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SPECIAL nth ORDER SURFACES WITH $(n-2)-$ple LINE

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ABSTRACT: In this paper, in Euclidean space $\mathbb{E}^3$, we treat the pedal surfaces of special line congruences $\mathcal{C}^1_{2k}$ which are of the 1st order and the $2k$th class. We derive the parametric and implicit equations of these surfaces which enable *Mathematica* visualizations and proving some properties such as their order is $2k+2$, they possess one $2k$-ple straight line and pass through the absolute conic of $\mathbb{E}^3$. The properties of their singularities, which do not lie on $2k$-ple line, and of the pinch points on the $2k$-ple line, are also shown.

Keywords: congruence of lines, inversion, pedal surfaces of congruence, multiple line, multiple point, pinch point

1. INTRODUCTION

Congruence $\mathcal{C}$ is a double infinite line system, i.e. it is the set of lines in the three-dimensional space (projective, affine or Euclidean) depending on two parameters. Line $z \in \mathcal{C}$ is said to be the *ray* of a congruence. The *order* of a congruence is the number of its rays which pass through an arbitrary point; the *class* of a congruence is the number of its rays which lie in an arbitrary plane. *m*th *order*, *n*th *class* congruence is signed $\mathcal{C}^m_n$. A point is called the *singular point* of a congruence if ∞^1 rays pass through it. A plane is called the *singular plane* of a congruence if it contains ∞^1 rays.

According to [6, p. 64], [10, pp. 1184-1185], there are only two types of the first order congruences: the first one are the congruences of nth class and their rays are transversals of one straight line d and nth order space curve c^n which cuts this straight line in $n-1$ points (see Fig. 1), and the second are congruences of 3rd class and its rays cut a twisted cubic twice. The properties of the first order congruences can be found in [1].

In Euclidean space E^3, the *pedal surface* of congruence $\mathcal{C}^m_n$ with respect to *pole* P is the locus of the feet of perpendiculars from finite point P to the rays of congruence $\mathcal{C}^m_n$, [5].

In [2] we define the transformation of three-dimensional projective space where corresponding points lie on the rays of congruence $\mathcal{C}^1_n$ and are conjugate with respect to proper quadric Ψ (see Fig. 2). This transformation we called the $(n+2)$ *degree inversion* with respect to congruence $\mathcal{C}^1_n$ and quadric Ψ and signed it by $i^{n+2}_\Psi : \mathbb{P}^3 \to \mathbb{P}^3$. We proved that it takes a straight line to the $(n+2)$ order space

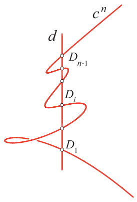

Figure 1: Directing lines of $\mathcal{C}^1_n$

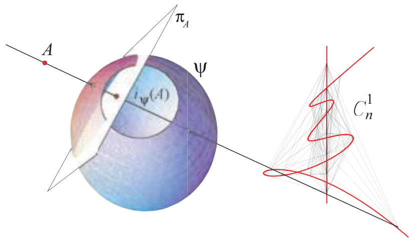

Figure 2: Inversion of degree $n+2$

curve and a plane to the $(n+2)$ order surface which contains n-ple straight line.

The class of such surfaces was elaborated in detail by Sturm [8, pp. 315-328].

The pedal surface of the first order congruence $\mathcal{C}_n^1$ is the image of the plane at infinity given by i_Ψ^{n+2}, where Ψ is any sphere with center P. According to the properties of i_Ψ^{n+2} it was shown that the pedal surface of congruence $\mathcal{C}_n^1$ is $(n+2)$ order surface with n-ple line straight line d which passes through the absolute conic of E^3 and the directing curve c^n, [2].

2. SPECIAL $\mathcal{C}_{2k}^1$ CONGRUENCES

A special class of $\mathcal{C}_n^1$ arises if all intersection points D_i (see Fig. 1) coincide. In this case c^n is a plane curve with one singular point of the highest multiplicity $n-1$, and line d passes through this point.

Here we will regard special $\mathcal{C}_n^1$ where n is an even number, i. e. $n=2k$, $k \in \mathbb{N}$, and directing curve c^{2k} is a plane curve with $(2k-1)$-ple singular point.

