Canonical active Brownian motion and bifurcation phenomena

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talk based on arXiv:0809.5011

5^{th} Vienna Central European Seminar on Particle Physics and QFT, Nov. 28–30, 2008

Simple (non-active) Brownian motion

Langevin picture in phase space, $H(q,p) = p^2/2 + U(q)$:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} = \delta^{ij} p_j$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} - \gamma_0\, \delta_{ij} \frac{\partial H}{\partial p_j} + \eta_i = -\frac{\partial U}{\partial q^i} - \gamma_0 p_i + \eta_i$$

where η_i is the random force, $\langle \eta_i(t) \rangle = 0$, $\langle \eta_i(t) \eta_j(t') \rangle = 2\gamma_0 \delta_{ij} \delta(t - t')$

Fokker–Planck picture: probability density has equilibrium distribution

$$\rho_{\mathrm{eq.}}(q,p) \propto \mathrm{e}^{-H(q,p)}$$

Active Brownian motion: original theory

Coupling of *internal energy* e to kinetic energy $K = p^2/2$:

$$\frac{\mathrm{d}e}{\mathrm{d}t} = c_1 - c_2\, e - c_3\, e\, \frac{p^2}{2}$$

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i} - g(e)\, \delta_{ij} \frac{\partial H}{\partial p_j} + \eta_i \qquad \text{where} \qquad g(e) = \gamma_0 - d_2\, e$$

Assuming $e = const$ in equilibrium, ABM is equivalent to SBM with *nonlinear friction:*

$$\gamma_0 \quad \longrightarrow \quad \gamma(p) = \gamma_0 - \frac{d_2 c_1}{c_2 + c_3 p^2/2}$$

Active Brownian motion: alternative formulation

Coupling of internal energy e to potential U:

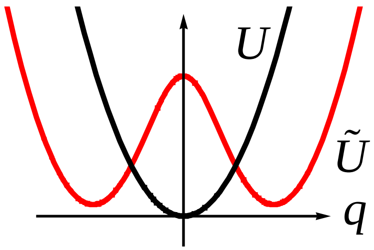
$$\frac{\mathrm{d}e}{\mathrm{d}t} = c_1 - c_2\, e - c_4\, e\, U(q)$$

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -f(e)\frac{\partial H}{\partial q^i} - \gamma_0\, \delta_{ij}\frac{\partial H}{\partial p_j} + \eta_i \qquad \text{where} \qquad f(e) = 1 - d_1\, e$$

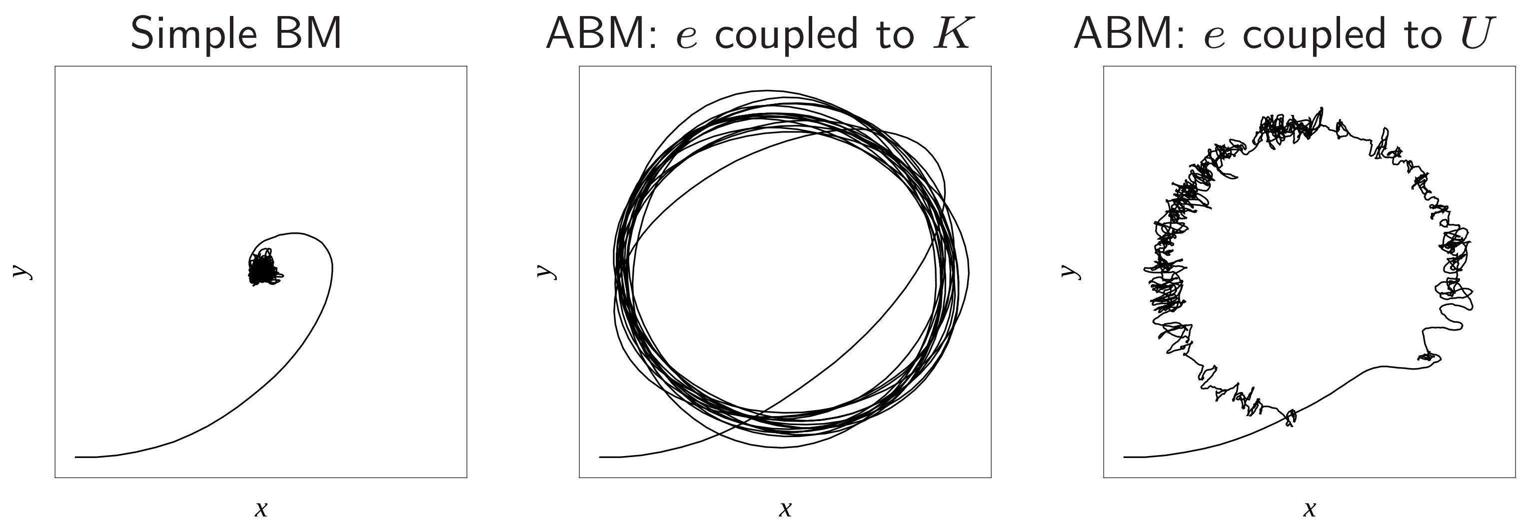
Assuming $e = const$, ABM becomes equivalent to SBM with *effective potential*:

$$\tilde{U}(q) = U(q) + \tfrac{d_1 c_1}{c_4} \ln(c_2 + c_4 U(q))$$



Summary

2D orbits in harmonic potential:



Applications:

- Coupling e to K: mobility of biological organisms, complex systems ...
- Coupling e to U: stochastic quantization in QFT, emergence of the Higgs potential ...

Active Brownian motion: generalization

Coupling of internal energy e to kinetic energy $K = p^2/2$ and potential U:

$$\frac{\mathrm{d}e}{\mathrm{d}t} = c_1 - c_2\, e - c_3\, e\, \frac{p^2}{2} - c_4\, e\, U$$

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -f(e)\, \frac{\partial H}{\partial q^i} - g(e)\, \delta_{ij} \frac{\partial H}{\partial p_j} + \eta_i$$

where

$$f(e) = 1 - d_1\, e \quad \text{and} \quad g(e) = \gamma_0 - d_2\, e$$

Canonical active Brownian motion

Special case: setting $c_3 = c_4$ couples e to the Hamiltonian $H = K + U$, assuming $e = const$ in equilibrium leads to a *"canonical dissipative"* system:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -F(H)\,\frac{\partial H}{\partial q^i} - G(H)\,\delta_{ij}\frac{\partial H}{\partial p_j} + \eta_i$$

where

$$F(H) = 1 - \frac{d_1 c_1}{c_2 + c_3 H} \quad \text{and} \quad G(H) = \gamma_0 - \frac{d_2 c_1}{c_2 + c_3 H}$$

Deterministic orbits in fully coupled ABM

Program for studying the nonlinearly coupled system of ODEs:

- assume harmonic potential $U = q^2/2$
- look for equilibria in the equations of motion
- linearize the equations around the equilibria
- obtain eigenvalues of the Jacobian
- study local stability and bifurcations
- special interest: formation of limit cycles

Routh Hurwitz stability criterion helps where eigenvalues of the Jacobian cannot be obtained in simple form. It formulates a procedure to test negativity of real parts of roots of a polynomial without having to compute the roots.

1D system: equilibria and stability

Notation:

$$\lambda_1 = 1 - \frac{c_1}{c_2}d_1, \quad \lambda_2 = \gamma_0 - \frac{c_1}{c_2}d_2, \quad \lambda_3 = \gamma_0 - \frac{d_2}{d_1}$$

- *Equilibrium (e1):* particle at rest at the minimum of U,

$$q_0 = 0, \quad p_0 = 0, \quad e_0 = \frac{c_1}{c_2}$$

stable (node or focus) if: $\lambda_{1,2} > 0$

- *Equilibrium (e2):* particle at rest outside of the minimum of U,

$$q_0 = \pm\sqrt{-2c_2\lambda_1/c_4}, \quad p_0 = 0, \quad e_0 = 1/d_1$$

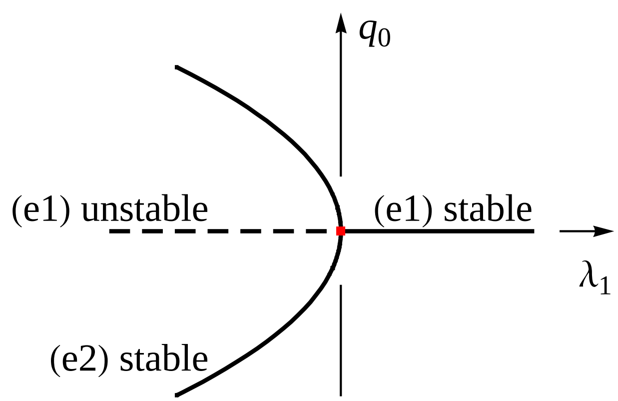
stable if: $\lambda_1 < 0, \quad \lambda_3 > 0, \quad \lambda_3 c_1 d_1(\lambda_3 + c_1 d_1) > -2c_2\lambda_1$

