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Deformed oscillator algebras and QFT in κ-Minkowski spacetime

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In this paper, we study the deformed statistics and oscillator algebras of quantum fields defined in κ-Minkowski spacetime. The twisted flip operator obtained from the twist associated with the star product requires an enlargement of the Poincaré algebra to include the dilatation generators. Here we propose a novel notion of a fully covariant flip operator and show that to the first order in the deformation parameter it can be expressed completely in terms of the Poincaré generators alone. The R matrices corresponding to the twisted and the covariant flip operators are compared up to first order in the deformation parameter and they are shown to be different. We also construct the deformed algebra of the creation and annihilation operators that arise in the mode expansion of a scalar field in κ-Minkowski spacetime. We obtain a large class of such new deformed algebras which, for certain choice of realizations, reduce to results known in the literature.

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I. INTRODUCTION

Noncommutative geometry as well as formulation and study of physical theories on noncommutative spaces have been attracting wide attention for quite some time now [1–10]. Unique features of the theories on such spaces and also the fact that noncommutative geometry provides one of the possible approaches for describing Planck scale physics, notably, quantum gravity are some of the motivations for the renewed interest in these studies [11–13]. A simple and by now reasonably well studied model of noncommutative space is the Moyal space. One of the interesting aspects brought out by these studies is the role of Hopf algebra (quantum group) [14] in analyzing the symmetries of field theories on Moyal space. Though, in the conventional sense, the Lorentz symmetry is lost in these theories, it is now well understood that using Hopf algebra approach, Lorentz invariance can be retained in these noncommutative models, enabling the conventional interpretation of field quanta [15].

In the Hopf algebra approach, the underlying symmetry algebra of the noncommutative theory acts on multiparticle states through the twisted coproducts of the symmetry generators. Alternate attempts to construct gravity theories on noncommutative spaces, where a compatibility between the so-called $*$ product and the action of diffeomorphism symmetry generators also led to the introduction of twisted Leibniz rule (i.e., coproducts) for these generators [13].

Noncommutative spaces which are more general than Moyal space are also possible [11], κ-deformed space being one such example, where the coordinates satisfy a Lie algebra type commutation relation [16–20]. Such a κ-deformed space has emerged in the attempts to construct special theory of relativity compatible with the existence of a dimensionful constant (Planck length) apart from the velocity of light in doubly special relativity [21–23]. Apart from the studies to understand the algebraic structure and symmetries, recently, field theory models have also been investigated on such spaces [24–28].

One of the notable features of field theories on noncommutative spaces with generalized symmetry is the notion of twisted statistics [29–34]. The twisted coproduct arising from the requirement of compatibility between the algebraic structures of the noncommutative geometry and the actions of symmetry generators lead to a notion of deformed statistics. This comes about when the compatibility between the action of flip operator on multiparticle states and the twisted action of symmetry generators is demanded, leading to a twisted flip operator [31,32]. It is this twisted flip operator that gives the definition of statistics which is invariant under the action of twisted symmetry generators. Most of these discussions reported are for

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the field theory models on the Moyal plane, though recently, some studies have been initiated, investigating the issue of statistics for theories on κ-deformed spaces [35–40]. Unlike the Moyal case, the twist operator in the κ-deformed space does not belong to the universal envelope of the underlying Poincaré algebra. Rather, the presence of the dilatation generator in the expression of the twist operator indicates that it belongs to the universal envelope of the corresponding general linear algebra [41–45]. The twisted flip operator associated with such a twisted coproduct for the κ-deformed space was constructed in [38]. In this paper, we present a different proposal for a covariant flip operator for the κ-Minkowski spacetime. For this, we consider the deformed coproduct for a fully covariant realization of the κ-Minkowski algebra [6,8]. This deformed coproduct is an element of the corresponding Hopf algebra and it is different from the twisted coproduct arising from the action of the twist operator. A covariant flip operator has to be compatible with the action of this deformed coproduct. We find an expression of such a covariant flip operator to first order in the deformation parameter, which is constructed from the generators of the Poincaré algebra alone. We show that the corresponding R matrix also shares the same property up to first order in the deformation parameter.

In quantum field theory (QFT), the action of the twisted flip-operator leads to a deformed algebra of the creation and annihilation operators. For the Moyal case, this first arose in the context of integrable models [46] and the consequences of such a deformed oscillator algebra in QFT are well studied [29–34]. For the κ-Minkowski case, there exists several proposals for such a deformed oscillator algebra [37,47,48]. In this paper, we construct a class of such deformed oscillator algebras corresponding to a family of realizations of the κ-Minkowski space. For particular choice of realizations, we recover the deformed oscillator algebra obtained in [37], although our construction leads to a much wider class of such oscillator algebras.

