Annals of Physics 324 (2009) 2179–2195

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Renormalization of Hamiltonian QCD

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ARTICLE INFO

Article history:
Received 27 March 2009
Accepted 24 April 2009
Available online 3 May 2009

PACS:
11.15.Bt
11.10.Gh

Keywords:
Coulomb gauge
Hamiltonian
Renormalization

ABSTRACT

We study to one-loop order the renormalization of QCD in the Coulomb gauge using the Hamiltonian formalism. Divergences occur which might require counter-terms outside the Hamiltonian formalism, but they can be cancelled by a redefinition of the Yang–Mills electric field.

1. Introduction

We study the renormalization of QCD in the Coulomb gauge Hamiltonian formalism. By Hamiltonian form, we mean that the Lagrangian contains only first order terms in time derivatives, and depends upon the conjugate momentum field E_i^a as well as the (transverse) gluon field A_i^a (here a is the colour index and $i = 1, 2, 3$ is a 3-vector index). This form has a number of attractive features:

(i) As a Hamiltonian exists, the theory is explicitly unitary, without the necessity to cancel unphysical degrees of freedom with ghosts.
(ii) The Lagrangian form of the Coulomb gauge has "energy divergences" in some of its Feynman integrals, that is integrals of the form (we use K for the spatial part of the 4-vector k)

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0003-4916/$ - see front matter
doi:10.1016/j.aop.2009.04.008

$$\int d^3K dk_0 f(K, k_0) \tag{1}$$

where f does not decrease as $k_0 \to \infty$ (for fixed K). These divergences cancel between different Feynman graphs [1], but this cancellation has to be organized "by hand". In the Hamiltonian form, each individual Feynman graph is free of such divergence. Formally 'energy divergent' integrals such as

$$\int \frac{d^3P}{(2\pi)^3} \int \frac{dp_0}{(2\pi)} \frac{p_0}{p_0^2 - P^2 + i\eta} \times \frac{1}{(P-K)^2} \tag{2}$$

are assigned the value zero.

(iii) It has been argued [2] that the Coulomb gauge throws light on confinement. Certainly it is known [3] that, in the Coulomb gauge, the source of asymptotic freedom lies in the Coulomb potential.

In spite of (i) above, to 2-loop order, mild energy-divergences remain [4–6] which result in ambiguities which have to be resolved by a prescription. This is connected with questions of operator ordering [7].

For other applications of the Coulomb gauge, for example to lattice QCD, see [8,9].

The question addressed here is the following. Ultra-violet divergences exist which seem to require the existence of counter-terms containing second order terms in time derivatives, $(\partial A_i^a/\partial t)^2$. Do these take us out of the Hamiltonian form? We argue that this does not happen because the divergences concerned can be cancelled by a redefinition of the E_m^a field.

We do not use quite the strict Hamiltonian formalism. We retain the auxiliary field A_0^a, which contains no time derivatives and should be integrated out to give a nonlocal Coulomb potential term in the real Hamiltonian. It seems to be convenient, for the purposes of renormalization, to retain A_0^a in the Lagrangian. Because of this, there is a ghost field, but it has an instantaneous propagator, and so is not relevant to unitarity. Its purpose is only to cancel out closed loops in the A_0^a field.

2. The Feynman rules

The Lagrangian for the Coulomb gauge is

$$\mathcal{L}' = \mathcal{L} - \frac{1}{2\alpha}(\partial_i A_i^a)^2 \tag{3}$$

(where α will eventually tend to zero to go to the Coulomb gauge),

$$\mathcal{L} = -\frac{1}{4}\mathbf{F}_{ij}\cdot\mathbf{F}_{ij} - \frac{1}{2}(\mathbf{E}_i)^2 + \mathbf{E}_i\cdot\mathbf{F}_{0i} + \partial_i\mathbf{c}^*\partial_i\mathbf{c} + g\partial_i\mathbf{c}^*\cdot(\mathbf{A}_i\wedge\mathbf{c}) + \mathbf{u}_i\cdot[\partial_i\mathbf{c} + g(\mathbf{A}_i\wedge\mathbf{c})]$$
$$+ \mathbf{u}_0\cdot[\partial_0\mathbf{c} + g(\mathbf{A}_0\wedge\mathbf{c})] - \frac{1}{2}g\mathbf{K}\cdot(\mathbf{c}\wedge\mathbf{c}) + g\mathbf{v}_i\cdot(\mathbf{E}_i\wedge\mathbf{c})$$

where we use a colour vector notation, and

$$F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + gf^{abc}A_i^b A_j^c$$

and

$$(\mathbf{A_i}\wedge\mathbf{c})^a = f^{abd}A_i^b c^d \tag{5}$$

Here $\mathbf{c}, \mathbf{c}^*$ are the ghost fields, and the sources $\mathbf{u}_i, \mathbf{v}_n$ and $\mathbf{K}$ are inserted for future use in formulating the BRST identities. The conjugate momentum (electric) field $\mathbf{E}_m$ could be integrated out to obtain the ordinary Lagrangian formalism, but for the Hamiltonian formalism it must be retained.

