vdash A \wedge (A \vee B) \equiv A$	(22)

A8	$\vdash \neg A \wedge A \equiv (\neg A \wedge A) \wedge B$	(23)
A9	$\vdash A \equiv \neg\neg A$	(24)
A10	$\vdash \neg(A \vee B) \equiv \neg A \wedge \neg B$	(25)
A11	$\vdash A \vee (\neg A \wedge (A \vee B)) \equiv A \vee B$	(26)
A12	$\vdash (A \equiv B) \equiv (B \equiv A)$	(27)
A13	$\vdash A \equiv B \to_0 (A \to_0 B)$	(28)
A14	$\vdash (A \to_0 B) \to_3 (A \to_3 (A \to_3 B))$	(29)
A15	$\vdash (A \to_3 B) \to_0 (A \to_0 B)$	(30)

Rule of Inference (*Modus Ponens*)

R1 $\quad \vdash A \quad \& \quad \vdash A \to_3 B \quad \Rightarrow \quad \vdash B$ (31)

In Kalmbach's presentation, the connectives $\vee$, $\wedge$, and $\neg$ are primitive. In the base set of any model (such as an OML or WOML model) that belongs to OL, $\cap$ can be defined in terms of $\cup$ and $'$, as justified by DeMorgan's laws, and thus the corresponding $\wedge$ can be defined in terms of $\vee$ and $\neg$ (Definition 20). We shall do this for simplicity. Regardless of whether we consider $\wedge$ primitive or defined, we can drop axioms A1, A11, and A15 because it has been proved that they are redundant, i.e., can be derived from the other axioms. [Pavičić and Megill, 1998b] Note that A11 is what we would expect to be *the* orthomodularity[20]—see Eq. (37) and the discussion following the equation.

DEFINITION 26. For $\Gamma \subseteq \mathcal{F}^\circ$ we say A is derivable from Γ and write $\Gamma \vdash_{\mathcal{QL}} A$ or just $\Gamma \vdash A$ if there is a sequence of formulas ending with A, each of which is either one of the axioms of $\mathcal{QL}$ or is a member of Γ or is obtained from its precursors with the help of a rule of inference of the logic.

3.2 Classical Logic

We make use of the PM classical logical system $\mathcal{CL}$ (Whitehead and Russell's *Principia Mathematica* axiomatization in Hilbert and Ackermann's presentation [Hilbert and Ackermann, 1950] but in schemata form so that we dispense with their rule of substitution). In this system, the connectives $\vee$ and $\neg$ are primitive, and the $\to_0$ connective shown in the axioms is implicitly understood to be expanded according to its definition. We will use $\vdash_{\mathcal{CL}}$ to denote provability from the axioms and rule of $\mathcal{CL}$, omitting the subscript when it is clear from context.

Axioms

A1	$\vdash A \vee A \to_0 A$	(32)
A2	$\vdash A \to_0 A \vee B$	(33)
A3	$\vdash A \vee B \to_0 B \vee A$	(34)
A4	$\vdash (A \to_0 B) \to_0 (C \vee A \to_0 C \vee B)$	(35)

[20] Cf. Definition (7), Eq. (11)

Rule of Inference (*Modus Ponens*)

$$\text{R1} \qquad \vdash A \qquad \& \qquad A \to_0 B \qquad \Rightarrow \qquad \vdash B \tag{36}$$

We assume that the only legitimate way of inferring theorems in $\mathcal{CL}$ is by means of these axioms and the Modus Ponens rule. We make no assumption about valuations of the primitive propositions from which wffs are built, but instead are interested in wffs that are valid in the underlying models. Soundness and completeness will show that those theorems that can be inferred from the axioms and the rule of inference are exactly those that are valid.

We define derivability in $\mathcal{CL}$, $\Gamma \vdash_{\mathcal{CL}} A$ or just $\Gamma \vdash A$, in the same way as we do for system $\mathcal{QL}$.

4 THE SOUNDNESS OF $\mathcal{QL}$: ORTHOMODULARITY LOST

In this section we show that the syntax of $\mathcal{QL}$ does not correspond to the syntax of an orthomodular lattice. We do this by proving the soundness of $\mathcal{QL}$ for WOML. To prove soundness means to prove that all axioms as well as the rules of inference (and therefore all theorems) of $\mathcal{QL}$ hold in its models. Since by Theorem 16 WOML properly includes OML, proving the soundness of $\mathcal{QL}$ for OML would not tell us anything new, and we can dispense with it.

DEFINITION 27. We call $\mathcal{M} = \langle \mathcal{L}, h \rangle$ a model if $\mathcal{L}$ is an algebra and $h : \mathcal{F}^\circ \longrightarrow \mathcal{L}$, called a valuation, is a morphism of formulas $\mathcal{F}^\circ$ into $\mathcal{L}$, preserving the operations $\neg, \vee$ while turning them into $', \cup$.