This paper is organized in the following way. In Sec. II, we briefly review the essential details of κ-Minkowski spacetime and present a particular class of realization of the associated coordinates in terms of commuting ones and corresponding derivatives. We also present the deformed coproducts of Poincaré generators in terms of the functions characterizing the realization [6,8,9,38]. In Sec. III, we discuss the $*$ product and the twist element and obtain the explicit expressions for the particular class of realization [6]. Our main results are discussed in Sec. IV. Here, we first briefly review the twisted flip operator for the κ-Minkowski spacetime [38]. We then discuss the construction of a novel, covariant flip operator and discuss its properties. We apply the twisted flip operator to multi-particle sector and obtain the novel, modified commutation relations between the creation and annihilation operators. We also show how to define a new product rule between the oscillator operators so as to express their commutation relations in the familiar form. We obtain a large class of novel deformed oscillator algebras which reduce to the one discussed in [37] for a special choice of the realization. We finally end in Sec. V with discussions. In the Appendix, we start with the $*$ product corresponding to the κ spacetime that can be defined using the commuting vectors fields and derive the twisted coproduct. Here, using this $*$ product, we first identify the twist element and using this we derive the twisted coproducts. We show that these twisted coproducts-products, for a specific realization, are exactly the same as the ones we derive in Sec. II.

II. κ-SPACE, ITS REALIZATIONS AND TWISTED COPRODUCTS

In this section, we review the results of earlier papers by some of the authors[6,8,9,38], which are used later. Similar results have been obtained in general Lie algebra type noncommutative spaces and, in particular, for κ space and quantum field theories on such spaces in [19,22,24–26,35,37,47,48]. Here we start with the generic Lie algebra type noncommutative spaces and then specialize to the case of κ-Minkowski space, for which we obtain a special class of realization of the noncommutative coordinates in terms of the coordinates and derivatives of the commuting space.

The coordinates of the generic Lie algebra type noncommutative space obey the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = iC_{\mu\nu\lambda}\hat{x}^\lambda, \qquad \hat{x}^\lambda = \eta^{\lambda\alpha}\hat{x}_\alpha \tag{1}$$

with the choice $C_{\mu\nu\lambda} = a_\mu\eta_{\nu\lambda} - a_\nu\eta_{\mu\lambda}$ and $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \ldots 1)$ and summation over repeated indices is understood. Here $a_\mu(\mu = 0, 1, 2, ..n-1)$ are real, dimensionful constants parametrizing the deformation of the Minkowski space. The κ space is defined by the choice $a_i = 0$, $i = 1, 2, \ldots n-1$; $a_0 = a = \frac{1}{\kappa}$. Thus we get the commutation relations between the coordinates of κ space as

$$[\hat{x}_i, \hat{x}_j] = 0, \qquad [\hat{x}_0, \hat{x}_i] = ia\hat{x}_i. \tag{2}$$

In terms of the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1 \ldots\ldots, 1)$, we can define $x^\mu = \eta^{\mu\alpha}x_\alpha$ and $\partial^\mu = \frac{\partial}{\partial x_\mu} = \eta^{\mu\alpha}\partial_\alpha$, which satisfy the relations

$$\begin{aligned} [x_\mu, x_\nu] &= 0, \qquad [\partial_\mu, \partial_\nu] = 0, \\ [\partial^\mu, x_\nu] &= \eta^\mu_\nu, \qquad [\partial_\mu, x_\nu] = \eta_{\mu\nu}. \end{aligned} \tag{3}$$

For later use, we also define $p_\mu = -i\partial_\mu$ so that $[p_\mu, x_\nu] = -i\eta_{\mu\nu}$.

We seek realizations of the noncommutative coordinates in terms of the commuting coordinates x_μ and corresponding derivatives ∂_μ as a power series. A class of such realizations is given by

$$\hat{x}_{\mu} = x^{\alpha}\Phi_{\alpha\mu}(\partial). \tag{4}$$

It is easy to see that these coordinates obey $[\partial_\mu, \hat{x}_\nu] = \Phi_{\mu\nu}(\partial)$. Such a realization defines a unique mapping between the functions on noncommutative space to functions on commutative space. This can be seen first by defining the vacuum $|0\rangle \equiv 1$ annihilated by ∂ and defining

$$F(\hat{x}_\varphi)|0\rangle = F_\varphi(x), \tag{5}$$

where the subscript φ specify the realization we work with. The functions of noncommutative coordinates are expanded as a power series in $\hat{x}_\mu$. Though there can be many monomials where $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{n-1}$ appear $m_0, m_1, \ldots, m_{n-1}$ times, respectively, all of them are related by the commutation relations given by Eqs. (2). Furthermore, to each φ realization there exists a corresponding ordering among noncommutative coordinates, such that