2.1 $(2k-1)$-folium

$(2k-1)$-folium is curve c^{2k} given by the following polar equation:

$$r(\varphi) = \cos(2k-1)\varphi, \quad \varphi \in [0, \pi). \qquad (1)$$

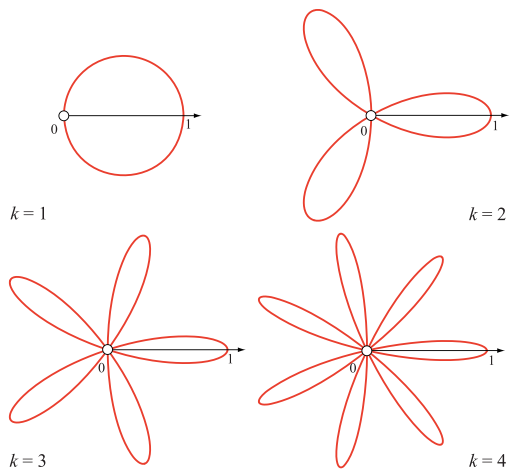

Figure 3: $(2k-1)$-folium

According to the multiple-angle formula, $\cos(2k-1)\varphi$ can be displayed as

$$\sum_{i=0}^{k} (-1)^i C_{2i}^{2k-1} (\cos\varphi)^{2k-1-2i} (\sin\varphi)^{2i} \qquad (2)$$

where C_{2i}^{2k-1} is a binomial coefficient.

Therefore, from eq. (1), by using the substitutions $r(\varphi) = \sqrt{x^2+y^2}$, $\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\varphi = \frac{y}{\sqrt{x^2+y^2}}$, we obtain the following implicit equation of $(2k-1)$-folium:

$$(x^2+y^2)^k - \tau^{2k-1} = 0, \quad \text{where,} \qquad (3)$$

$$\tau^{2k-1} = \sum_{i=0}^{k} (-1)^i C_{2i}^{2k-1} x^{2k-1-2i} y^{2i}. \qquad (4)$$

From eq. (3) it is clear that $(2k-1)$-folium is $2k$-order curve c^{2k}, with $(2k-1)$-ple point at the origin, where $2k-1$ tangent lines at it are given by equation $\tau^{2k-1} = 0$, [7, p. 27]. The line at infinity is the k-ple tangent line of c^{2k} which touches it at the absolute points.

2.2 Congruence $\mathcal{C}_{2k}^1$

Let axis z and $(2k-1)$-folium c^{2k} in plane $z=0$ be the directing lines of congruence $\mathcal{C}_{2k}^1$.

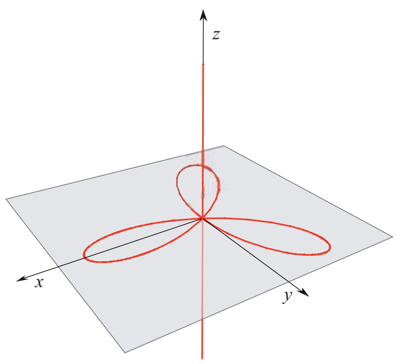

Figure 4: Directing lines of $\mathcal{C}_{2k}^1$ for $k=2$

All singular points of $\mathcal{C}_{2k}^1$ (the points which contain ∞^1 rays of $\mathcal{C}_{2k}^1$) lie on its directing lines c^{2k} and z. If point C lies on curve c^{2k} and $C \neq O$, then the rays of $\mathcal{C}_{2k}^1$ which pass trough C form pencil of lines (C) in plane $\zeta \in [z]$ which contains C and z, see Fig. 5a. If point Z lies on axis z and $Z \neq O$, then all the lines which join Z with the points of curve c^{2k} are

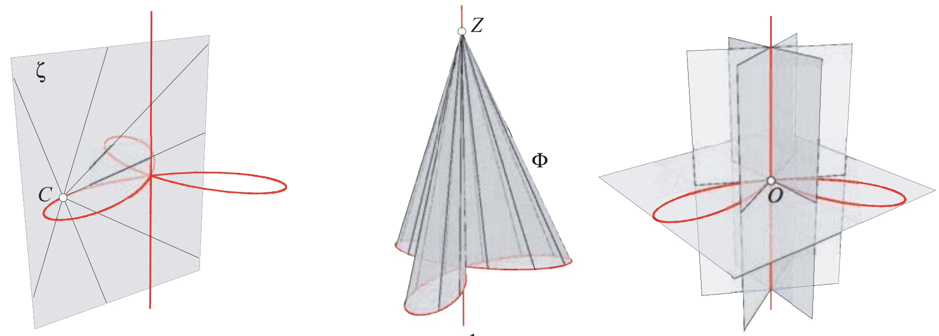

Figure 5: Singular points and planes of $\mathcal{C}^1_{2k}$ for $k = 2$.