Whenever the base set $\mathcal{L}$ of a model belongs to WOML (or another class of algebras), we say (informally) that the model belongs to WOML (or the other class). In particular, if we say "for all models in WOML" or "for all WOML models," we mean for all base sets in WOML and for all valuations on each base set. The term "model" may refer either to a specific pair $\langle \mathcal{L}, h \rangle$ or to all possible such pairs with the base set $\mathcal{L}$, depending on context.

DEFINITION 28. We call a formula $A \in \mathcal{F}^\circ$ valid in the model $\mathcal{M}$, and write $\vDash_{\mathcal{M}} A$, if $h(A) = 1$ for all valuations h on the model, i.e. for all h associated with the base set $\mathcal{L}$ of the model. We call a formula $A \in \mathcal{F}^\circ$ a consequence of $\Gamma \subseteq \mathcal{F}^\circ$ in the model $\mathcal{M}$ and write $\Gamma \vDash_{\mathcal{M}} A$ if $h(X) = 1$ for all X in Γ implies $h(A) = 1$, for all valuations h.

For brevity, whenever we do not make it explicit, the notations $\vDash_{\mathcal{M}} A$ and $\Gamma \vDash_{\mathcal{M}} A$ will always be implicitly quantified over all models of the appropriate type, in this section for all WOML models $\mathcal{M}$. Similarly, when we say "valid" without qualification, we will mean valid in all models of that type.

We now prove the soundness of quantum logic by means of WOML, i.e., that if A is a theorem in $\mathcal{QL}$, then A is valid in any WOML model.

THEOREM 29. [Soundness] $\qquad \Gamma \vdash A \quad \Rightarrow \quad \Gamma \vDash_{\mathcal{M}} A$

Proof. We must show that any axiom A1–A15, given by Eqs. (16–30), is valid in any WOML model $\mathcal{M}$, and that any set of formulas that are consequences of Γ in the model are closed under the rule of inference R1, Eq. (31).

Let us put $a = h(A)$, $b = h(B)$, ...

By Theorem 16, we can prove that WOML is equal to OL restricted to all orthomodular lattice conditions of the form $t \equiv s = 1$, where t and s are terms (polynomials) built from the ortholattice operations and $t = s$ is an equation that holds in all OMLs. ■

Hence, mappings of $\mathcal{QL}$ axioms and its rule of inference can be easily proved to hold in WOML. Moreover, mappings of A1,A3,A5–A13,A15 and R1 hold in any ortholattice. In particular, the

$$\text{A11 mapping}: \qquad (a \cup (a' \cap (a \cup b))) \equiv (a \cup b) = 1 \tag{37}$$

holds in every ortholattice and A11 itself is redundant, i.e., can be be inferred from other axioms. Notice that by Corollary 17, $a \equiv b = 1$ does not imply $a = b$. In particular, Eq. (37) does not imply $(a \cup (a' \cap (a \cup b))) = (a \cup b)$

5 THE SOUNDNESS OF $\mathcal{CL}$: DISTRIBUTIVITY LOST

In this section we show that the syntax of $\mathcal{CL}$ does not correspond to the syntax of a Boolean algebra. In a way analogous to the $\mathcal{QL}$ soundness proof, we prove the soundness of $\mathcal{CL}$ only by means of WDOL.

Recall Definitions 27 and 28 for "model," "valid," and "consequence."

We now prove the soundness of classical logic by means of WDOL, i.e., that if A is a theorem in $\mathcal{CL}$, then A is valid in any WDOL model.

THEOREM 30. [Soundness] $\qquad \Gamma \vdash A \quad \Rightarrow \quad \Gamma \vDash_{\mathcal{M}} A$

Proof. We must show that any axiom A1–A4, given by Eqs. (32–35), is valid in any WDOL model $\mathcal{M}$, and that any set of formulas that are consequences of Γ in the model are closed under the rule of inference R1, Eq. (36).

Let us put $a = h(A)$, $b = h(B)$, ...

By Theorem 18, we can prove that WDOL is equal to OL restricted to all Boolean algebra conditions of the form $t \equiv_0 s = 1$, where t and s are terms and $t = s$ is an equation that holds in all Boolean algebras. Notice that according to Corollary 19, $t \equiv_0 s = 1$ is not generally equivalent to $t = s$ in WDOL. For example, the mappings of A1–A3 and R1 hold in every ortholattice, and the ortholattice mapping of A4 does not make the ortholattice even orthomodular let alone distributive. In other words,

$$(a \cap (b \cup c)) \equiv_0 ((a \cap b) \cup (a \cap c)) = 1 \tag{38}$$

does not imply $(a \cap (b \cup c)) = ((a \cap b) \cup (a \cap c))$, and therefore we cannot speak of distributivity within $\mathcal{CL}$. ■

6 THE COMPLETENESS OF $\mathcal{QL}$ FOR WOML MODELS: NON-ORTHOMODULARITY CONFIRMED

Our main task in proving the soundness of $\mathcal{QL}$ in the previous section was to show that all axioms as well as the rules of inference (and therefore all theorems) from $\mathcal{QL}$ hold in WOML. The task of proving the completeness of $\mathcal{QL}$ is the opposite one: we have to impose the structure of WOML on the set $\mathcal{F}^\circ$ of formulas of $\mathcal{QL}$.