$$:F(\hat{x}_\varphi):_\varphi|0\rangle = F(x)$$

(and vice versa). Thus we can define left, right, totally symmetric (Weyl) ordering, respectively, as

$$\begin{aligned} :e^{ik_\mu\hat{x}^\mu}:_L &\equiv e^{-ik_0\hat{x}_0+ik_i\hat{x}_i\varphi_s(-ak_0)e^{-iak_0}} = e^{-ik_0\hat{x}_0}e^{ik_i\hat{x}_i},\\ :e^{ik_\mu\hat{x}^\mu}:_R &\equiv e^{-ik_0\hat{x}_0+ik_i\hat{x}_i\varphi_s(-ak_0)} = e^{ik_i\hat{x}_i}e^{-ik_0\hat{x}_0},\\ :e^{ik_\mu\hat{x}^\mu}:_S &\equiv e^{ik_\mu\hat{x}^\mu}\end{aligned} \tag{6}$$

here $\varphi_s(A) = \frac{A}{e^A-1}$, $A = ia\partial^0 = -ia\partial_0$.

In this paper, we work with a specific class of realization satisfying $[\partial_\mu, \hat{x}_\nu] = \Phi_{\mu\nu}(\partial)$ given by

$$[\partial_i, \hat{x}_j] = \delta_{ij}\varphi(A), \qquad [\partial_i, \hat{x}_0] = ia\partial_i\gamma(A), \tag{7}$$

$$[\partial_0, \hat{x}_i] = 0, \qquad [\partial_0, \hat{x}_0] = \eta_{00} = -1, \tag{8}$$

where $A = -ia\partial_0$. An explicit solution of this realization is

$$\hat{x}_i = x_i\varphi(A) \qquad \hat{x}_0 = x_0\psi(A) + iax_i\partial_i\gamma(A). \tag{9}$$

Using the above realization in Eq. (2) we get

$$\frac{\varphi'}{\varphi}\psi = \gamma - 1, \tag{10}$$

where φ' is the derivative of φ with respect to its argument $ia\partial_0$ and these functions satisfy the boundary conditions $\varphi(0) = 1$, $\psi(0) = 1$, and $\gamma(0) = \varphi'(0) + 1$ is finite and all are positive functions. Further demanding that the commutators of the Lorentz generators with the κ-space coordinates be linear in $\hat{x}_\mu$ as well as in the generators and have smooth commutative limit as the deformation parameter $a \to 0$, imposes further requirements on these functions and one can easily see that there are only two class of realizations possible, viz., one where $\psi = 1$ and a second one where $\psi = 1 + 2A$. We restrict ourselves to the case $\psi = 1$.

It may be noted that for $\varphi_S(A) = \frac{A}{e^A-1}$ there exists a covariant realization corresponding to Weyl-symmetric ordering. This realization is given by

$$\begin{aligned}\hat{x}_\mu &= x_\mu\varphi_S(A) + ia_\mu x_\alpha\partial^\alpha\gamma_S(A),\\ x_\alpha\partial^\alpha &= \eta^{\alpha\beta}x_\alpha\partial_\beta,\end{aligned} \tag{11}$$

where $A = ia_0\partial^0 = -ia\partial_0$. Here we choose $a_\mu = (a, 0, \ldots, 0)$ to be timelike, which can be chosen to be spacelike or lightlike as well.

For the realization given in Eq. (9) defined by $\varphi(A) = e^{-\Lambda A/2}$, there also exists a one-parameter family of ordering prescriptions

$$:e^{ik_\mu\hat{x}^\mu}:_\Lambda = e^{-i\Lambda k_0\hat{x}_0}e^{ik_i\hat{x}_i}e^{-i(1-\Lambda)k_0\hat{x}_0}, \tag{12}$$

which interpolate between right, time symmetric, and left corresponding to $\Lambda = 0, \frac{1}{2}$, and 1, respectively. Note that what we call here as totally symmetric ordering [6,8,9] corresponding to the realization $\varphi_S(A) = \frac{A}{e^A-1}$ is completely different from the time-symmetric ordering corresponding to $\Lambda = \frac{1}{2}$ [49].