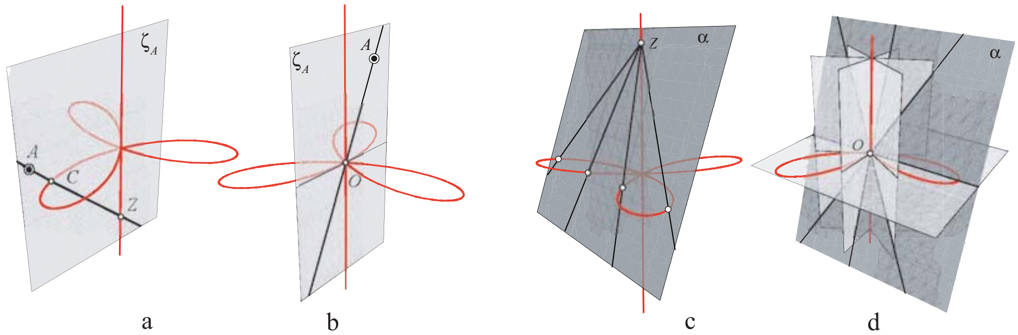

Figure 6: Rays of $\mathcal{C}^1_{2k}$ (for $k = 2$) through non singular points and non singular planes.

the rays of $\mathcal{C}^1_n$. They form $2k$- degree cone Φ^{2k}_Z with vertex Z. Axis z is $(2k-1)$-ple generatrix of Φ^{2k}_Z, see Fig. 5b. The rays through point O form $2k-1$ pencil of lines (O) in the planes determined by axis z and $2k-1$ tangent lines of c^{2k} at O, and pencil (O) in the plane of c^{2k}, see Fig. 5c. Singular planes of $\mathcal{C}^1_{2k}$ (the planes which contain ∞^1 rays) are the planes of the pencil $[z]$ and plane of $(2k-1)$-folium, see Fig. 5c.

Every point A, which is not the singular point of $\mathcal{C}^1_{2k}$, determines plane $\zeta_A \in [z]$ which cuts c^{2k} in only one point C beside the origin. Line AC, which cuts z in one point Z, is the unique ray of $\mathcal{C}^1_{2k}$ through point A, see Fig. 6a. If plane ζ_A contains one of the tangent lines of c^{2k} at O, then points C and Z coincide with O and line AO is the unique ray of $\mathcal{C}^1_{2k}$ through A, see Fig. 6b.

Every plane α which is not the singular plane of $\mathcal{C}^2_{2k}$, contains $2k$ rays of $\mathcal{C}^1_{2k}$. Plane α cuts axis z in point Z and curve c^{2k} in points $C_j, j = 1, ..., 2k$. Lines ZC_j are $2k$ rays of $\mathcal{C}^1_n$ in plane α. They are the intersection of plane α and $2k$-degree cone Φ^{2k}_Z, and can be real and different, coinciding or imaginary, see Fig. 6c.. If α passes through point O, then $2k-1$ rays are the intersections of α with the planes through z and the tangent lines of c^{2k} at O, and one ray lies in the plane of c^{2k}, see Fig. 6d.

3. PEDAL SURFACES OF $\mathcal{C}^1_{2k}$

As we mentioned in the Introduction: in Euclidean space $\mathbb{E}^3$, the pedal surface of congruence $\mathcal{C}$ with respect to pole P is the locus of the feet of perpendiculars from finite point P to the rays of congruence $\mathcal{C}$. According to [2], the pedal surfaces of $\mathcal{C}^1_{2k}$ is $2k+2$ order surface with $2k$-ple line z, and we will denote it $\mathcal{P}^{2k+2}_{2k}$.