We start with a relation of congruence, i.e., a relation of equivalence compatible with the operations in $\mathcal{QL}$. We make use of an equivalence relation to establish a correspondence between formulas of $\mathcal{QL}$ and formulas of WOML. The resulting equivalence classes stand for elements of a WOML and enable the completeness proof of $\mathcal{QL}$ by means of WOML.

Our definition of congruence involves a special set of valuations on lattice O6 (shown in Figure 1 in Section 2) called $\mathcal{O}6$ and defined as follows. Its definition is the same for both the quantum logic completeness proof in this section and the classical logic completeness proof in Section 7.

DEFINITION 31. Letting *O6* represent the lattice from Definition 11, we define $\mathcal{O}6$ as the set of all mappings $o : \mathcal{F}^\circ \longrightarrow \mathrm{O6}$ such that for $A, B \in \mathcal{F}^\circ$, $o(\neg A) = o(A)'$, and $o(A \vee B) = o(A) \cup o(B)$.

The purpose of $\mathcal{O}6$ is to let us refine the equivalence classes used for the completeness proof, so that the Lindenbaum algebra will be a proper WOML, i.e. one that is not orthomodular. This is accomplished by conjoining the term $(\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \Rightarrow o(A) = o(B)]$ to the equivalence relation definition, meaning that for equivalence we require also that (whenever the valuations o of the wffs in Γ are all 1) the valuations of wffs A and B map to the same point in the lattice O6. For example, the two wffs $A \vee B$ and $A \vee (\neg A \wedge (A \vee B))$ will become members of two separate equivalence classes by Theorem 37 below. Without the conjoined term, these two wffs would belong to the same equivalence class. The point of doing this is to provide a completeness proof that is not dependent in any way on the orthomodular law, to show that completeness does not require that the underlying models be OMLs.

THEOREM 32. *The relation of* equivalence $\approx_{\Gamma,\mathcal{QL}}$ *or just* $\approx$, *defined as*

$$A \approx B \stackrel{\mathrm{def}}{=} \Gamma \vdash A \equiv B \ \& \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \Rightarrow o(A) = o(B)], \tag{39}$$

is a relation of congruence in the algebra $\mathcal{F}$, *where* $\Gamma \subseteq \mathcal{F}^\circ$

Proof. Let us first prove that $\approx$ is an equivalence relation. $A \approx A$ follows from A1 [Eq. (16)] of system $\mathcal{QL}$ and the identity law of equality. If $\Gamma \vdash A \equiv B$, we can detach the left-hand side of A12 to conclude $\Gamma \vdash B \equiv A$, through the use of A13 and repeated uses of A14 and R1. From this and commutativity of equality, we conclude $A \approx B \Rightarrow B \approx A$. (For brevity we will not usually mention further uses of A12, A13, A14, and R1 in what follows.) The proof of transitivity runs as follows.

$$
\begin{aligned}
A \approx B \qquad & \& \quad B \approx C \\
& \Rightarrow\ \Gamma \vdash A \equiv B \quad \& \quad \Gamma \vdash B \equiv C \\
& \& \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(A) = o(B)] \\
& \& \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(B) = o(C)] \\
& \Rightarrow\ \Gamma \vdash A \equiv C \\
& \& \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(A) = o(B) \ \&\ o(B) = o(C)].
\end{aligned} \tag{40}
$$

In the last line above, $\Gamma \vdash A \equiv C$ follows from A2, and the last metaconjunction reduces to $o(A) = o(C)$ by transitivity of equality. Hence the conclusion $A \approx C$ by definition.

In order to be a relation of congruence, the relation of equivalence must be compatible with the operations $\neg$ and $\vee$. These proofs run as follows.

$$
\begin{aligned}
A \approx B & \\
& \Rightarrow \Gamma \vdash A \equiv B \\
& \& \ \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(A) = o(B)] \\
& \Rightarrow \Gamma \vdash \neg A \equiv \neg B \\
& \& \ \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(A)' = o(B)'] \\
& \Rightarrow \Gamma \vdash \neg A \equiv \neg B \\
& \& \ \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(\neg A) = o(\neg B)] \\
& \Rightarrow \neg A \approx \neg B
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
A \approx B & \\
& \Rightarrow \Gamma \vdash A \equiv B \\
& \& \ \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(A) = o(B)] \\
& \Rightarrow \Gamma \vdash (A \vee C) \equiv (B \vee C) \\
& \& \ \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \ \Rightarrow\ o(A) \cup o(C) = o(B) \cup o(C)] \\
& \Rightarrow (A \vee C) \approx (B \vee C)
\end{aligned} \tag{42}
$$