We will use indices $m, n, \ldots = 1, 2, 3$ to denote the (spatial) components of $\mathbf{E}$, so the seven fields are (A_i^a, A_0^a, E_n^a). We will use indices $I, J, ..$ to denote the seven indices $(i, 0, n)$. The bilinear part of the Lagrangian in momentum space is a 7×7 matrix

$$S_{IJ}\delta_{ab} = \begin{pmatrix} -K^2(T_{ij} + L_{ij}/\alpha) & 0 & -ik_0\delta_{in} \\ 0 & 0 & iK_n \\ ik_0\delta_{mj} & -iK_m & -\delta_{mn} \end{pmatrix}$$

where

$$T_{ij} \equiv \delta_{ij} - L_{ij}, \quad L_{ij} \equiv K_iK_j/K^2, \quad k^2 = k_0^2 - K^2. \tag{6}$$

For the propagators, we need the inverse

$$S_{IJ}^{-1}\delta_{ab} = \begin{pmatrix} T_{ij}/k^2 - \alpha L_{ij}/K^2 & \alpha k_0K_i/(K^2)^2 & -ik_0T_{in}/k^2 \\ \alpha k_0K_j/(K^2)^2 & 1/K^2 + \alpha k_0^2/(K^2)^2 & iK_n/K^2 \\ ik_0T_{mj}/k^2 & -iK_m/K^2 & T_{mn}K^2/k^2 \end{pmatrix}. \tag{7}$$

We can now let $\alpha \to 0$, to obtain the Coulomb gauge. From this, and the interaction terms in the Lagrangian (4), we can read off the Feynman rules. We represent the $\mathbf{A}_i$ field by dashed lines, the $\mathbf{E}_n$ field by continuous lines, and the $\mathbf{A}_0$ field by dotted lines. With this notation, we now list the rules (a factor of $\frac{1}{(2\pi)^4 i}$ is to be included for each propagator, and a factor of $(2\pi)^4 i$ for each vertex). If we choose the propagators in Fig. 1 to be the negative of the matrix (7), the extra factors of $\frac{1}{(2\pi)^4 i}$ for the propagator and $(2\pi)^4 i$ for the vertices cancel (see Figs. 1–3).

3. The ultra-violet divergences

The divergent graphs with 2 and with 3 external lines are shown in Figs. 4–31. Examples of the method of evaluation of divergent parts are given in Appendices A and B.

The ultra-violet divergent parts of these graphs are, in terms of the divergent constant (using dimensional regularization in $4 - \epsilon$ dimensions)

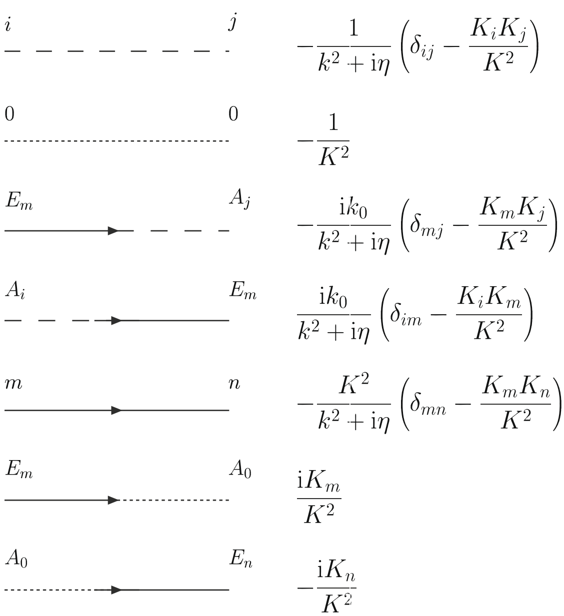

Fig. 1. Feynman rules for the propagators in the Coulomb gauge.

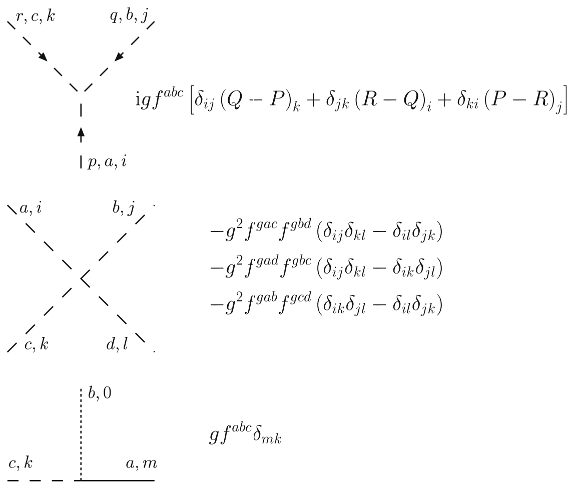

Fig. 2. Feynman rules for the vertices in the Coulomb gauge. The arrows denote the directions of the momenta.

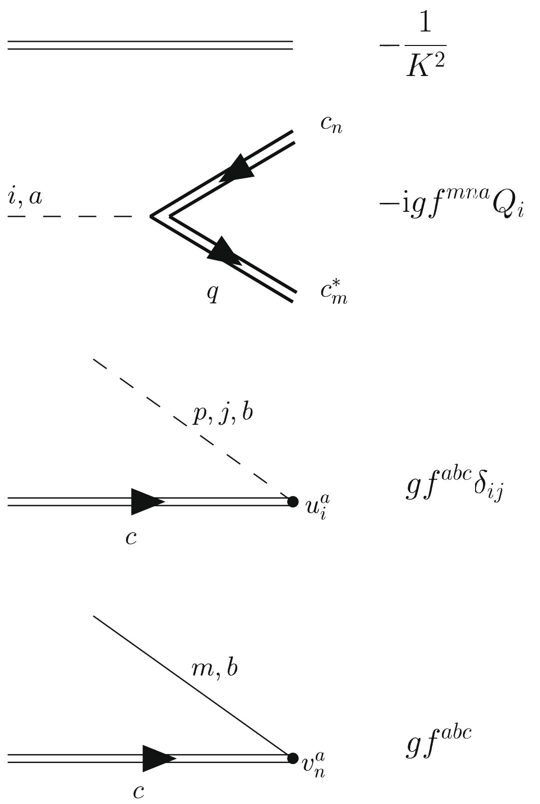

Fig. 3. Feynman rules for ghosts and sources in the Coulomb gauge. Doubled lines denote ghosts. The black arrows distinguish between ghosts and anti-ghosts. Momenta flow into the vertex.