1D system: bifurcation of equilibria

Two types of bifurcation are found:

- Pitchfork bifurcation:

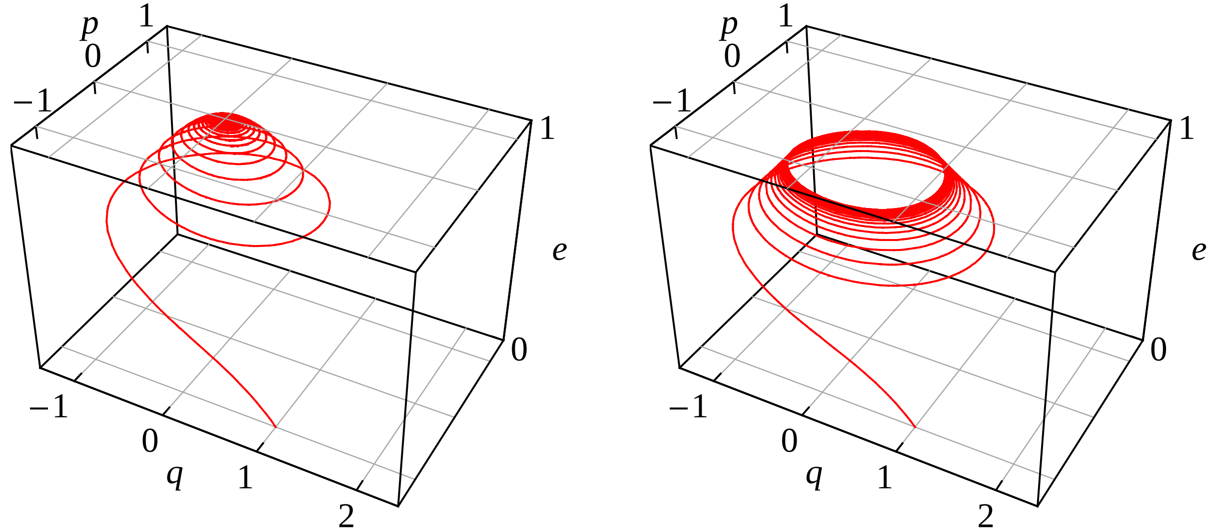
  $\lambda_1 > 0$: (e1) stable, (e2) does not exist
  $\lambda_1 < 0$: (e1) unstable, (e2) stable



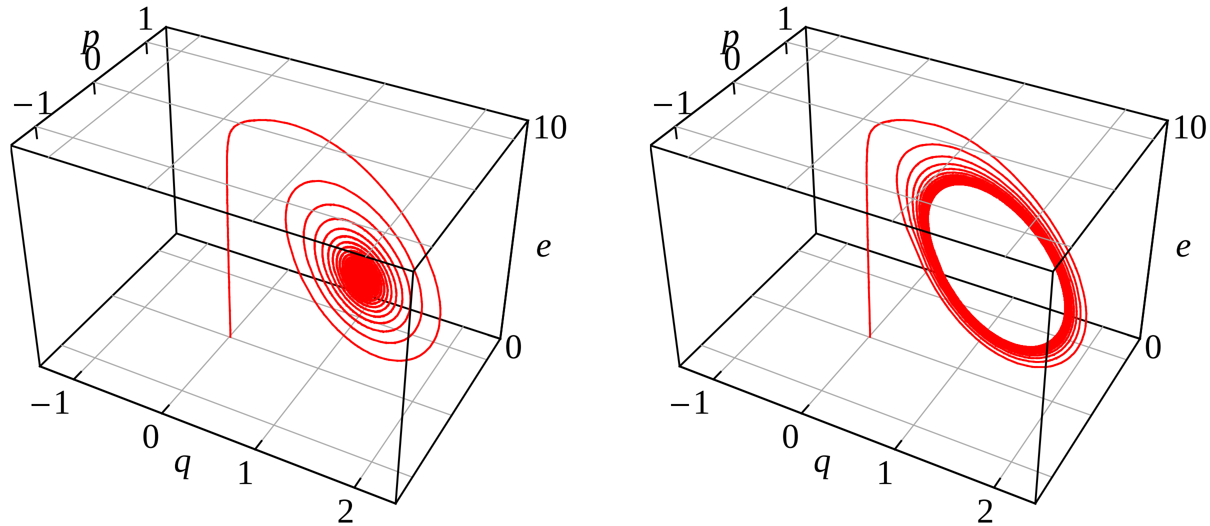
- Hopf bifurcations (pair of eigenvalues $= \pm i\omega$):