In the second step of Eq. 41, we used A3. In the second step of Eq. 42, we used A4 and A10. For the quantified part of these expressions, we applied the definition of $\mathcal{O}6$. ■

DEFINITION 33. The equivalence class for wff A under the relation of equivalence $\approx$ is defined as $|A| = \{B \in \mathcal{F}^\circ : A \approx B\}$, and we denote $\mathcal{F}^\circ/\approx\, = \{|A| : A \in \mathcal{F}^\circ\}$. The equivalence classes define the natural morphism $f : \mathcal{F}^\circ \longrightarrow \mathcal{F}^\circ/\approx$, which gives $f(A) \ =^{\mathrm{def}}\ |A|$. We write $a = f(A)$, $b = f(B)$, etc.

LEMMA 34. *The relation $a = b$ on $\mathcal{F}^\circ/\approx$ is given by:*

$$|A| = |B| \qquad \Leftrightarrow \qquad A \approx B \tag{43}$$

LEMMA 35. *The Lindenbaum algebra $\mathcal{A} = \langle \mathcal{F}^\circ/\approx, \neg/\approx, \vee/\approx\rangle$ is a* WOML, *i.e., Eqs. (1)–(6) and Eq. (9) hold for $\neg/\approx$ and $\vee/\approx$ as $'$ and $\cup$ respectively* [where—for simplicity—we use the same symbols ($'$ and $\cup$) as for O6, since there are no ambiguous expressions in which the origin of the operations would not be clear from the context].

Proof. For the $\Gamma \vdash A \equiv B$ part of the $A \approx B$ definition, the proofs of the ortholattice conditions, Eqs. (1)–(6), follow from A5, A6, A9, the dual of A8, the dual of A7, and DeMorgan's laws respectively. (The duals follow from DeMorgan's laws, derived from A10, A9, and A3.) A11 gives us an analog of the OML law for the $\Gamma \vdash A \equiv B$ part, and the WOML law Eq. (9) follows from the OML law in an ortholattice. For the quantified part of the $A \approx B$ definition, lattice O6 is a WOML by Theorem 14. ■

LEMMA 36. *In the Lindenbaum algebra $\mathcal{A}$, if $f(X) = 1$ for all X in Γ implies $f(A) = 1$, then $\Gamma \vdash A$.*

Proof. Let us assume that $f(X) = 1$ for all X in Γ implies $f(A) = 1$ i.e. $|A| = 1 = |A| \cup |A|' = |A \vee \neg A|$, where the first equality is from Definition 33, the second equality follows from Eq. (7) (the definition of 1 in an ortholattice), and the third from the fact that $\approx$ is a congruence. Thus $A \approx (A \vee \neg A)$, which by definition means $\Gamma \vdash A \equiv (A \vee \neg A)\ \&\ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \Rightarrow o(A) = o((A \vee \neg A))]$. This implies, in particular, $\Gamma \vdash A \equiv (A \vee \neg A)$. In any ortholattice, $a \equiv (a \cup a') = a$ holds. By analogy, we can prove $\Gamma \vdash (A \equiv (A \vee \neg A)) \equiv A$ from $\mathcal{QL}$ axioms A1–A15. Detaching the left-hand side (using A12, A13, A14, and R1), we conclude $\Gamma \vdash A$. ■

THEOREM 37. *The orthomodular law does not hold in $\mathcal{A}$.*

Proof. This is Theorem 3.27 from [Pavičić and Megill, 1999], and the proof provided there runs as follows. We assume $\mathcal{F}^\circ$ contains at least two elementary (primitive) propositions $p_0, p_1, \ldots$. We pick a valuation o that maps two of them, A and B, to distinct nodes $o(A)$ and $o(B)$ of O6 that are neither 0 nor 1 such that $o(A) \le o(B)$ [i.e. $o(A)$ and $o(B)$ are on the same side of hexagon O6 in Figure 1 in Section 2]. From the structure of O6, we obtain $o(A) \cup o(B) = o(B)$ and $o(A) \cup (o(A)' \cap (o(A) \cup o(B))) = o(A) \cup (o(A)' \cap o(B)) = o(A) \cup 0 = o(A)$. Therefore $o(A) \cup o(B) \ne o(A) \cup (o(A)' \cap (o(A) \cup o(B))$, i.e., $o(A \vee B) \ne o(A \vee (\neg A \wedge (A \vee B)))$. This falsifies $(A \vee B) \approx (A \vee (\neg A \wedge (A \vee B))$. Therefore $a \cup b \ne a \cup (a' \cap (a \cup b))$, providing a counterexample to the orthomodular law for $\mathcal{F}^\circ/\approx$. ■

LEMMA 38. *$\mathcal{M} = \langle \mathcal{F}/\approx, f\rangle$ is a* WOML *model.*

Proof. Follows from Lemma 35. ■

Now we are able to prove the completeness of $\mathcal{QL}$, i.e., that if a formula A is a consequence of a set of wffs Γ in all WOML models, then $\Gamma \vdash A$. In particular, when $\Gamma = \varnothing$, all valid formulas are provable in $\mathcal{QL}$. (Recall from the note below Definition 28 that the left-hand side of the metaimplication below is implicitly quantified over all WOML models $\mathcal{M}$.)