$$c = \frac{g^2}{16\pi^2} C_G \Gamma(\epsilon/2), \tag{8}$$

(where the superfix (4) and (5) etc. refers to the corresponding figure and $\Pi_{ij}, \Pi_{0i} \ldots \Pi_{mn}$ denote self-energies, $V_{ijk}, V_{0in} \ldots V_{0in}$ vertices and Λ stands for diagrams with external ghost lines), are:

$$\Pi^{(4)ab}_{ij} = ic\left[\frac{1}{3}k_0^2\delta_{ij} + K^2\delta_{ij} - K_iK_j\right]\delta_{ab} \tag{9}$$

$$\Pi^{(5)ab}_{i0} = -\frac{1}{3}ick_0K_i\delta_{ab} \tag{10}$$

$$\Pi^{(6)ab}_{00} = \frac{1}{3}icK^2\delta_{ab} \tag{11}$$

$$\Pi^{(7)ab}_{mi} = 0 \tag{12}$$

$$\Pi^{(8)ab}_{m0} = -\frac{4}{3}ic[iK_i\delta_{ab}] \tag{13}$$

$$\Pi^{(9)ab}_{mn} = -\frac{4}{3}ic\delta_{mn}\delta_{ab} \tag{14}$$

$$V^{(10)abc}_{ijk}(p,q,r) = -\frac{1}{3}cgf^{abc}[(Q-P)_k\delta_{ij} + (R-Q)_i\delta_{jk} + (P-R)_j\delta_{ik}] \tag{15}$$

$$V^{(11)abc}_{ijk}(p,q,r) = -\frac{5}{6}cgf^{abc}[(Q-P)_k\delta_{ij} + (R-Q)_i\delta_{jk} + (P-R)_j\delta_{ik}] \tag{16}$$

$$V^{(12)abc}_{ijk}(p,q,r) = -\frac{2}{3}cgf^{abc}[(Q-P)_k\delta_{ij} + (R-Q)_i\delta_{jk} + (P-R)_j\delta_{ik}] \tag{17}$$

$$V^{(13)abc}_{ijk}(p,q,r) = \frac{3}{2}cgf^{abc}[(Q-P)_k\delta_{ij} + (R-Q)_i\delta_{jk} + (P-R)_j\delta_{ik}] \tag{18}$$

$$V^{(14)abc}_{i00}(p,q,r) = \frac{1}{4}cgf^{abc}(R-Q)_i \tag{19}$$

$$V^{(15)abc}_{i00}(p,q,r) = -\frac{1}{3}cgf^{abc}(R-Q)_i \tag{20}$$

$$V^{(16)abc}_{i00}(p,q,r) = \frac{1}{3}cgf^{abc}(R-Q)_i \tag{21}$$

$$V^{(17)abc}_{i00}(p,q,r) = \frac{1}{12}cgf^{abc}(R-Q)_i \tag{22}$$

$$V^{(18)abc}_{0jl}(p,q,r) = 0 \tag{23}$$

$$V^{(19)abc}_{0jl}(p,q,r) = \frac{2}{3}cgf^{abc}(R-Q)_0 \tag{24}$$

$$V^{(20)abc}_{0jl}(p,q,r) = -\frac{1}{3}cgf^{abc}(R-Q)_0 \tag{25}$$

$$V^{(21)abc}_{0jl}(p,q,r) = 0. \tag{26}$$

Graphs involving external $\mathbf{E}_m$ line are

$$V^{(29)abc}_{im0}(p,q,r) = \frac{1}{3}icgf^{abc}\delta_{im} \tag{27}$$

$$V^{(30)abc}_{im0}(p,q,r) = -\frac{1}{3}icgf^{abc}\delta_{im} \tag{28}$$

$$V^{(31)abc}_{im0}(p,q,r) = 0. \tag{29}$$

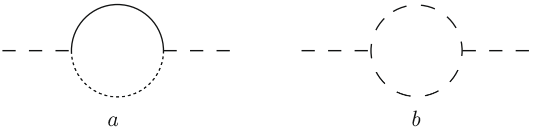

Fig. 4. The transverse gluon self-energy graphs.

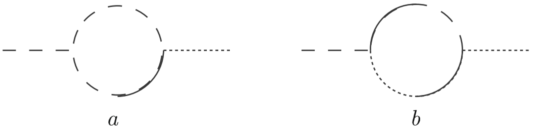

Fig. 5. The $\mathbf{A}_i\mathbf{A}_0$ two-point function.

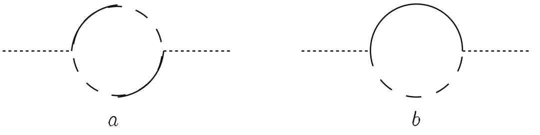

Fig. 6. The time-time component of the gluon self-energy.

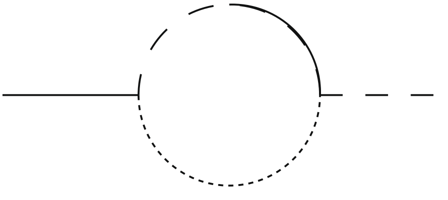

Fig. 7. The transition between the transverse gluon field and its conjugate field $\mathbf{E}_i$.

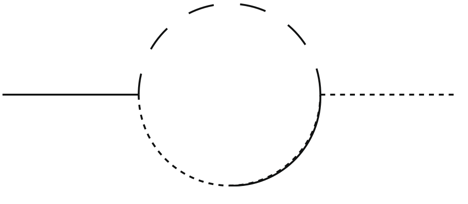

Fig. 8. The transition between the Coulomb field $\mathbf{A}_0$ and the conjugate field $\mathbf{E}_i$.

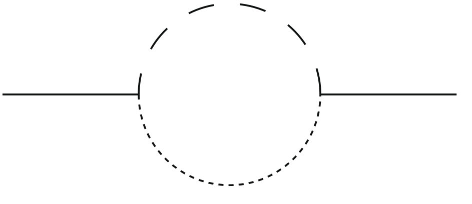

Fig. 9. The conjugate field self-energy.