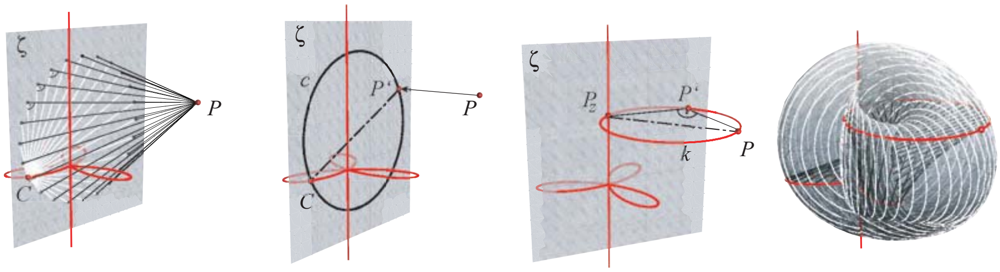

Figure 7: Construction of circles $c \in \zeta \in [z]$

3.1 Construction

In plane ζ trough axis z, the rays of $\mathcal{C}^1_{2k}$ form pencil of lines (C), where $C \neq O$ is the intersection of ζ and c^{2k}. In $2k-1$ planes, determined by the tangent lines of c^{2k} at O and axis z, point C coincides with O. If finite pole P is in the general position to the directing lines of $\mathcal{C}^1_{2k}$, the feet of perpendiculars from P to the rays of pencil (C) form circle c with diameter $\overline{CP'}$, where P' is the orthogonal projection of P to ζ, see Fig. 7a,b. The proof of this statement is elementary.

For given pole P, the path of point P', with respect to the moving plane ζ, is the circle, denoted by k, which lies in the plane through P, perpendicular to axis z. The diameter of k is $\overline{PP_z}$, where P_z is the normal projection of P to z. Thus, we can regard surface $\mathcal{P}^{2k+2}_{2k}$ as the system of circles in the planes through axis z with the end points of diameters on (2k-1)-folium c^{2k} and circle k, see Fig. 7c,d.

The diameters of circles c lie on ruled surface with directing lines c^{2k}, k and z. According to the formula [6, p. 90], the degree of this surface is: $2{\cdot}2k{\cdot}2{\cdot}1-2{\cdot}1-(2k-1){\cdot}2-1{\cdot}2k = 2k$.

3.2 Parametric equations of $\mathcal{P}^{2k+2}_{2k}$ and *Mathematica* visualizations

Let $(p_x, p_y, p_z) \in \mathbb{R}^3$ be the coordinates of pole P and let $(2k-1)$-folium is given by eq. (1). Let (r, z), where $|r| = \sqrt{x^2+y^2}$, be the coordinates of the points in plane $\zeta(\varphi)$, which is given by equation $y = x\tan\varphi$ if $\varphi \in [0,\pi)$, $\varphi \neq \pi/2$, and $x = 0$ if $\varphi = \pi/2$, see Fig. 8.

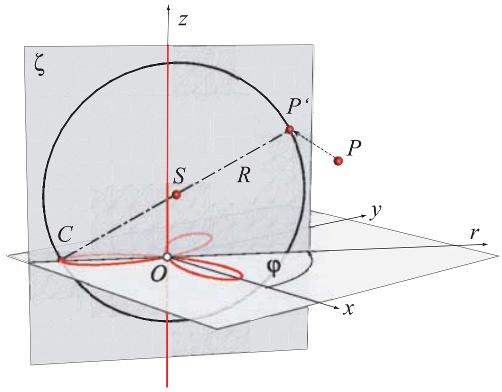

Figure 8

The coordinates of points $P', C \in \zeta(\varphi)$ are

$$\begin{aligned}(r_{P'}, z_{P'})(\varphi) &= (p_x\cos\varphi + p_y\sin\varphi, p_z)\\ (r_C, z_C)(\varphi) &= (\cos(2k-1)\varphi, 0). \qquad (5)\end{aligned}$$

$R(\varphi)$ is the radius and $S(r_S(\varphi), z_S(\varphi))$ is the center of circle c in plane $\zeta(\varphi)$.

$$\begin{aligned}R(\varphi) &= \frac{1}{2}\sqrt{(r_C(\varphi) - r_{P'}(\varphi))^2 + p_z^2}\\ r_S(\varphi) &= \frac{r_C(\varphi) + r_{P'}(\varphi)}{2}\\ z_S(\varphi) &= \frac{p_z}{2}. \qquad (6)\end{aligned}$$