THEOREM 39. [Completeness] $\qquad \Gamma \vDash_{\mathcal{M}} A \qquad \Rightarrow \qquad \Gamma \vdash A.$

Proof. $\Gamma \vDash_{\mathcal{M}} A$ means that in all WOML models $\mathcal{M}$, if $f(X) = 1$ for all X in Γ, then $f(A) = 1$ holds. In particular, it holds for $\mathcal{M} = \langle \mathcal{F}/\approx, f \rangle$, which is a WOML model by Lemma 38. Therefore, in the Lindenbaum algebra $\mathcal{A}$, if $f(X) = 1$ for all X in Γ, then $f(A) = 1$ holds. By Lemma 36, it follows that $\Gamma \vdash A$. ■

7 THE COMPLETENESS OF $\mathcal{CL}$ FOR WDOL MODELS: NON-DISTRIBUTIVITY CONFIRMED

In this section we will prove the completeness of $\mathcal{CL}$, i.e., we will impose the structure of WDOL on the set $\mathcal{F}^\circ$ of formulas of $\mathcal{CL}$.

We start with a relation of congruence, i.e., a relation of equivalence compatible with the operations in $\mathcal{CL}$. We have to make use of an equivalence relation to establish a correspondence between formulas from $\mathcal{CL}$ and formulas from WDOL. The resulting equivalence classes stand for elements of a WDOL and enable the completeness proof of $\mathcal{CL}$.

THEOREM 40. *The relation of* equivalence $\approx_{\Gamma,\mathcal{CL}}$ *or just* $\approx$*, defined as*

$$A \approx B \tag{44}$$
$$\stackrel{\text{def}}{=} \Gamma \vdash A \equiv_0 B \ \& \ (\forall o \in \mathcal{O}6)[(\forall X \in \Gamma)(o(X) = 1) \Rightarrow o(A) = o(B)],$$

is a relation of congruence in the algebra $\mathcal{F}$*.*

Proof. The axioms and rules of $\mathcal{QL}$, A1–A15 and R1, i.e., Eqs. (16)–(31), are theorems of $\mathcal{CL}$, A1–A4 and R1, i.e. Eqs. (32)–(36). To verify this we refer the reader to *Principia Mathematica* by Alfred Whitehead and Bertrand Russell [Whitehead and Russell, 1910], where the $\mathcal{QL}$ axioms either will be found as theorems or can easily be derived from them. For example, axiom A1 of $\mathcal{QL}$ is given as Theorem *4.2 [Whitehead and Russell, 1910, p. 116] after using Theorem *5.23 [Whitehead and Russell, 1910, p. 124] to convert from $\equiv_0$ to $\equiv$. This will let us take advantage of parts of the completeness proof for $\mathcal{QL}$, implicitly using Theorem *5.23 [Whitehead and Russell, 1910, p. 124] in either direction as required.

With this in mind, the proof that $\approx$ is an equivalence and congruence relation becomes exactly the proof of Theorem 32. ■

DEFINITION 41. The equivalence class for wff A under the relation of equivalence $\approx$ is defined as $|A| = \{B \in \mathcal{F}^\circ : A \approx B\}$, and we denote $\mathcal{F}^\circ/\approx = \{|A| \in \mathcal{F}^\circ\}$. The equivalence classes define the natural morphism $f : \mathcal{F}^\circ \longrightarrow \mathcal{F}^\circ/\approx$, which gives $f(A) \ =^{\rm def}\ |A|$. We write $a = f(A)$, $b = f(B)$, etc.