All other graphs involving external $\mathbf{E}_m$-lines are convergent. The divergent parts of graphs with open ghost line are

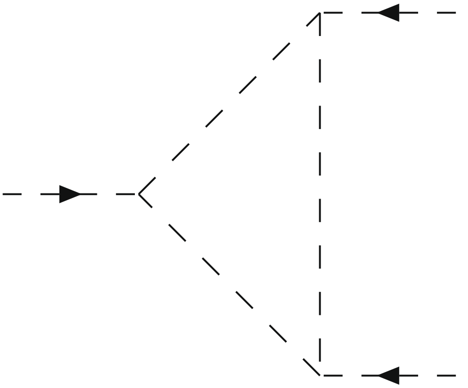

Fig. 10. Graph contributing to the three-gluon vertex function.

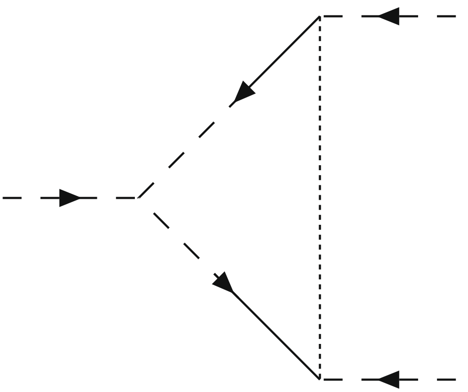

Fig. 11. There are three graphs in this class with permutations of the vertices.

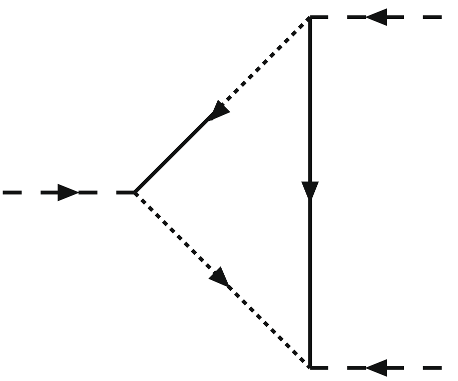

Fig. 12. Graph representing a class of six diagrams.

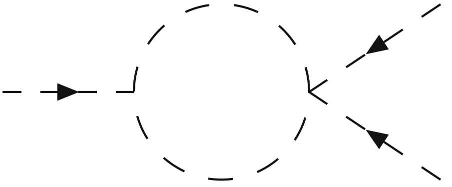

Fig. 13. There are three graphs in this class of diagrams.

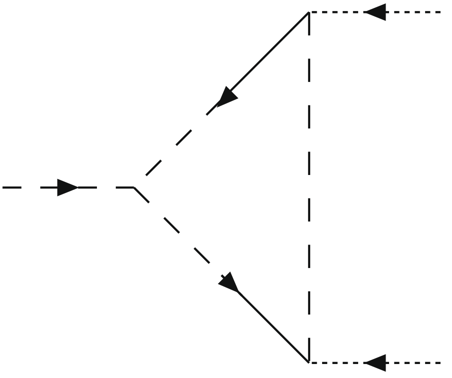

Fig. 14. Graph with two external Coulomb lines (there are three diagrams in this class).

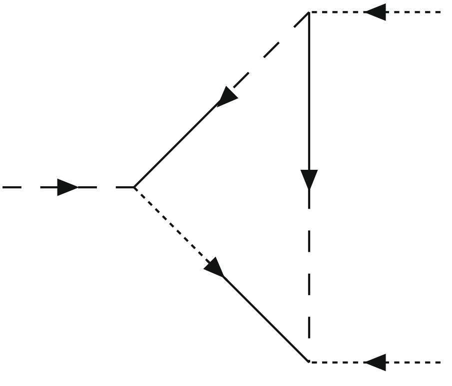

Fig. 15. There are two graphs in this class.

$$\Lambda^{(22)ab}(q) = -\frac{4}{3} icQ^2 \delta_{ab} \tag{30}$$

$$\Lambda_i^{(23)ab}(q) = -\frac{4}{3} cQ_i \delta_{ab} \tag{31}$$

$$\Lambda^{(24)abc}(p,q) = 0 \tag{32}$$

$$\Lambda_k^{(25)abc}(p,q,r) = 0 \tag{33}$$

$$\Lambda_0^{(26)abc}(p,q) = 0 \tag{34}$$

$$\Lambda_i^{(27)abc}(p,q) = 0 \tag{35}$$

$$\Lambda_n^{(28)abc}(p,q) = 0. \tag{36}$$

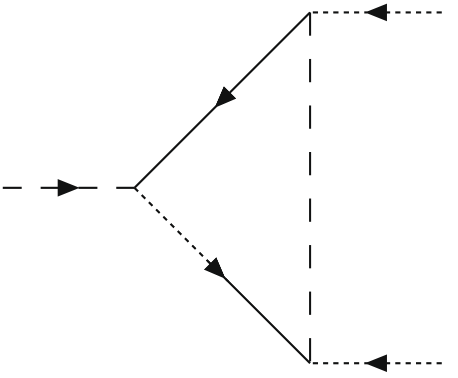

Fig. 16. There are two graphs in this class.

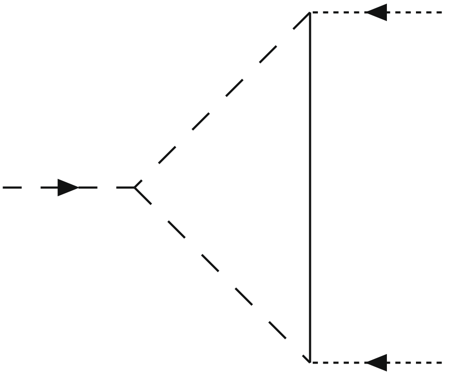

Fig. 17. The graph with two external Coulomb lines and one three-gluon vertex.

