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CONCENTRIC CIRCLES AND THE GENERALIZED GERGONNE POINT

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ABSTRACT: The well-known triangle center, the Gergonne point is generalized for circles concentric to the inscribed circle in this paper. Given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_1, q_2, q_2, we consider a circle with radius r and center I, for which finding $r = IQ_1 = IQ_2 = IQ_3$ along the given directions, the three cevians V_iQ_i are concurrent. Solutions can be obtained by the common intersection points of three conics. Geometric properties and open problems related to the solutions are discussed in a synthetic way.

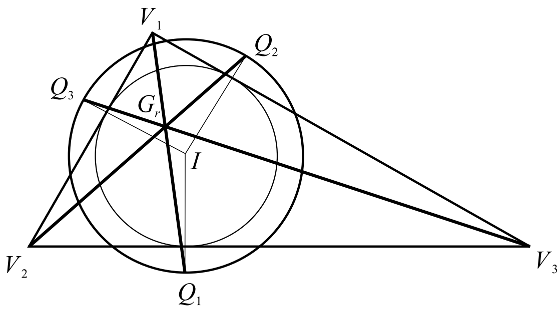
Keywords: Gergonne point, conics, pencil of conics

1. INTRODUCTION

Gergonne point, the intersection of three cevians defined by the touching points of the inscribed circle is a well-known center of triangle. This point is first generalized by Konecny [1], applying circles concentric to the inscribed circle (see Figure 1).

Boyd et al. computed the convex coordinates of G_r in [2], which are functions of the radius of the circle. Altering the radius, the path of the point G_r is a hyperbola. Further comments on Gergonne points can be found in [3].

The basic idea of the further generalization is that instead of the inscribed circle we consider an inscribed ellipse: V_iQ_i are still concurrent, according to the corresponding affine transformation which can transform the ellipse into the inscribed circle of a triangle, preserving the incidence relations. This point of concurrence is called the Brianchon point of the ellipse (c.f. [4]). There are infinitely many inscribed ellipses, center I and directions q_i can be chosen in many ways, but not arbitrarily. The locus of the possible centers I of the inscribed ellipses in a triangle is the interior of the medial triangle. If the center and one direction q_1 is given then the other two directions are determined. This follows from the fact that five points or tangents uniquely determine the conic (Q_1 and the tangent line V_2V_3 in it, the antipodal point of Q_1, the tangent line V_1V_2 and the tangent line V_1V_3).



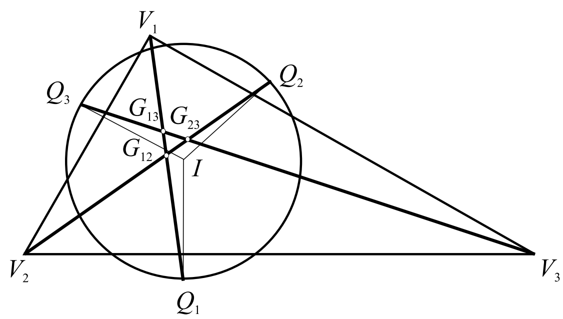


Figure 1: Lines V_iQ_i are also concurrent for circles concentric to the inscribed circle, but for arbitrary directions and distance, cevians V_iQ_i are generally meet at three different points

Using these directions we can generalize the concept of concentric circles: given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_1, q_2, q_3, find a distance $r = IQ_1 = IQ_2 = IQ_3$ along these directions, for which the three cevians V_iQ_i are concurrent. In general these lines will not meet in one point (see Fig. 1): instead of one single center G we have three different intersection points G_{12},G_{13} and G_{23}, naturally all depend on r.

Note, that there are no further restrictions for the positions of the center I and the directed lines. The center can even be out of the reference triangle.

2. SOLUTION OF THE GENERAL PROBLEM

The basic solution of the general problem mentioned above has been provided in [5]. In that paper the authors proved that altering the value r, the points G_{12},G_{13} and G_{23} will move on three conics, c_1, c_2 and c_3, respectively. If there is a solution to our generalized problem, it would mean that these conics have to meet in one common point. It is easy to observe that each pairs of conics have two common points at I and at one of the vertices of the triangle. The final statement is that the other two intersection points can be common for all the three conics.