  equilibrium (e1): formation of limit cycles at $\lambda_2 = 0$,
  equilibrium (e2): limit cycles at $\lambda_3 c_1 d_1 (\lambda_3 + c_1 d_1) + 2 c_2 \lambda_1 = 0$

Hopf bifurcation of stable focus (e1) at $\lambda_2 = 0$ into a limit cycle:



Hopf bifurcation of stable focus (e2) into a limit cycle:



Arbitrary dimensions

New variables:

$$S = q\,p, \qquad U = q^2/2, \qquad K = p^2/2, \qquad e$$

Equations of motion:

$$\dot{S} = -g(e)S - 2f(e)U + 2K$$

$$\dot{U} = S$$

$$\dot{K} = -f(e)S - 2g(e)K$$

$$\dot{e} = c_1 - c_2 e - c_3 e K - c_4 e U$$

Arbitrary dimensions: equilibria

- *Equilibrium (E1):* corresponds to (e1) in 1D,

$$S_0 = U_0 = K_0 = 0, \quad e_0 = \frac{c_1}{c_2}$$

- *Equilibrium (E2):* corresponds to (e2) in 1D,

$$S_0 = 0, \quad U_0 = -\frac{c_2}{c_4}\lambda_1, \quad K_0 = 0, \quad e_0 = \frac{1}{d_1}$$

- *Equilibrium (E3):* possible if $D > 1$,

$$S_0 = 0, \quad U_0 = \frac{c_2 d_2 \lambda_2/\gamma_0}{c_3 d_1 \lambda_3 - c_4 d_2}, \quad K_0 = -\frac{d_1}{d_2}\lambda_3 U_0, \quad e_0 = \frac{\gamma_0}{d_2}$$

existence: $\lambda_{2,3} < 0$, stability: no results in simple form

Arbitrary dimensions: bifurcations

- Collision of (E3) with (E1) at $\lambda_2 = 0$:

  $\lambda_2 > 0$: (E1) stable, (E3) doesn't exist
  $\lambda_2 < 0$: (E1) unstable, (E3) stable

- Collision of (E3) with (E2) at $\lambda_3 = 0$:

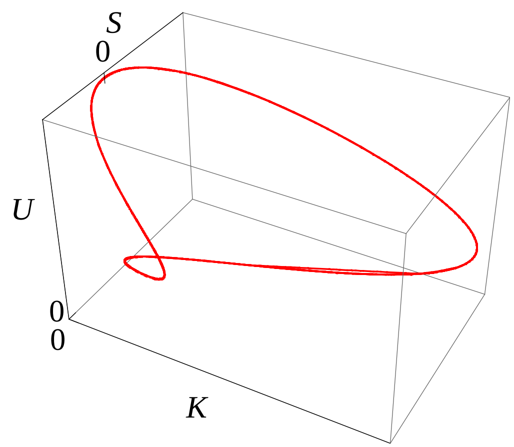
  $\lambda_3 > 0$: (E2) stable, (E3) doesn't exist
  $\lambda_3 < 0$: (E2) unstable, (E3) stable

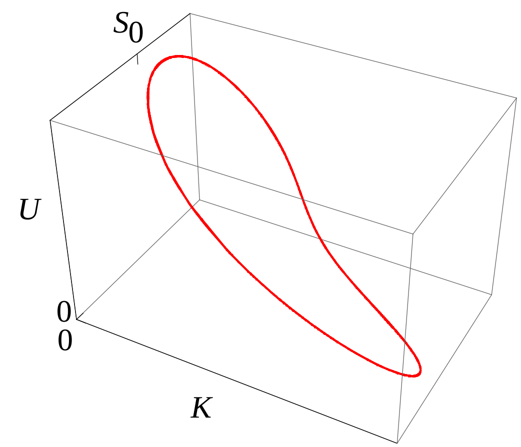
(E1) and (E2) have one eigenvalue in addition relative to (e1) and (e2) which vanishes at the above collisions ... (fold–Hopf bifurcation)