LEMMA 42. *The relation $a = b$ on $\mathcal{F}^\circ/\approx$ is given as:*

$$|A| = |B| \qquad \Leftrightarrow \qquad A \approx B \tag{45}$$

LEMMA 43. *The Lindenbaum algebra $\mathcal{A} = \langle \mathcal{F}^\circ/\approx, \neg/\approx, \vee/\approx, \wedge/\approx\rangle$ is a* WDOL*, i.e., Eqs. (1)–(6) and Eq. (13), hold for $\neg/\approx$ and $\vee/\approx$ as $'$ and $\cup$ respectively.*

Proof. For the $\Gamma \vdash A \equiv_0 B$ part of the $A \approx B$ definition, the proofs of the ortholattice axioms are identical to those in the proof of Lemma 35 (after using using Theorem *5.23 on p. 124 of Ref. [Whitehead and Russell, 1910] to convert between $\equiv_0$ and $\equiv$). The WDOL law Eq. (13) for the $\Gamma \vdash A \equiv_0 B$ part can be derived using Theorems *5.24, *4.21, *5.17, *3.2, *2.11, and *5.1 [Whitehead and Russell, 1910, pp. 101–124]. For the quantified part of the $A \approx B$ definition, lattice O6 is a WDOL by Theorem 14. ■

LEMMA 44. *In the Lindenbaum algebra $\mathcal{A}$, if $f(X) = 1$ for all X in Γ implies $f(A) = 1$, then $\Gamma \vdash A$.*

Proof. Identical to the proof of Lemma 36. ■

THEOREM 45. *Distributivity does not hold in $\mathcal{A}$.*

Proof. $(a \cap (b \cup c)) = ((a \cap b) \cup (a \cap c))$ fails in O6. Cf. the proof of Theorem 37. ■

LEMMA 46. *$\mathcal{M} = \langle \mathcal{F}/\approx, f\rangle$ is a* WDOL *model.*

Proof. Follows Lemma 43. ■

Now we are able to prove the completeness of $\mathcal{CL}$, i.e., that if a formula A is a consequence of a set of wffs Γ in all WDOL models, then $\Gamma \vdash A$. In particular, when $\Gamma = \varnothing$, all valid formulas are provable in $\mathcal{QL}$.

THEOREM 47. [Completeness] $\qquad \Gamma \vDash_{\mathcal{M}} A \quad \Rightarrow \quad \Gamma \vdash A$

Proof. Analogous to the proof of Theorem 39. ■

8 THE COMPLETENESS OF $\mathcal{QL}$ FOR OML MODELS: ORTHOMODULARITY REGAINED

Completeness proofs for $\mathcal{QL}$ carried out in the literature so far—with the exception of Pavičić and Megill [Pavičić and Megill, 1999]—do not invoke Definition 11 and Theorem 14, and instead of Theorem 32 one invokes the following one:

THEOREM 48. *Relation* $\approx$ *defined as*

$$A \approx B \quad \stackrel{\text{def}}{=} \quad \Gamma \vdash A \equiv B \tag{46}$$

is a relation of congruence in the algebra $\mathcal{F}$.

Instead of Definition 33 one has:

DEFINITION 49. The equivalence class under the relation of equivalence is defined as $|A| = \{B \in \mathcal{F}^\circ : A \approx B\}$, and we denote $\mathcal{F}^\circ/\approx\, = \{|A| \in \mathcal{F}^\circ\}$ The equivalence classes define the natural morphism $f : \mathcal{F}^\circ \longrightarrow \mathcal{F}^\circ/\approx$, which gives $f(A) \ =^{\text{def}} \ |A|$. We write $a = f(A)$, $b = f(A)$, etc.

And instead of Lemma 34 one is able to obtain:

LEMMA 50. *The relation* $a = b$ *on* $\mathcal{F}^\circ/\approx$ *is given as:*

$$a = b \quad \Leftrightarrow \quad |A| = |B| \quad \Leftrightarrow \quad A \approx B \quad \Leftrightarrow \quad \Gamma \vdash A \equiv B \tag{47}$$

Hence, from the following easily provable theorem in $\mathcal{QL}$:

$$\vdash (A \equiv B) \equiv (C \vee \neg C) \quad \Rightarrow \quad \vdash A \equiv B \tag{48}$$

one is also able to get:

$$a \equiv b = 1 \quad \Rightarrow \quad a = b \tag{49}$$

in the Lindenbaum algebra $\mathcal{A}$, which is the orthomodularity as given by Definition 6. [Pavičić, 1998]

The point here is that Eq. (49) has nothing to do with any axiom or rule of inference from $\mathcal{QL}$—it is nothing but a consequence of the definition of the relation of equivalence from Theorem 48. Hence, the very definition of the standard relation of equivalence introduces a hidden axiom—the orthomodularity—into the Lindenbaum algebra $\mathcal{A}$, thus turning it into an orthomodular lattice. Without this hidden axiom, the Lindenbaum algebra stays WOML as required by the $\mathcal{QL}$ syntax. With it the Lindenbaum algebra turns into OML as follows.

LEMMA 51. *In the Lindenbaum algebra* $\mathcal{A}$*, if* $f(X) = 1$ *for all* X *in* Γ *implies* $f(A) = 1$*, then* $\Gamma \vdash A$.

Proof. In complete analogy to the proof of Theorem 36. ■

THEOREM 52. *The orthomodular law holds in* $\mathcal{A}$.