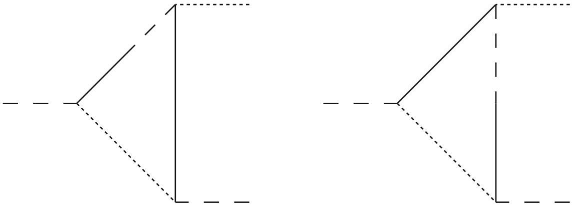

Fig. 18. Graphs contributing to the $(\mathbf{A}_i\mathbf{A}_j\mathbf{A}_0)$ three-point function.

4. Counter-terms

Let

$$\Gamma_0 = \int d^4x\mathcal{L}(x) \tag{37}$$

be the original action, Γ be the complete effective action, and let Γ_1 be the effective action to one-loop order. The complete BRST identities are

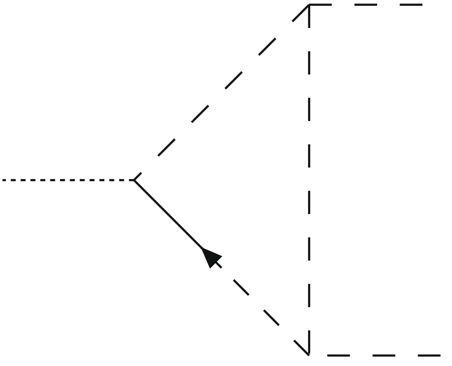

Fig. 19. Graph contributing to the $(\mathbf{A}_i\mathbf{A}_j\mathbf{A}_0)$ three-point function which contains a three-gluon vertex.

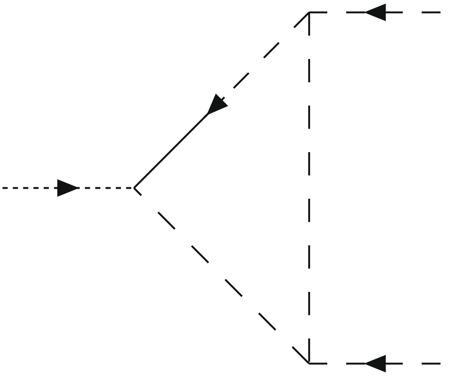

Fig. 20. The $(\mathbf{A}_i\mathbf{A}_j\mathbf{A}_0)$ graph with a three-gluon vertex.

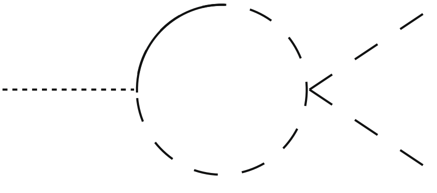

Fig. 21. The $(\mathbf{A}_i\mathbf{A}_j\mathbf{A}_0)$ graph with a four-gluon vertex.

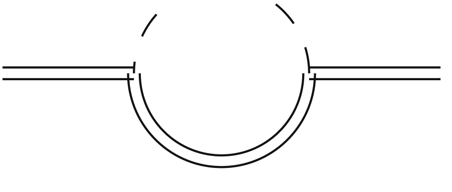

Fig. 22. The ghost self-energy.

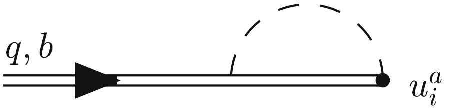

Fig. 23. Ghost and the $\mathbf{u}_i$ source graph.

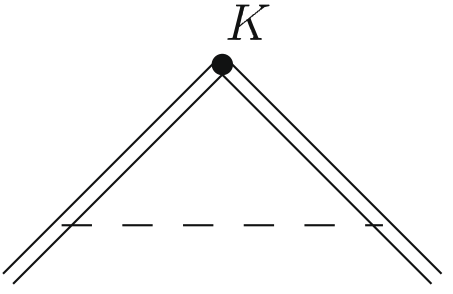

Fig. 24. The ghost vertex graph with a **K** source.

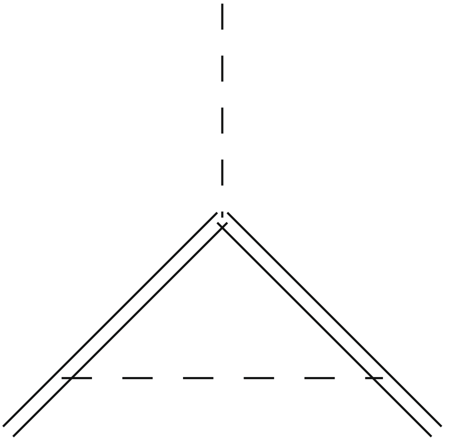

Fig. 25. Graph with external $\mathbf{A}_i$, ghost and anti-ghost lines.

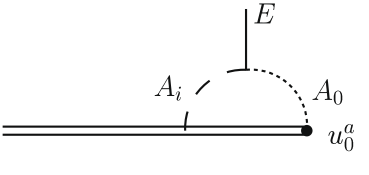

Fig. 26. Graph with $\mathbf{u}_0$ source, $\mathbf{E}_i$ and **c** lines.

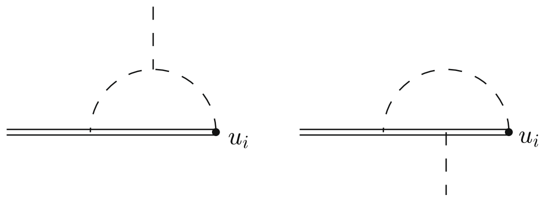

Fig. 27. Graph with $\mathbf{u}_i$ source, $\mathbf{A}_i$ and $\mathbf{c}$ lines.

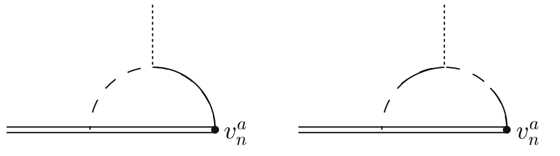

Fig. 28. Diagram with $\mathbf{v}_n$ source, $\mathbf{A}_0$ and $\mathbf{c}$ lines.