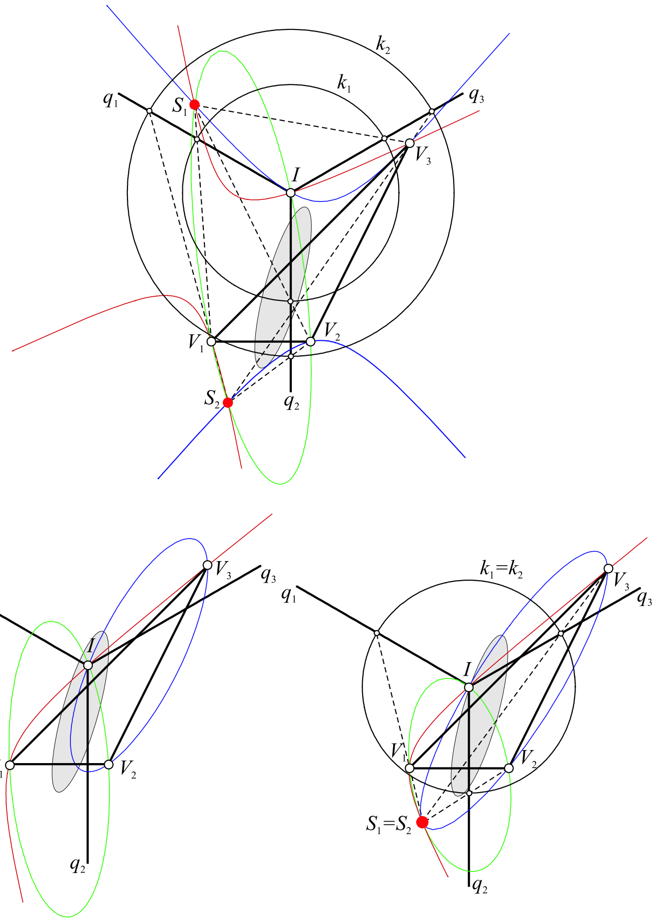


Figure 2: Given a triangle $V_1V_2V_2$ and directions $q_i(i = 1, 2, 3)$ there can be two different real solutions (upper figure), two coinciding solutions (bottom right) and two imaginary solutions (bottom left). Cevians are plotted by dashed lines. The type of solutions depends on the relative position of I to the shaded conic. The three conic paths of G_{12} (green), G_{13} (red) and G_{23} (blue) are also shown.(This figure is computed and plotted by the software *Mathematica*)

Theorem 1 *Let V_1, V_2, V_3 and I are four points in the plane in general position. Let q_1, q_2, q_3 are three different oriented lines through I ($V_i \notin q_i$). Then there exist at most two values $r \in \mathbb{R}\backslash\{0\}$ such that finding points Q_i along the lines q_i as $IQ_1 = IQ_2 = IQ_3 = r$, the lines V_iQ_i are concurrent.*

The two possible solutions yield two points of concurrency S_1 and S_2, which can be real and different, real and coinciding or imaginary in pair. Fig. 2 shows the three different possibilities. If the triangle and the directions $q_i (i = 1, 2, 3)$ are fixed, then the radius of the future circle can be obtained by the solutions of a quadratic equation in which the only unknown is the point I. The type of the solutions depends on the discriminant, which is a quadratic function of I. This means that for every triangle and triple of directions there exists a conic which separates the possible positions of I in the following way: if I is outside the conic (discriminant > 0)then there are two different real solutions, if I is on the conic (discriminant $= 0$) then there are two coinciding real solutions, while if I is inside the conic (discriminant < 0) then there are two imaginary solutions. This conic is also shown in Fig. 2.

3. PENCIL OF CONICS

As we have seen, the solution can be obtained by the common intersection points of three conics c_1, c_2 and c_3. Each of these conics pass through the given point I and two vertices of the triangle V_1, V_2, V_3. But each of them has one more fixed point: consider the lines $\bar{q}_i$ parallel to the given directions q_i through $V_i, i = 1, 2, 3$. Denote the intersection points of these lines by

$$C_1 = \bar{q}_2 \cap \bar{q}_3$$
$$C_2 = \bar{q}_1 \cap \bar{q}_3$$
$$C_3 = \bar{q}_1 \cap \bar{q}_2.$$

Now the conic c_i passes through C_i as well, because this point corresponds to the point at infinity of the line q_i (c.f. Fig. 3).

According to the projective principles in the proof, the statement remains valid if we substitute the condition

$$IQ_1 = IQ_2 = IQ_3 = r$$

with the more general case

$$k_1IQ_1 = k_2IQ_2 = k_3IQ_3 = r,$$

i.e. only the ratios of these lengths have to be preserved. This way for each point I and directions q_i the conics c_1, c_2, c_3 belong to three pencils of conics $(IC_1V_2V_3), (IC_2V_1V_3), (IC_3V_1V_2)$, respectively, where for every given ratio $k_1 : k_2 : k_3$ we can determine the fifth point P_i on c_i.

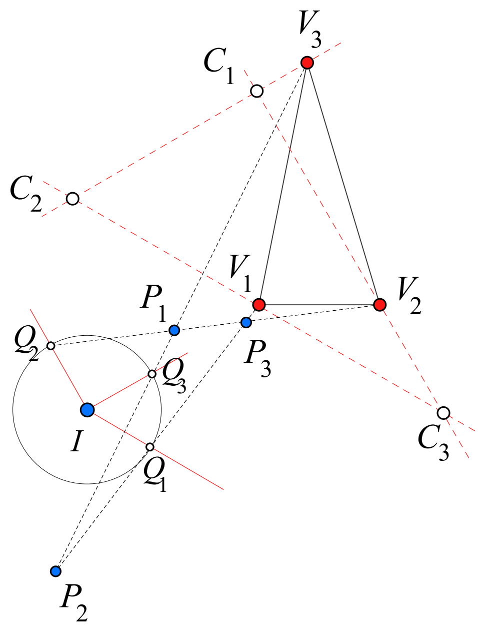


Figure 3: Points C_i correspond to the points at infinity of the lines $IQ_i = q_i$, respectively

4. FURTHER RESEARCH AND OPEN PROBLEMS

There are numerous questions left untouched in this note. First of all, experiments suggest that all the solutions S_i are on a single conic, but this assumption has not been proved yet. We have seen another conic (the gray conic in Figure 2) containing the positions of the given point I for which the two solutions are coinciding. This conic may be an inscribed conic into the triangle $C_1C_2C_3$. Moreover the affine types of all these conics must be the topic of our future investigations as well.

References

[1] Konecny, V., Problem department, *Math. Mag.*, **69** (1990), 130-131.

[2] Boyd, J.N. and Raychowdhury, P.N.: The Gergonne point generalized through convex coordinates, *Int. J. Math. Math. Sci.*, **22** (1999), 423-430.

[3] Wojtowicz, B.: Desargues' Configuration in a Special Layout, *Journal for Geometry and Graphics*, **7** (2003), 191-199.

[4] Veblen, O. and Young, J. W.: Projective Geometry, Boston, MA, Ginn, 1938.

[5] Hoffmann, M. and Gorjanc, S.: On the generalized Gergonne point and beyond,*Forum Geom.* (to appear)

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