Proof. $a \cup (a' \cap (a \cup b)) = a \cup b$ follows from A11, Eq. (26) and Eq. (49). ■

LEMMA 53. $\mathcal{M} = \langle \mathcal{F}/\approx, f \rangle$ *is an* OML *model.*

Proof. Follows from Lemma 51. ■

Now we are able to prove the completeness of $\mathcal{QL}$, i.e., that if a formula A is a consequence of a set of wffs Γ in all OML models, then $\Gamma \vdash A$.

THEOREM 54. [Completeness] $\quad \Gamma \vDash_{\mathcal{M}} A \quad \Rightarrow \quad \Gamma \vdash A$

Proof. Analogous to the proof of Theorem 39. ■

9 THE COMPLETENESS OF $\mathcal{CL}$ FOR BOOLEAN ALGEBRA MODELS: DISTRIBUTIVITY REGAINED

The completeness proof carried out in almost all logic books and textbooks do not invoke Definition 11, Theorem 14, and Theorem 40. The exception is the *Classical and Nonclassical Logics* by Eric Schechter [Schechter, 2005, p. 272] who adopted them from Pavičić and Megill [Pavičić and Megill, 1999] and presented in a reduced approach which he called the *hexagon interpretation*. Other books, though, are based on:

THEOREM 55. *Relation* $\approx$ *defined as*

$$A \approx B \quad \stackrel{\text{def}}{=} \quad \Gamma \vdash A \equiv_0 B \tag{50}$$

is a relation of congruence in the algebra $\mathcal{F}$.

Instead of Definition 41 one has:

DEFINITION 56. The equivalence class under the relation of equivalence is defined as $|A| = \{B \in \mathcal{F}^\circ : A \approx B\}$, and we denote $\mathcal{F}^\circ/\approx\, = \{|A| \in \mathcal{F}^\circ\}$ The equivalence classes define the natural morphism $f : \mathcal{F}^\circ \longrightarrow \mathcal{F}^\circ/\approx$, which gives $f(A) \; =^{\text{def}} \; |A|$. We write $a = f(A)$, $b = f(A)$, etc.

And instead of Lemma 42 one is able to obtain:

LEMMA 57. *The relation* $a = b$ *on* $\mathcal{F}^\circ/\approx$ *is given as:*

$$a = b \quad \Leftrightarrow \quad |A| = |B| \quad \Leftrightarrow \quad A \approx B \quad \Leftrightarrow \quad \Gamma \vdash A \equiv_0 B \tag{51}$$

Hence, from the following easily provable theorem in $\mathcal{CL}$:

$$\vdash (A \equiv_0 B) \equiv_0 (C \vee \neg C) \quad \Rightarrow \quad \vdash A \equiv_0 B \tag{52}$$

one is also able to get:

$$a \equiv_0 b = 1 \quad \Rightarrow \quad a = b \tag{53}$$

in the Lindenbaum algebra $\mathcal{A}$, which is the distributivity as given by Definition 9. [Pavičić, 1998] The point here is that Eq. (53) has nothing to do with any axiom or rule of inference from $\mathcal{CL}$—it is nothing but a consequence of the definition of the relation of equivalence from Theorem 55. Hence, the very definition of the standard relation of equivalence introduces the distributivity as a hidden axiom into the Lindenbaum algebra $\mathcal{A}$ and turns it into a Boolean algebra.

THEOREM 58. [Completeness] $\qquad \Gamma \vDash_{\mathcal{M}} A \quad \Rightarrow \quad \Gamma \vdash A$

Proof. Analogous to the proof of Theorem 47. ■

10 DISCUSSION

In the above sections, we reviewed the historical results that we considered relevant to decide whether quantum logic can be considered a logic or not. In the Introduction, we showed that many authors in the past thirty years tried to decide on this question by starting with particular models and their syntax—the orthomodular lattice for the quantum logic and Boolean algebra for the classical. They compared the models and often came to a conclusion that since they are so different, quantum logic should not be considered a logic. This was, however, in obvious conflict with the growing number of well-formulated quantum logic systems over the same period. We mentioned some of them in the Introduction.

Orthomodular lattices and Boolean algebras *are* very different. As reviewed in the Introduction, in any orthomodular lattice all operations, variables, and constants are sixfold defined (five *quantum* and one *classical*) and in a Boolean algebra they all merge to classical operations, variables, and constants (0,1). Both an orthomodular lattice and a Boolean algebra can be formulated as algebras—as reviewed in Section 2. Such algebras can mimic both quantum and classical logics and show that one can formulate the Deduction Theorem in a special orthomodular lattice—a distributive one, i.e., a Boolean algebra—but cannot in a general one. As a consequence, the operation of implication—which the Deduction Theorem[21] is based on—plays a special unique role in classical logic and does not in quantum logic. Also, the Boolean algebra used as a model for classical logic is almost always two-valued, i.e., it consists of only two elements 0 and 1, and an orthomodular lattice, according to the Kochen-Specker theorem, cannot be given a $\{0, 1\}$ valuation.[22]