The coproducts Δ_φ of the derivative operators in the realization given in Eq. (9) are

$$\Delta_\varphi(\partial_0) = \partial_0 \otimes I + I \otimes \partial_0 \equiv \partial_0^x + \partial_0^y, \tag{13}$$

$$\Delta_\varphi(\partial_i) = \varphi(A \otimes I + I \otimes A)\left[\frac{\partial_i}{\varphi(A)} \otimes I + e^A \otimes \frac{\partial_i}{\varphi(A)}\right]. \tag{14}$$

A. κ-Poincaré algebra and Casimir

Let $M_{\mu\nu}$ denote the rotation and boost generators satisfying the undeformed $so(n-1, 1)$ algebra. We require that their commutators with the κ-space coordinates be linear functions of $\hat{x}_\mu$ and $M_{\mu\nu}$. In addition, the requirement that these commutators have a smooth commutative limit leads to

$$[M_{i0}, \hat{x}_0] = -\hat{x}_i + iaM_{i0} \tag{15}$$

$$[M_{i0}, \hat{x}_j] = -\delta_{ij}\hat{x}_0 + iaM_{ij}. \tag{16}$$

We note here that ∂_0, ∂_i defined in Eq. (7) along with $M_{\mu\nu}$ given above generates the κ-deformed Poincaré algebra [6,8,9]. Note that the Lorentz algebra is undeformed and the commutator $[M_{\mu\nu}, \partial_\lambda]$ is deformed and depends on the realization. We also note that the twisted coproducts of $M_{\mu\nu}$ can also be obtained from Eqs. (15) and (16) [6,8,9]. They are

$$\Delta_\varphi(M_{ij}) = M_{ij} \otimes I + I \otimes M_{ij} \equiv \Delta_0(M_{ij}) \tag{17}$$

$$\Delta_\varphi(M_{i0}) = M_{i0} \otimes I + e^A \otimes M_{i0} + ia\partial_j \frac{1}{\varphi(A)} \otimes M_{ij}. \tag{18}$$

For $\psi = 1$ class of realizations we are interested in, the explicit form of $M_{\mu\nu}$ are

$$M_{ij} = x_i\partial_j - x_j\partial_i, \tag{19}$$

$$M_{i0} = x_i\partial_0\varphi\frac{e^{2A}-1}{2A} - x_0\partial_i\frac{1}{\varphi} + iax_i\Delta\frac{1}{2\varphi} - iax_k\partial_k\partial_i\frac{\gamma}{\varphi}, \tag{20}$$

where $\Delta = \partial_k\partial_k$. Note here that M_{0i} involves $x_i\partial_i$ and so does $\Delta_\varphi(M_{i0})$. The $\Delta(M_{i0})$ is expressed in terms of enveloping algebra of κ-deformed Poincaré algebra generated by ∂_μ, $M_{\mu\nu}$.

We can also define

$$\tilde{M}_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu = i(x_\mu p_\nu - x_\nu p_\mu), \tag{21}$$

which generate an undeformed Poincaré algebra.

The generalized Klein-Gordon equation, which is invariant under the action of the undeformed Poincaré algebra generated by ∂_0, ∂_i, $\tilde{M}_{\mu\nu}$, is given as

$$(\partial_\mu\partial^\mu - m^2)\Phi(x) = 0. \tag{22}$$

Here we note that the above field equation is not invariant under the κ-deformed Poincaré transformations generated by ∂_0, ∂_i, $\tilde{M}_{\mu\nu}$ defined above. This can be seen easily by noticing that the derivatives do not transform like a vector under the transformations generated by $M_{\mu\nu}$. A possible way to avoid this is to introduce the (Dirac) derivatives D_μ, for which there exists coordinates X_μ satisfying the conditions

$$[D^\mu, X_\nu] = \eta^\mu_\nu = \delta^\mu_\nu, \qquad [D_\mu, X_\nu] = \eta_{\mu\nu}. \tag{23}$$

Then, we have

$$\begin{aligned} \hat{x}_\mu &= X_\mu Z^{-1} + i(X_\nu a^\nu)D_\mu, \\ M_{\mu\nu} &= X_\mu D_\nu - X_\nu D_\mu = i(X_\mu P_\nu - X_\nu P_\mu) \\ &= (\hat{x}_\mu D_\nu - \hat{x}_\nu D_\mu)Z, \\ P_\mu &= -iD_\mu. \end{aligned} \tag{24}$$

Generators $M_{\mu\nu}$, D_λ generate the undeformed Poincaré algebra.