Since the parametric equations of circle c in plane $\zeta(\varphi)$ are

$$\begin{aligned}r(\theta) &= R(\varphi)\sin\theta + r_S(\varphi)\\ z(\theta) &= R(\varphi)\cos\theta + z_S(\varphi),\\ &\quad \theta \in [0, 2\pi), \qquad (7)\end{aligned}$$

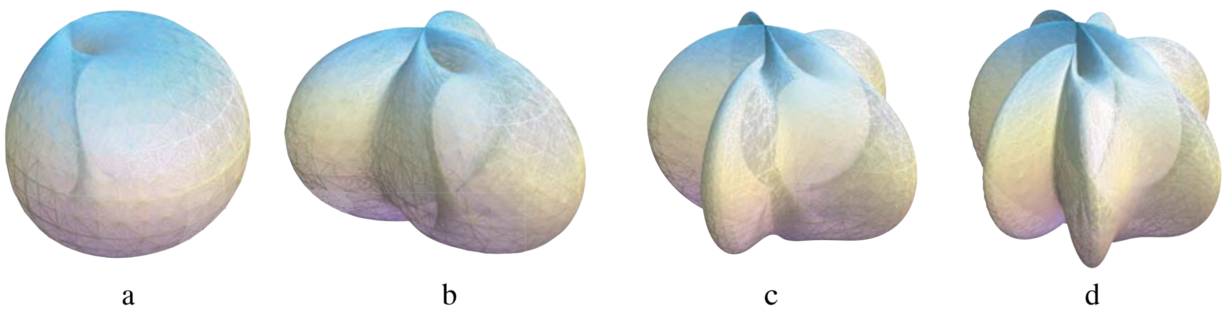

Figure 9: $\mathcal{P}_{2k}^{2k+2}$ for $P(1,0,2)$ and $k=1,2,3,4$, respectively in figures a, b, c and d

therefore the parametric equations of surface $\mathcal{P}_{2k}^{2k+1}$ are the following

$$\begin{aligned} x(\varphi,\theta) &= \cos\varphi\,(R(\varphi)\sin\theta + r_S(\varphi)) \\ y(\varphi,\theta) &= \sin\varphi\,(R(\varphi)\sin\theta + r_S(\varphi)) \\ z(\varphi,\theta) &= R(\varphi)\cos\theta + z_S(\varphi), \\ & \quad \varphi\in[0,\pi),\ \theta\in[0,2\pi). \end{aligned} \tag{8}$$

Equations (8) enable *Mathematica* visualizations of surfaces $\mathcal{P}_{2k}^{2k+1}$. See Fig. 9.

3.3 Implicit equation of $\mathcal{P}_{2k}^{2k+2}$

In plane $\zeta(u)$ through axis z, in coordinates (r,z), the equation of circle c is

$$\begin{aligned} (r-r_S(\varphi))^2 + (z-p_z/2)^2 = R(\varphi)^2, \\ \varphi\in[0,\pi). \end{aligned} \tag{9}$$

From equations (5), by using the formula (2) and the substitutions $\cos u = \frac{x}{\sqrt{x^2+y^2}}$, $\sin u = \frac{y}{\sqrt{x^2+y^2}}$, we obtain the following

$$\begin{aligned} r_C(\varphi) =& \frac{\tau^{2k-1}}{\sqrt{(x^2+y^2)^{2k-1}}} \\ r_{P'}(\varphi) =& \frac{p\,x+qy}{\sqrt{x^2+y^2}} \end{aligned} \tag{10}$$

where τ^{2k-1} is given by eq. (4).

Now, we can express $r_S(\varphi)$ and $R(\varphi)$, given by formulas (6), as the functions of x and y. If we put these functions and $r=\sqrt{x^2+y^2}$ into equation (9) and multiply it by $(x^2+y^2)^k$, we obtain the implicit equation of $\mathcal{P}_{2k}^{2k+2}$ which can be written in the following form

$$\begin{aligned} & (x^2+y^2)^k(x^2+y^2+z^2) \\ +\ & H^{2k+1}(x,y) + H_1^{2k}(x,y)z \\ +\ & H_2^{2k}(x,y) = 0, \end{aligned} \tag{11}$$

where $H^i(x,y)$ are homogeneous polynomials in x and y of degree i, given by the formulas:

$$\begin{aligned} & H^{2k+1}(x,y) = \\ & -(x^2+y^2)^k(p_x x+p_y y) - (x^2+y^2)\tau^{2k-1} \\ & H_1^{2k}(x,y) = -p_z(x^2+y^2)^k \\ & H_2^{2k}(x,y) = (p_x x+p_y y)\tau^{2k-1}. \end{aligned} \tag{12}$$