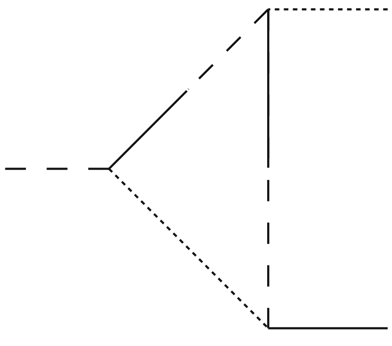

Fig. 29. Graph contributing to the $(\mathbf{A}_i\mathbf{E}_j\mathbf{A}_0)$ vertex function.

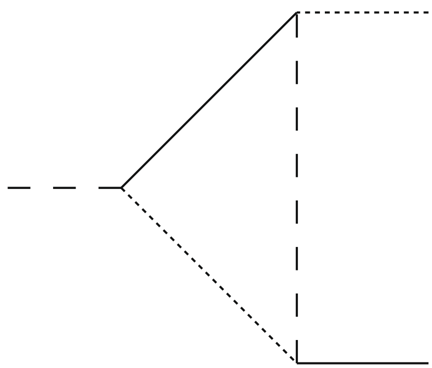

Fig. 30. Graph with external gluon, Coulomb and **E**-field.

$$\Gamma * \Gamma \equiv \frac{\partial \Gamma}{\partial \mathbf{A}_i} \cdot \frac{\partial \Gamma}{\partial \mathbf{u}_i} + \frac{\partial \Gamma}{\partial \mathbf{A}_0} \cdot \frac{\partial \Gamma}{\partial \mathbf{u}_0} + \frac{\partial \Gamma}{\partial \mathbf{c}} \cdot \frac{\partial \Gamma}{\partial \mathbf{K}} + \frac{\partial \Gamma}{\partial \mathbf{E}_i} \cdot \frac{\partial \Gamma}{\partial \mathbf{v}_i} = 0. \tag{38}$$

So to one-loop order

$$\Gamma_1 * \Gamma_0 + \Gamma_0 * \Gamma_1 \equiv \Delta\Gamma_1 = 0 \tag{39}$$

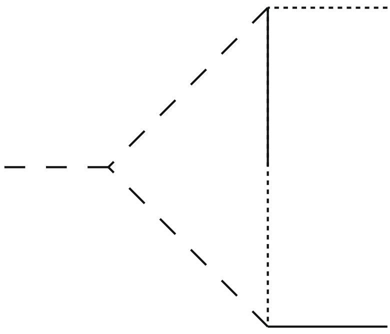

Fig. 31. Graph in the $(\mathbf{A}_i\mathbf{E}_j\mathbf{A}_0)$ vertex function.

where

$$\Delta = \frac{\partial\Gamma}{\partial\mathbf{A}_i}\cdot\frac{\partial}{\partial\mathbf{u}_i}+\frac{\partial\Gamma}{\partial\mathbf{u}_i}\cdot\frac{\partial}{\partial\mathbf{A}_i}+\frac{\partial\Gamma}{\partial\mathbf{A}_0}\cdot\frac{\partial}{\partial\mathbf{u}_0}+\frac{\partial\Gamma}{\partial\mathbf{u}_0}\cdot\frac{\partial}{\partial\mathbf{A}_0}+\frac{\partial\Gamma}{\partial\mathbf{c}}\cdot\frac{\partial}{\partial\mathbf{K}}+\frac{\partial\Gamma}{\partial\mathbf{K}}\cdot\frac{\partial}{\partial\mathbf{c}}+\frac{\partial\Gamma}{\partial\mathbf{E}_i}\cdot\frac{\partial}{\partial\mathbf{v}_i}+\frac{\partial\Gamma}{\partial\mathbf{v}_i}\cdot\frac{\partial}{\partial\mathbf{E}_i} \tag{40}$$

and

$$\Delta^2 = 0. \tag{41}$$

One class of solutions to this equation is of the form

$$\Gamma_1^{(i)} = \Delta G, \tag{42}$$

where the allowed form of G is, in terms of constants $a_5,\dots a_{11}$,

$$G = a_5\mathbf{A}_i\cdot(\mathbf{u}_i+\partial_i\mathbf{c}^*)+a_6\mathbf{A}_0\cdot\mathbf{u}_0+a_7\mathbf{c}\cdot\mathbf{K}+a_8\mathbf{E}_i\cdot\mathbf{v}_i+a_9\mathbf{v}_i\cdot\partial_i\mathbf{A}_0+a_{10}\mathbf{v}_i\cdot\partial_0\mathbf{A}_i+a_{11}\mathbf{v}_i\cdot(\mathbf{A}_0 \wedge\mathbf{A}_i). \tag{43}$$

Other solutions of Eq. (39) are the explicitly gauge-invariant terms

$$\Gamma_1^{(ii)} = a_1(\mathbf{F}_{ij})^2+a_2\mathbf{E}_i\cdot\mathbf{F}_{0i}+a_3(\mathbf{F}_{0i})^2+a_4(\mathbf{E}_i)^2. \tag{44}$$

Finally, by differentiating (38) with respect to the coupling constant g and specialising to one-loop order, we see that

$$\Delta\Gamma_1^{(iii)} = 0 \tag{45}$$

where (a_0 being another divergent constant)

$$\Gamma_1^{(iii)} = a_0 g\partial\Gamma_0\frac{\partial}{g}. \tag{46}$$

Combining these three contributions, we obtain

$$\Gamma_1 = \Gamma_1^{(i)}+\Gamma_1^{(ii)}+\Gamma_1^{(iii)} = \int d^4x\mathcal{L}_1(x) \tag{47}$$