The Dirac derivatives transform like a vector under $M_{\mu\nu}$. The undeformed Poincaré algebra is defined through the relations

$$[M_{\mu\nu}, D_\lambda] = \eta_{\nu\lambda}D_\mu - \eta_{\mu\lambda}D_\nu, \tag{25}$$

$$[D_\mu, D_\nu] = 0, \qquad [M_{\mu\nu}, \Box] = 0, \qquad [\Box, \hat{x}_\mu] = 2D_\mu, \tag{26}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = \eta_{\mu\rho}M_{\nu\lambda} + \eta_{\nu\lambda}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\lambda}M_{\nu\rho}, \tag{27}$$

which were obtained in [6,8]. Note that D_μ and $M_{\mu\nu}$ given above generate undeformed Poincaré algebra. These Dirac derivatives are different from usual derivatives as can be seen easily from their action on $\hat{x}_\mu$, i.e.,

$$[D_\mu, \hat{x}_\nu] = \eta_{\mu\nu}Z^{-1} + ia_\mu D_\nu, \tag{28}$$

where $Z^{-1} = iaD_0 + \sqrt{1 + a^2D_\alpha D^\alpha}$. Using Eq. (28), we get the twisted Leibniz rule for $D\mu$ as

$$\Delta(D_\mu) = D_\mu \otimes Z^{-1} + I \otimes D_\mu + ia_\mu(D^\alpha Z) \otimes D_\alpha - \frac{ia_\mu}{2}\Box Z \otimes ia_\alpha D^\alpha. \tag{29}$$

Similarly, we also get the covariant form of the twisted Leibniz rule for $M_{\mu\nu}$ as

$$\Delta(M\mu\nu) = M_{\mu\nu} \otimes I + I \otimes M_{\mu\nu} + ia_\mu\Big(D^\alpha - \frac{ia^\alpha}{2}\Box\Big)Z \otimes M_{\alpha\nu} - ia_\nu\Big(D^\alpha - \frac{ia^\alpha}{2}\Box\Big)Z \otimes M_{\alpha\mu}, \tag{30}$$

where $M_{\mu\nu}$ is as given in Eq. (24).

B. Dispersion relations

For arbitrary realizations characterized by φ, these Dirac derivatives and the $\Box$ operator are

$$D_i = \partial_i\frac{e^{-A}}{\varphi}, \tag{31}$$

$$D_0 = \partial_0\frac{\sinh A}{A} - ia_0\Delta\frac{e^{-A}}{2\varphi^2}, \tag{32}$$

$$\begin{aligned} \Box &= \Delta\frac{e^{-A}}{\varphi^2} + 2\partial_0^2\frac{(1-\cosh A)}{A^2} \\ &= \frac{2}{a^2}(\cosh(ap_0) - 1) - p_ip_i\frac{e^{-ap_0}}{\varphi^2(ap_0)}. \end{aligned} \tag{33}$$

The relation between Dirac D_μ and ∂_μ derivatives corresponding to $\varphi_S(A)$ (Weyl-symmetric ordering) is given by

$$D_\mu = \partial_\mu\frac{Z^{-1}}{\varphi_S(A)} + \frac{ia_\mu}{2}\Box \tag{34}$$

and $\Box = \partial_0\partial^0\frac{e^{-A}}{(\varphi_S(A))^2}$. It is clear that the coalgebra of the undeformed Poincaré algebra generated by D_μ, $M\mu\nu$ is closed in the enveloping algebra of Poincaré generators and it is a Hopf algebra.

It is also clear that the Casimir, $D_\mu D^\mu$ has a vanishing commutator with $M_{\mu\nu}$ and has the correct commutative limit. The Casimir can be expressed in terms of the $\Box$ operator [6,8,9,20,24] as

$$D_\mu D^\mu = \Box\Big(1 + \frac{a^2}{4}\Box\Big). \tag{35}$$

Here note that the $\Box$ operator is quadratic in space derivatives and thus the Casimir has quartic terms in space derivatives.

Generalizing the notions from commutative space, it is natural to write the equation of motion for the scalar particle, i.e., the generalized Klein-Gordon equation using the Casimir. Thus the generalized Klein-Gordon equation on κ space is

$$\left(\Box\left(1+\frac{a^4}{4}\Box\right)-m^2\right)\Phi(x)=0 \tag{36}$$

and has the correct commutative limit. But since the Casimir as well as the $\Box$ operator have the same commutative limit, the requirement of the correct Klein-Gordon equation in the commutative limit does not rule out other possible generalizations in the κ space. Thus, one can equally well start with

$$(\Box-m^2)\Phi(x)=0 \tag{37}$$

as the equation for scalar theory on κ space. Other choices were also considered [16,50] for effective scalar Lagrangians in κ space.