3.4 Properties of $\mathcal{P}_{2k}^{2k+2}$

Proposition 1 *The plane at infinity cuts surface $\mathcal{P}_{2k}^{2k+2}$ at the absolute conic of $\mathbb{E}^3$ and the rays of congruence $\mathcal{C}_{2k}^1$.*

PROOF: In the Cartesian homogeneous coordinates $(x:y:z:w)$, where $w=0$ means that the point lies in the plane at infinity, the equation of surface $\mathcal{P}_{2k}^{2k+2}$ takes the form

$$\begin{aligned} (x^2+y^2)^y(x^2+y^2+z^2) + H^{2k+1}(x,y)w \\ +H_1^{2k}(x,y)zw + H_2^{2k}(x,y)w^2 = 0. \end{aligned} \tag{13}$$

Therefore, the intersection of $\mathcal{P}_{2k}^{2k+2}$ and the plane at infinity splits into the absolute conic, given by equations $x^2+y^2+z^2=0$, $w=0$, and the pair of imaginary lines through the point $(0:0:1:0)$, counted k times, which are given by equations $(x^2+y^2)^k=0, w=0$.

It is clear from eq. (3) that curve c^{2k} touches the plane at infinity k times at the absolute points. Thus, $\mathcal{C}^1_{2k}$ has k-ple pair of isotropic rays through $(0\!:\!0\!:\!1\!:\!0)$ at infinity. $\square$

Proposition 2 *Axis z is the $2k$-ple line of surface $\mathcal{P}^{2k+2}_{2k}$.*

PROOF: According to [4, p. 251]: If the nth order surface in $\mathbb{E}^3$, which passes through the origin, is given by equation $F(x,z,y)=f_m(x,y,z)+f_{m+1}(x,y,z)+\cdots \cdots + f_n(x,y,z) = 0$, where $f_k(x,y,z)$ $(1\le k\le n)$ is homogeneous polynomial of degree k, then the tangent cone at the point $(0,0,0)$ is given by equation $f_m(x,y,z)=0$.

If we move the origin to any point $Z_0=(0,0,z_0)$ on axis z, from eq. (11) we obtain the following equation for the tangent cone $\mathcal{T}_{Z_0}$ of $\mathcal{P}^{2k+2}_{2k}$ at point Z_0

$$(x^2+y^2)^k z_0^2 + H_1^{2k}(x,y)z_0 + H_2^{2k}(x,y)=0. \tag{14}$$

Since it is the homogeneous equation in x and y of degree $2k$, in the general case $\mathcal{T}_{Z_0}$ always splits into $2k$ planes through axis z. $\square$

There are many possibilities for the type of $2k$-ple singular point Z_0 on line z. It depends on how the homogeneous polynomial from eq. (14) can be factorized, i. e. how tangent cone $\mathcal{T}_{Z_0}$ splits (how many real and imaginary planes, how many coinciding planes, and so on). For example, for point O, $\mathcal{T}_O$ is given by equation $(p_xx+p_yy)\tau^{2k-1}=0$ and, in general, splits into $2k$ real and different planes. But, if the line in plane xy which is given by $p_xx+p_yy=0$ coincides with one of the tangent lines of c^{2k} through O, $\mathcal{P}^{2k+2}_{2k}$ has the *pinch point* in O. Pinch points are the points on multiple line in which two or more tangent planes coincide.

Proposition 3 *Surface $\mathcal{P}^{2k+2}_{2k}$ has $4(2k-1)$ pinch points on $2k$-ple axis z (real or complex). Among them one is always the point at infinity and it is the pinch-point counted k times.*

PROOF: The proof that nth order surface with $(n-2)$-ple line always possesses $4(n-3)$ pinch-points, is given in [8, p. 317]. We give here only its interpretation for this $2k$-order case: Every plane ζ through axes z cuts $\mathcal{P}^{2k+2}_{2k}$ into the $2k$-ple line and one conic c which cuts $2k$-ple line in two points. These points are the touching points of plane ζ and surface $\mathcal{P}^{2k+2}_{2k}$. The correspondence between the planes of pencil $[z]$, where corresponding planes have the same touching point, is the involution of the order $2(2k-1)$, because that through each touching point of plane ζ another $2k-1$ tangent planes pass. This involution has $2\cdot 2(2k-1)$ double elements which are the coinciding tangent planes through the points on $2k$-ple line and their touching points are the pinch-points of $\mathcal{P}^{2k+2}_{2k}$.