where

$$\begin{aligned}\mathcal{L}_1 = {} & a_1(\mathbf{F}_{ij})^2+(a_2+a_8+a_9)\mathbf{E}_i\cdot\mathbf{F}_{0i}+(a_3-a_9)(\mathbf{F}_{0i})^2+(a_4-a_8)(\mathbf{E}_i)^2+a_5\mathbf{F}_{ij}\cdot\partial_j\mathbf{A}_i-(a_5\\ & +\frac{1}{2}a_0)g\mathbf{F}_{ij}\cdot(\mathbf{A}_i\wedge\mathbf{A}_j)-(a_0+a_5+a_6)g\mathbf{E}_i\cdot(\mathbf{A}_i\wedge\mathbf{A}_0)+\mathbf{E}_i\cdot(a_5\partial_0\mathbf{A}_i-a_6\partial_i\mathbf{A}_0)-a_5(\mathbf{u}_i\\ & +\partial_i\mathbf{c}^*)\cdot\partial_i\mathbf{c}+a_0g\partial_i\mathbf{c}^*\cdot(\mathbf{A}_i\wedge\mathbf{c})-a_6\mathbf{u}_0\cdot\partial_0\mathbf{c}+a_0g\mathbf{u}_0\cdot(\mathbf{A}_0\wedge\mathbf{c})-a_7(\mathbf{u}_i+\partial_i\mathbf{c}^*)\cdot\{\partial_i\mathbf{c}\\ & +g(\mathbf{A}_i\wedge\mathbf{c})\}+a_0g\mathbf{u}_i\cdot(\mathbf{A}_i\wedge\mathbf{c})-a_7\mathbf{u}_0\cdot\{\partial_0\mathbf{c}+g(\mathbf{A}_0\wedge\mathbf{c})\}+\frac{1}{2}g(a_7-a_0)\mathbf{K}\cdot(\mathbf{c}\wedge\mathbf{c})\\ & +(a_0-a_7)g\mathbf{v}_i\cdot(\mathbf{E}_i\wedge\mathbf{c}).\end{aligned} \tag{48}$$

The conditions coming from the vanishing ghost graphs Figs. 24–28 are particularly simple. They fix

$$\begin{aligned} &a_9 = -a_{10} \\ &a_{11} = -g a_9 \\ &a_0 = a_7 = -a_6. \end{aligned} \tag{49}$$

In order for the counter-terms to cancel the divergences in the other graphs, we require the conditions

$$\begin{aligned} &4a_1 - 2a_5 = -c \\ &4a_1 - 3a_5 - a_0 = \frac{1}{3}c \\ &a_3 - a_9 = -\frac{1}{6}c \\ &a_6 - a_5 = \frac{4}{3}c \\ &a_5 + a_7 = -\frac{4}{3}c \\ &a_4 - a_8 = \frac{2}{3}c \\ &a_2 + a_5 + a_8 + a_9 = 0. \end{aligned} \tag{50}$$

These equations do not fix the constants uniquely. We are free to make some choices. The term $(\mathbf{F}_{0i})^2$ in $\Gamma_1^{(ii)}$ Eq. (44) is not present in the original Hamiltonian form of the Lagrangian (4), so we choose

$$a_3 = 0. \tag{51}$$

We can also arrange for the combination

$$-\frac{1}{2}(\mathbf{E}_i)^2 + \mathbf{E}_i \cdot \mathbf{F}_{0i} \tag{52}$$

to appear in $\mathcal{L}_1^{(ii)}$ as it does in $\mathcal{L}_0$. This requires (from (50))

$$\begin{aligned} &a_1 = -\frac{1}{4}c + \frac{1}{2}a_5 \\ &a_2 = c - 2a_5 \\ &a_4 = -\frac{1}{2}c + a_5 \\ &a_6 = \frac{4}{3}c + a_5 \\ &a_7 = -\frac{4}{3}c - a_5 \\ &a_8 = -\frac{7}{6}c + a_5 \\ &a_9 = \frac{1}{6}c \\ &a_0 = -\frac{4}{3}c - a_5 \end{aligned} \tag{53}$$

and so

$$\mathcal{L}_1^{(ii)} = -4a_1\left[-\frac{1}{4}(\mathbf{F}_{ij})^2 - \frac{1}{2}(\mathbf{E}_i)^2 + \mathbf{E}_i \cdot \mathbf{F}_{0i}\right] \tag{54}$$

proportional to the non-ghost part of the original Lagrangian (3).

Eq. (54) does not come from the BRST identities, it just emerges from the numerical values of the divergent integrals. It may be a consequence of some hidden Lorentz invariance.

The constants $a_0, a_1, \ldots$ are still not uniquely fixed. There are two particularly simple choices.

(i) Choose $a_0 = 0$ with $a_5 = -\frac{4}{3}c$. Then we find

$$\begin{aligned}
a_1 &= -\frac{11}{12}c \\
a_2 &= \frac{11}{3}c \\
a_4 &= -\frac{11}{6}c \\
a_6 &= a_7 = 0 \\
a_8 &= -\frac{5}{2}c \\
a_9 &= \frac{1}{6}c.
\end{aligned} \tag{55}$$

(ii) The second choice is $a_1 = 0$ with $a_5 = \frac{1}{2}c$. Then

$$\begin{aligned}
a_0 &= -\frac{11}{6}c \\
a_2 &= 0 \\
a_4 &= 0 \\
a_6 &= \frac{11}{6}c \\
a_7 &= -\frac{11}{6}c \\
a_8 &= -\frac{2}{3}c \\
a_9 &= \frac{1}{6}c.
\end{aligned} \tag{56}$$

Note that a_0 has the expected value for coupling constant renormalization.
The counter-terms in either case are

$$\begin{aligned}
\mathcal{L}_1 = &-\frac{11}{12}c(\mathbf{F}_{ij})^2 - \frac{4}{3}c\mathbf{F}_{ij}\cdot\partial_j\mathbf{A}_i + \frac{4}{3}cg\mathbf{F}_{ij}\cdot(\mathbf{A}_i \wedge \mathbf{A}_j) - \frac{1}{6}c(\mathbf{F}_{0i})^2 + \frac{2}{3}c(\mathbf{E}_i)^2 - \frac{4}{3}cgE_i\cdot\partial_i A_0 \\
&+ \frac{4}{3}c(\mathbf{u}_i + \partial_i\mathbf{c}^*)\cdot\partial_i\mathbf{c}.
\end{aligned} \tag{57}$$