So, recently research was carried out on whether a logic could have more than one model of the same type, e.g., an ortholattice, with the idea of freeing logics of any semantics and valuation. The result was affirmative, and a consequence was that quantum logic can be considered a logic in the same sense in which classical logic can be considered a logic. The details are given in Sections 3–9, where we chose Kalmbach's system to represent quantum logic in Section 3.1 and Hilbert and Ackermann's presentation of *Principa Mathematica* to represent classical logic in Section 3.2 (although we could have chosen any other system mentioned in the Introduction or from the literature).[23]

In Sections 4 and 6, we then proved the soundness and completeness, respectively, of quantum logic $\mathcal{QL}$ for a non-orthomodular model WOML and in Sections

[21] See footnote No. 14.

[22] In 2004 Mladen Pavičić, Jean-Pierre Merlet, Brendan McKay, and Norman Megill gave exhaustive algorithms for generation of Kochen-Specker vector systems with arbitrary number of vectors in Hilbert spaces of arbitrary dimension. [Pavičić *et al.*, 2004; Pavičić *et al*, 2005; Pavičić, 2005] The algorithms use MMP (McKay-Megill-Pavičić) diagrams for which in 3-dim Hilbert space a direct correspondence to Greechie and Hasse diagrams can be established. Thus, we also have a constructive proof within the lattice itself.

[23] Quantum logics given by Mladen Pavičić [Pavičić, 1989] and by Mladen Pavičić and Norman Megill [Pavičić and Megill, 1999] are particulary instructive since they contain only axioms designed so as to directly map into WOML conditions.

5 and 7 the soundness and completeness, respectively, of classical logic $\mathcal{CL}$ for a non-distributive model WDOL. Hence, with respect to these models, quantum logic $\mathcal{QL}$ cannot be called orthomodular and classical logic $\mathcal{CL}$ cannot be called distributive or Boolean. Also, neither $\mathcal{QL}$ nor $\mathcal{CL}$ can have a numerical valuation in general, since the truth table method is inapplicable within their OML, WOML, and WDOL models.

One might be tempted to "explain" these results in the following way. "It is true that WOML and WDOL obviously contain lattices that violate the orthomodularity law, for example the O6 hexagon (shown in Figure 1 in Section 2) itself, but most probably they also *must* contain lattices that pass the law and that would, with reference to Theorem 16, explain why we were able to prove the completeness of quantum and classical logic for WOML and WDOL." This is, however, not the case. We can prove the soundness and completeness of quantum and classical logics using a class of WOML lattices none of which pass the orthomodularity law. [Megill and Pavičić, 2006] Moreover, Eric Schechter has simplified the results of Pavičić and Megill [Pavičić and Megill, 1999] to the point of proving the soundness and completeness of classical logic for nothing but O6 itself. [Schechter, 2005, p. 272]

One of the conclusions Eric Schechter has drawn from the unexpected non-distributivity of the WDOL models, especially when reduced to the O6 lattice alone, is that all the axioms that one can prove by means of $\{0, 1\}$ truth tables, one can also prove by any Boolean algebra, and by O6. So, logics are, first of all, axiomatic deductive systems. Semantics are a next layer that concern models and valuations. Quantum and classical logics can be considered to be two such deductive systems. There are no grounds for considering any of the two logics more "proper" than the other. As we have shown above, semantics of the logics that consider their models show bigger differences between the two aforementioned classical models than between two corresponding quantum and classical models.

Whether we will ever use O6 semantics of classical logic or WOML semantics of quantum logic remains an open question, but these semantics certainly enrich our understanding of the role of logics in applications to mathematics and physics. We cannot make use of bare axiomatics of logic without specifying semantics (models and valuations) for the purpose. By making such a choice we commit ourselves to a particular model and disregard the original logical axioms and their syntax. Thus we do not use quantum logic itself in quantum mechanics and in quantum computers but instead an orthomodular lattice, and we do not use classical logic in our computers today but instead a two-valued Boolean algebra (we even hardly ever use more complicated Boolean algebras). We certainly cannot use O6 semantics to build a computer or an arithmetic; however, one day we might come forward with significant applications of these alternative semantics, and then it might prove important to have a common formal denominator for all the models—logics they are semantics of. We can also imagine an alternative scenario—searching for different semantics of the same logics.

Whatever strategy we choose to apply, we should always bear in mind that the syntaxes of the logics correspond to WOML, WDOL, and O6 semantics (models) while OML and Boolean algebra semantics (models) are imposed on the logics with the help of "hidden" axioms, Eqs. (49) and (53), that emerge from the standard way of defining the relation of equivalence in the completeness proofs, Theorems 48 and 55, of the logics for the latter models.

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