For the above choices of equations, we get the deformed dispersion relations as

$$\frac{4}{a^2}\sinh^2\left(\frac{ap_0}{2}\right)-p_ip_i\frac{e^{-ap_0}}{\varphi^2(ap_0)}-m^2 +\frac{a^2}{4}\left[\frac{4}{a^2}\sinh^2\left(\frac{ap_0}{2}\right)-p_ip_i\frac{e^{-ap_0}}{\varphi^2(ap_0)}\right]^2=0 \tag{38}$$

$$\frac{4}{a^2}\sinh^2\left(\frac{ap_0}{2}\right)-p_ip_i\frac{e^{-ap_0}}{\varphi^2(ap_0)}-m^2=0, \tag{39}$$

respectively, and here φ characterizes the realizations. Thus with $\varphi=e^{-A}$, 1, $\frac{A}{e^A-1}$ one gets left, right, and Weyl-symmetric orderings, respectively.

III. $*$ PRODUCT

The mapping between the functions on κ space to that of commutative space [see Eq. (5)] also defines a $*$ product, which naturally depends on the realization φ. Thus the $*$ product is defined as

$$F_\varphi(\hat{x}_\varphi)G_\varphi(\hat{x}_\varphi)|0\rangle=F_\varphi *_\varphi G_\varphi. \tag{40}$$

For the realizations we are interested in, i.e., the one given in Eq. (9), this implies the following $*$-product rules

$$\begin{aligned} x_i *_\varphi f(x)&=(\hat{x}_\varphi)_if(\hat{x}_\varphi)|0\rangle=x_i\varphi(A)f(x)\\ x_0 *_\varphi f(x)&=(\hat{x}_\varphi)_0f(\hat{x}_\varphi)|0\rangle=[x_0\psi(A)+iax_i\partial_i\gamma(A)]f(x)\end{aligned} \tag{41}$$

and similarly

$$\begin{aligned} f(x)*_\varphi x_i&=x_i\varphi(A)e^Af(x)\\ f(x)*_\varphi x_0&=[x_0\psi(A)+iax_i\partial_i(\gamma(A)-1)]f(x).\end{aligned} \tag{42}$$

For any realization φ, the $*$ product can be expressed in terms of the twist element $\mathcal{F}_\varphi$ as

$$f *_\varphi g=m_0(\mathcal{F}_\varphi f\otimes g)=m_\varphi(f\otimes g), \tag{43}$$

where f and g are functions of the commutative coordinates and m_0 is the usual pointwise multiplication map in the commutative algebra of smooth functions. This can be reexpressed as

$$(f *_\varphi g)(x)=m_0(e^{x_i(\Delta_\varphi-\Delta_0)\partial_i}f(u)g(t))|_{u=t=x_i}, \tag{44}$$

where Δ_φ is the twisted coproduct given in Eq. (14) and the undeformed coproducts is given by $\Delta_0=\partial\otimes I+I\otimes\partial$. Comparing Eqs. (43) and (44), we find the twist element as

$$\mathcal{F}_\varphi=e^{x_i(\Delta_\varphi-\Delta_0)\partial_i} \tag{45}$$

and then it is easy to find

$$\Delta_\varphi=\mathcal{F}_\varphi^{-1}\Delta_0\mathcal{F}_\varphi. \tag{46}$$

Thus we find that by applying the twist element obtained in Eq. (45) to the undeformed coproduct of ∂_0 and ∂_i, we get the twisted coproducts which are exactly the same as the one obtained in Eqs. (13) and (14). But $\mathcal{F}_\varphi^{-1}\Delta_0(\tilde{M}_{\mu\nu})\mathcal{F}_\varphi$ do not give the twisted coproducts of the deformed Poincaré algebra obtained in Eqs. (17) and (18), which can be easily checked using Eq. (48) below. Also we note that the $\tilde{M}_{\mu\nu}$ along with p_μ generate undeformed Poincaré algebra. The corresponding coalgebra does not close in enveloping Poincaré algebra, but in enveloping algebra of $igl(n)\times igl(n)$.

Using Eqs. (14) and (44), we find that the $*$ product can be written as [9,51]

$$(f *_\varphi g)(x)=e^{x_i\partial_i^u(((\varphi(A_u+A_t))/(\varphi(A_u)))-1)+x_i\partial_i^t(((\varphi(A_u+A_t))/(\varphi(A_t)))e^{A_u}-1)}f(u)g(t)|_{u=t=x_i}. \tag{47}$$

The explicit form of the corresponding twist element is now given by

$$\mathcal{F}_\varphi=e^{N_x\ln((\varphi(A_x+A_y))/(\varphi(A_x)))+N_y(A_x+\ln(((\varphi(A_x+A_y))/(\varphi(A_y))))}, \tag{48}$$

where $N_x=x_i\frac{\partial}{\partial x_i}$ [6,8,9].