The counter-terms in a_5, a_6, a_7, a_8 and a_9 are involved in a rescaling of the fields. Defining

$$\begin{aligned}
\mathbf{A}'_i &= (1 + a_5)\mathbf{A}_i \\
\mathbf{A}'_0 &= (1 + a_6)\mathbf{A}_0 \\
\mathbf{E}'_m &= (1 + a_8)\mathbf{E}_m - a_9\mathbf{F}_{0m} \\
\mathbf{u}'_i &= (1 - a_5)\mathbf{u}_i \\
\mathbf{u}'_0 &= (1 - a_6)\mathbf{u}_0 \\
\mathbf{c}' &= (1 - a_7)\mathbf{c} \\
\mathbf{K}' &= (1 + a_7)\mathbf{K} \\
g' &= (1 + a_0)g \\
\mathbf{c}'^* &= (1 - a_5)\mathbf{c}^* \\
\mathbf{v}' &= (1 - a_8)\mathbf{v},
\end{aligned} \tag{58}$$

we have from (48) that

$$\mathcal{L}_0 + \mathcal{L}_1 = (1 - 4a_1)\mathcal{L}_0(g', \mathbf{A}'_i, \mathbf{A}'_0, \mathbf{E}', \mathbf{c}', \mathbf{c}'^*, \mathbf{u}'_i, \mathbf{u}'_0, \mathbf{K}'). \tag{59}$$

Note that a_6 which determines the renormalization of the Coulomb field A_0^a has the same numerical value as a_0.

We have not calculated the divergences in graphs with four external lines. We assume they will be cancelled by the same counter-terms.

5. Comments

We conclude that there is no difficulty to one-loop order in renormalizing the Hamiltonian form of the Coulomb gauge. We guess that the renormalization would formally go through to higher orders, but then there is the problem mentioned in [4–6] of combining the renormalization of ultra-violet divergences with the resolution of energy-divergence ambiguities.

It is not quite obvious how the renormalization would be formulated if the A_0^a field had been eliminated to give the non-local colour Coulomb potential (note the non-zero value of the A_0^a field renormalization constant a_6 in (56)).

Acknowledgments

A.A. thank the Royal Society for a grant and DAMTP for hospitality. We are grateful to Dr. G. Duplančić for drawing the figures. The work was supported by the Ministry of Science and Technology of the Republic of Croatia under Contract No. 098-0000000-2865.

Appendix A. Here we give as an example the evaluation of the ultra-violet divergent part of the graph in Fig. 20.

$$V_{0jk}^{(20)abc}(q,-q,0) = ig^3 C_G f^{abc} \int d^4p \frac{p_0 P_k}{(p^2+i\eta)^2} \cdot \frac{1}{(q+p)^2+i\eta} \times T_{rz}(P)T_{zv}(P)T_{ru}(Q+P)$$
$$\times [(-2Q-P)_v \delta_{uj} + (Q-P)_u \delta_{jv} + (2P+Q)_j \delta_{vu}]. \quad (A.1)$$

Applying the integral

$$\int dp_0 \frac{p_0}{(p^2+i\eta)^2} \frac{1}{(q+p)^2+i\eta}$$
$$= i\pi^{\frac{1}{2}} \Gamma\left(\frac{5}{2}\right) q_0 \int_0^1 dy y(1-y)\left\{(P+yQ)^2 + y(1-y)(-q^2-i\eta)\right\}^{-\frac{5}{2}} \quad (A.2)$$

and power counting to (A.1)

$$V_{0jk}^{(20)abc}(q,-q,0) = -4g^3 C_G f^{abc} \sqrt{\pi} q_0 \Gamma\left(\frac{5}{2}\right) \int_0^1 dy y(1-y)$$
$$\times \int d^{3-\epsilon} P P_j P_k \left\{(P+yQ)^2 + y(1-y)(-q^2-i\eta)\right\}^{-\frac{5}{2}}, \quad (A.3)$$

leading to

$$V_{0jk}^{(20)abc}(q,-q,0) = -\frac{1}{3} cg f^{abc} q_0 \delta_{jk}. \quad (A.4)$$

Appendix B. Example of self-energy evaluation $\Pi_{00}^{(6)ab}$ in Eq. (11). Let p, q be internal and k external momentum, $p - q = k$. The sum of two graphs is

$$(2\pi)^{-4} \frac{T_{ij}(P)T_{ji}(Q)}{p^2 q^2} \left[\frac{1}{2}(P^2+Q^2) - (ip_0)(iq_0)\right] \delta_{ab} \quad (B.1)$$

where we have symmetrized the first term in P, Q. the minus sign in the second term comes from the opposite order of the f^{abc} factors at the two vertices. Doing the p_0 integration by Cauchy, we get

$$(2\pi)^{-4}(2\pi i)\frac{T_{ij}T_{ji}}{4PQ}\frac{1}{(P+Q)^2-k_0^2}(P+Q)[P^2+Q^2-2PQ]\delta_{ab}. \tag{B.2}$$

The last factor $(P-Q)^2$ is approximately $(P\cdot K)^2/P^2$. With this factor, the integral is only logarithmically divergent, and to get the divergent part we can put $Q=P$ everywhere. We use $T_{ij}(P)T_{ji}(P)=2$. Then we get

$$(2\pi)^{-4}\frac{2\pi i}{4}K_iK_j\int d^{3-\epsilon}P\frac{P_iP_j}{(P^2+m^2)^{5/2}}. \tag{B.3}$$

So the divergent part is[1]

$$\frac{1}{3}icK^2\delta_{ab}. \tag{B.4}$$

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[1] Note that there was an error of sign in Eur. Phys. J. C37 (2004) 307–313 which however did not influence the final result.