Arbitrary dimensions: limit cycles

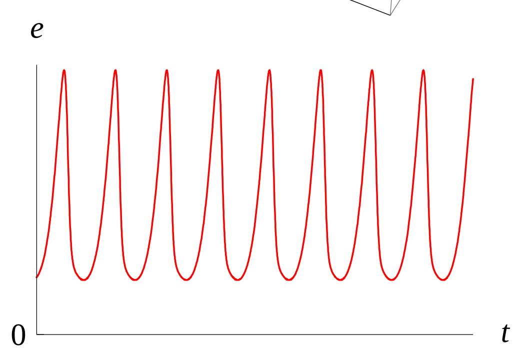
Examples of limit cycles following the loss of stability of equilibrium (E3):

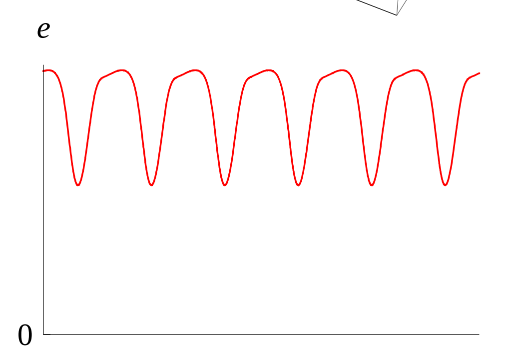
Limit cycle in (S, U, K, e) coordinates, internal energy not shown:





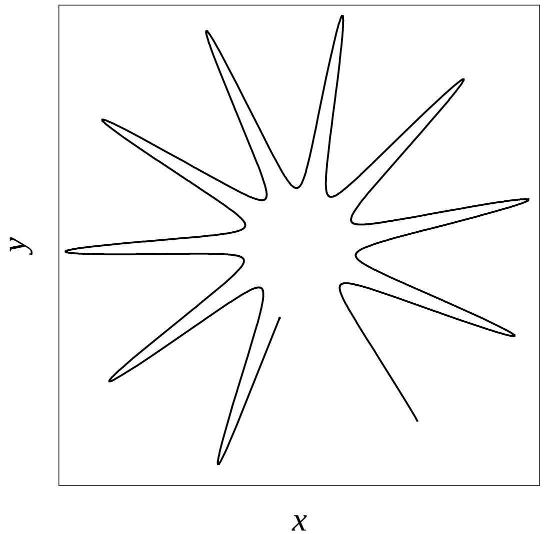
Oscillation of internal energy:

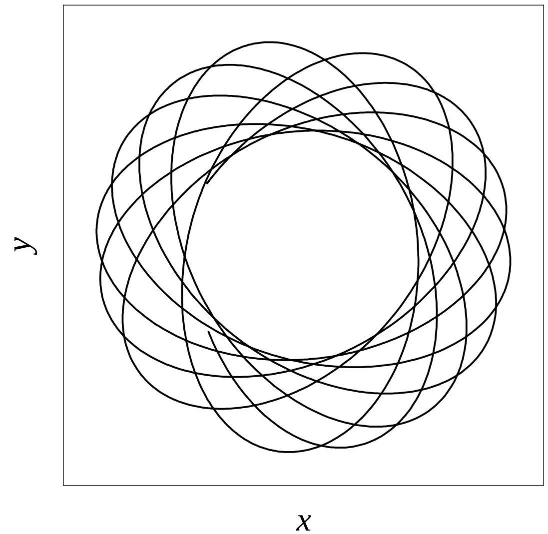




Limit cycles in 2D

Limit cycles as shown on preceding slide can also be viewed in position space:





Conclusion

- Original formulation of active Brownian motion can be generalized by coupling of the internal energy to the Hamiltonian
- Rich dynamical structure showing limit cycles and bifurcations
- $e = const$ assumption is generally not justified

Outlook

- Complex systems: bifurcation analysis for *swarms* of active particles
- Stochastic quantization: Is there an effect of limit cycles on the Higgs mechanism?

References

- AG & HH, *"Nonlinear Brownian motion and Higgs mechanism"*, Phys. Lett. B 659 (2008) 447
- AG, HH & SI, *"Canonical active Brownian motion"*, arXiv:0809.5011