According to eq. (13), the tangent cone at point $Z_0^\infty(0:0:1:0)$ is given by equation $(x^2+y^2)^k=0$, thus Z_0^∞ is the pinch-point counted k times. $\square$

Proposition 4 *$\mathcal{P}^{2k+2}_{2k}$ contains curve c^{2k}.*

PROOF: If $z=0$. eq. (11) takes the form

$$(x^2+y^2-p_xx-p_yy)((x^2+y^2)^k-\tau^{2k+1})=0.$$

Thus, plane $z=0$ cuts $\mathcal{P}^{2k+2}_{2k}$ through curve c^{2k} and circle with diameter OP', where P' is the normal projection of pole P on plane $z=0$. $\square$

Proposition 5 *If pole P lies on axis z, $\mathcal{P}^{2k+2}_{2k}$ splits into the pair of isotropic planes through z and $2k$-order surface.*

PROOF: If $p_x=p_y=0$, eq. (11) takes the form

$$(x^2+y^2)P^{2k}(x,y,z)=0,$$

where $P^{2k}(x,y,z)$ is $(x^2+y^2)^{k-1}(x^2+y^2+z^2-p_zz)-\tau^{2k-1}$. $\square$

See figures 10 and 11.

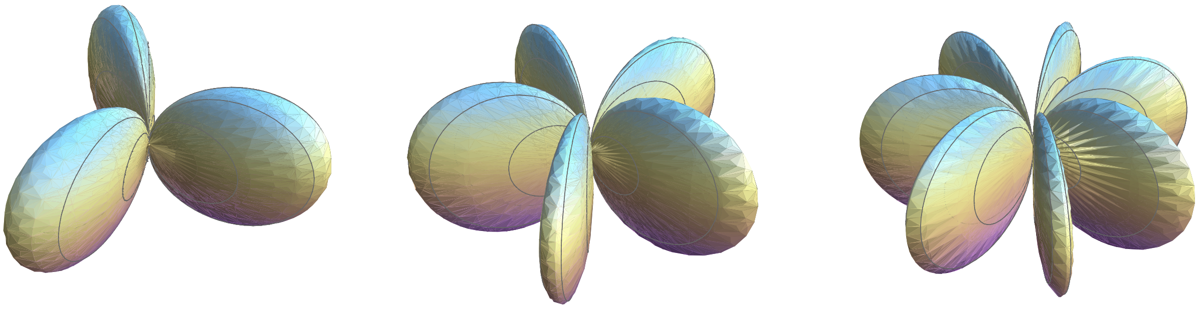

Figure 10: Surfaces $\mathcal{P}^{2k}$, given by equations $P^{2k}(x, y, z) = 0$, for $k = 2, 3, 4$ and $P(0, 0, 0)$

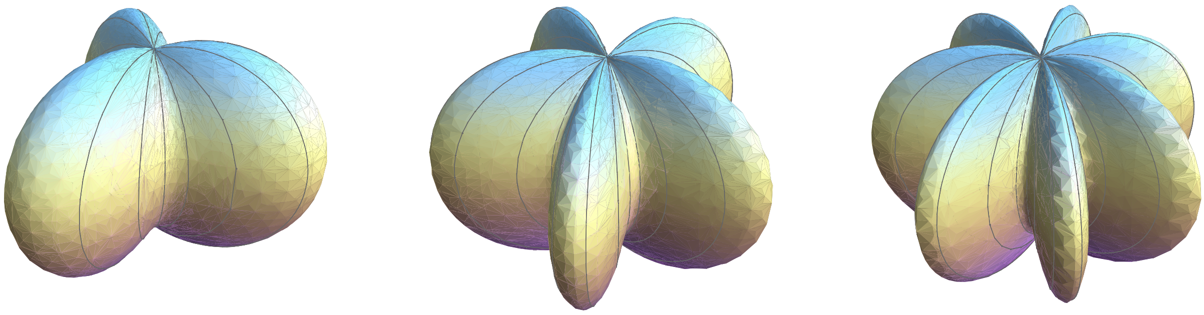

Figure 11: Surfaces $\mathcal{P}^{2k}$, given by equations $P^{2k}(x, y, z) = 0$, for $k = 2, 3, 4$ and $P(0, 0, 2)$