Since the $*$ product depends on the ordering (or equivalently on realization) as has been seen from Eqs. (41) and (42), it is natural to have different twist elements depending on the ordering. Indeed, we get the twist element for left ordering as

$$\mathcal{F}_L=e^{-N_xA_y}=e^{N\otimes A} \tag{49}$$

and corresponding to right ordering we get

$$\mathcal{F}_R = e^{A_x N_y} = e^{A\otimes N}, \tag{50}$$

with $A_x = -ia\partial_0^x$ and $N_x = x_i\partial_i^x$. One can combine the above two to write down an interpolating twist element

$$\mathcal{F}_\Lambda = e^{-\Lambda N\otimes A + (1-\Lambda)A\otimes N}, \tag{51}$$

which reduces to $\mathcal{F}_L$ and $\mathcal{F}_R$ when $\Lambda = 1$ and $\Lambda = 0$, respectively. This twist element satisfies the cocycle condition

$$(\mathcal{F}_\Lambda \otimes I)(\Delta \otimes I)\mathcal{F}_\Lambda = (I \otimes \mathcal{F}_\Lambda)(I \otimes \Delta)\mathcal{F}_\Lambda. \tag{52}$$

One can now get the modified momentum addition rules for κ space from the coproducts given in Eqs. (13) and (14) also. Thus going to momentum space we find

$$\begin{aligned} [K_\varphi(p,q)]_\mu &= -i\Delta_\varphi(\partial_\mu), \\ K_\varphi(p,q)x &= -(p_0+q_0)x_0 + \varphi(-ap_0 - aq_0) \\ &\quad\times\left[\frac{p_i x_i}{\varphi(-ap_0)} + \frac{e^{-ap_0}}{\varphi(-aq_0)}q_i x_i\right]. \end{aligned} \tag{53}$$

Similarly, we can also obtain the twist element in the momentum space, denoted by $\mathcal{F}$, which tells how the $*$ product acts on the momentum space [by expressing the operators A and N in the momentum space in Eqs. (49)–(51), we get the explicit form for $\mathcal{F}$, for different ordering]. Starting from

$$\mathcal{F} f(x)g(y) \equiv \mathcal{F}\int d^4k d^4q e^{ikx}\tilde{f}(k)e^{iqy}\tilde{g}(q) \tag{54}$$

and using the action of $\mathcal{F}$ on plane waves, we can easily get

$$\mathcal{F}\tilde{f}(k)\otimes\tilde{g}(q) = \mathcal{F}\left(i\frac{\partial}{\partial k}, k, i\frac{\partial}{\partial q}, q\right)\tilde{f}(k)\tilde{g}(q). \tag{55}$$

The above result will be of use to obtain the twisted commutation relations between the Fourier coefficients, necessary to discuss the twisted oscillators.

IV. DEFORMED STATISTICS AND OSCILLATORS IN κ-MINKOWSKI SPACE

It is known that for the QFT's defined on the Moyal plane, the twisted coproduct rules affect the statistics [29–34,38]. This is natural as the physical theory has to be invariant under the action of the underlying symmetry group of the space (or spacetime) and the definition of statistics should also be invariant under this group action. This ensures that the statistics are superselected. Such a superselection rule is implemented by demanding that the flip operator commutes with the coproduct. As the coproduct rule is now changed, we do expect a corresponding change in the definition of flip operator also. Such a twisted flip operator for the κ-deformed space was constructed in [38]. In the first part of this section, we briefly review that construction which requires us to consider a larger general linear algebra. Next we introduce the concept of a covariant flip operator, which preserves the algebraic structure of the κ-Minkowski space, which is a new result. We give an explicit expression of this covariant flip operator to the first order in the deformation parameter in terms of the generators of the Poincaré algebra alone. We also obtain an expression of the corresponding R matrix to the first order. We find that up to first order in the deformation parameter, the expression for the R matrix obtained using the covariant flip operator is different from that obtained using the twisted flip operator.

Our main results are given in the second part of this section, where we obtain novel twisted commutation relations between the creation and annihilation operators appearing in the mode decomposition of the scalar field satisfying the generalized Klein-Gordon equation. This leads to a large class of such deformed algebras depending on the family of realizations of the κ-Minkowski space. For a certain choice of realization, we explicitly obtain the deformed algebra obtained in [37]. Our analysis however indicates the possibility of a much wider class of deformed oscillator algebras.

A. Twisted flip operators

In this subsection, we discuss the twisted flip operators compatible with the coproducts of the deformed Poincaré algebra defined by the generators in Eqs. (7), (15), and (16) and for the undeformed Poincaré algebra generated by D_μ and $M_{\mu\nu}$, respectively.