Proposition 6 *Surface $\mathcal{P}^{2k+2}_{2k}$ has real double points out of axis z iff $p_z = 0$. There are at the most $2k-1$ and at least 1 such real points on $\mathcal{P}^{2k+2}_{2k}$.*

PROOF:

Except the points of $2k$-ple line z, the highest singularity which $\mathcal{P}^{2k+2}_{2k}$ can possess is a double point. Namely, if $\mathcal{P}^{2k+2}_{2k}$ had a higher multiple point out of z, the line through that point which cuts z would cut $\mathcal{P}^{2k+2}_{2k}$ in more than $2k+2$ points, which is impossible.

If D is the double point of $\mathcal{P}^{2k+2}_{2k}$ it is the double point of every section of $\mathcal{P}^{2k+2}_{2k}$ through D. Thus, circle c in the plane ζ through D and axis z splits into the pair of isotropic lines through D. It is the case when the end points of diameter $\overline{CP'}$ coincide, i. e. circle k intersects curve c^{2k}. If $p_z \neq 0$ circle k and curve c^{2k} intersect only into the absolute points of plane $z = 0$. Curves k and c^{2k} can possess real intersection points only in the case when $p_z = 0$. In this case they have $4k$ intersection points, where $2k-1$ points coincide with O, 2 points are the absolute points of plane $z = 0$, thus only $2k-1$ intersection points can lie out of axis z and be real. Since $2k-1$ is an odd number, at least one real double point exists on $\mathcal{P}^{2k+2}_{2k}$, if p_z=0. □

See figures 12 and 13.

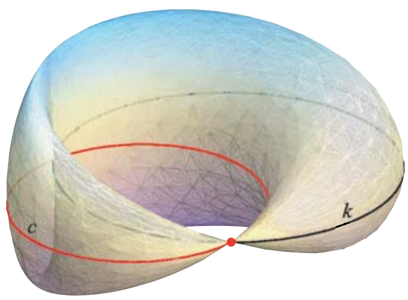

Figure 12: Surface $\mathcal{P}^4_2$ with 1 real double point out of its double line.

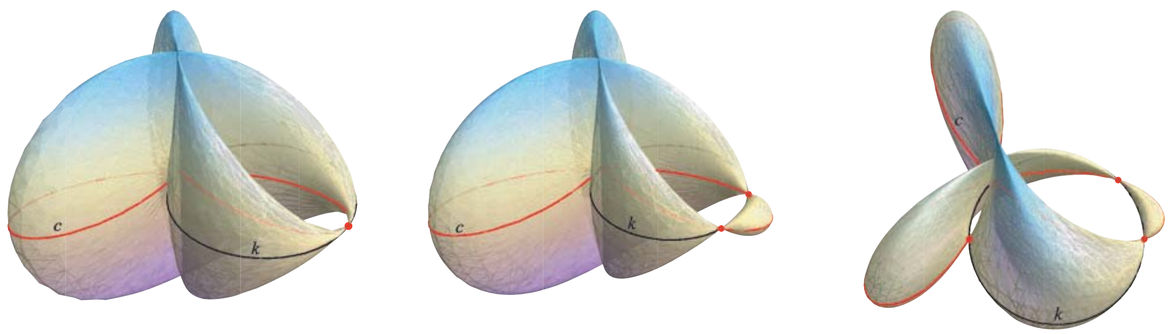

Figure 13: Surfaces $\mathcal{P}_4^6$ with 1, 2 and 3 real double points out of their quadruple lines

4. CONCLUSIONS

The objective of this paper is to visualize numerous forms and properties of special class of surfaces in $\mathbb{E}^3$. Surfaces treated in this paper form only a small subclass of $\mathbb{P}_n^{n+2}$ which is the class of the $(n+2)$th surfaces with n-ple straight line. It may be assumed that the whole class $\mathbb{P}_n^{n+2}$ could be obtained by inversion i_Ψ^{n+2} and visualized by the program *Mathematica*..

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