In the commutative case, the flip operator is defined through its action on multiparticle states. Without loss of generality, let us consider a two-particle state $f\otimes g \in \mathcal{A}_0\otimes\mathcal{A}_0$. The action of the flip operator on this (tensor-product) state is given by

$$\tau_0(f\otimes g) = g\otimes f. \tag{56}$$

It is easy to see that $(\tau_0)^2 = I$. Symmetric and antisymmetric states of the physical Hilbert space are projected from the tensor-product state as

$$\tfrac{1}{2}(1\pm\tau_0)(f\otimes g) = \tfrac{1}{2}(f\otimes g \pm g\otimes f), \tag{57}$$

respectively. Since this definition of (anti)symmetric states should remain invariant under the action of the underlying symmetry, its clear that the flip operator must commute with the symmetry generator. Since Λ, a typical element of the symmetry group acts on the tensor-product state through some representation D as

$$\Lambda\colon f\otimes g = (D\otimes D)\Delta(\Lambda)f\otimes g, \tag{58}$$

this requirement implies that the coproduct $\Delta(\Lambda)$ commutes with the flip operator τ_0. Thus in the commutative space the flip operator τ_0 is superselected so as to have vanishing commutators with all observables. In this case of noncommutative theories, as we have seen, the coproducts get twisted and the twisted coproducts do not satisfy

$$[\Delta_\varphi, \tau_0] \neq 0. \quad (59)$$

Thus, the meaning of (anti)symmetric states defined using τ_0 are no longer invariant. We are, thus forced to define a new twisted flip operator which commutes with the coproduct action. Since $\Delta_\varphi = \mathcal{F}_\varphi^{-1}\Delta_0\mathcal{F}_\varphi$, where Δ_0 is the coproducts of the undeformed Poincaré algebra, we are immediately led to the twisted flip-operator

$$\tau_\varphi = \mathcal{F}_\varphi^{-1}\tau_0\mathcal{F}_\varphi \quad (60)$$

which satisfies

$$[\Delta_\varphi, \tau_\varphi] = 0. \quad (61)$$

Using this twisted flip operator, we can define an invariant definition of symmetric and antisymmetric states as $\frac{1}{2}\times(1 \pm \tau_\varphi)(f \otimes g)$, respectively. The twisted flip operator for a generic φ realization can be easily obtained using Eq. (48) in Eq. (60) as

$$\tau_\varphi = e^{i(x_i p_i \otimes A - A \otimes x_i p_i)}\tau_0, \quad (62)$$

where $A = -ia\partial_0$. In the limit $a \to 0$, we get back the familiar commutative flip operator, smoothly. It is interesting to note that the τ_φ given above is independent of φ. It may be noted that the twisted flip operator τ_φ is not covariant and involves operators belonging to the universal enveloping algebra of $GL(d-1,1)$. The R matrix corresponding to the flip operator τ_φ, denoted by R_τ, is defined as

$$R_\tau = I \otimes I + iN \wedge A = I \otimes I - a(x_i\partial^i) \wedge \partial_0 \quad (63)$$

and it satisfies the classical Yang-Baxter equation since $[N, A] = 0$. Note that in the above $ax_i\partial^i \wedge \partial_0 = a(\overleftrightarrow{x_i\partial^i}\,\overrightarrow{\partial_0} - \overleftarrow{\partial_0}\,\overleftrightarrow{x_i\partial^i})$.

Alternately, we can define another deformed flip operator τ_c which is covariant. This new covariant flip operator is compatible with the symmetries implemented by the covariant twisted coproducts of D_μ and $M_{\mu\nu}$ given in Eqs. (29) and (30). It is defined by the conditions

$$[\Delta(D_\mu), \tau_c] = 0, \qquad [\Delta(M_{\mu\nu}), \tau_c] = 0, \quad (64)$$

where $\tau_c = R_c\tau_0$. Expanding the R_c matrix in powers of the deformation parameters a_μ as $R_c = I \otimes I + \sum \Gamma(a, \Lambda)$, where Λ stands for the generators of the κ-Poincaré algebra and using the twisted coproducts [see Eqs. (29) and (30)] in the above condition, we get the R_c matrix (to the first order in the deformation parameter) as

$$R_c = I \otimes I + I[M^{\mu\nu} \otimes a_\mu D_\nu - a_\mu D_\nu \otimes M^{\mu\nu}]. \quad (65)$$

The explicit form of $M_{\mu\nu}$ appearing above is given in Eq. (24). We note here that the above R matrix, up to first order in the parameter, involves only the generators of the κ-Poincaré algebra, namely $M_{\mu\nu}$ and D_μ. This has to be contrasted with the one in Eq. (63) for the noncovariant, twisted flip operator, which involves the (space) dilation operator which is not in the k-Poincaré algebra (R matrix, as an expansion in inverse powers of κ, was studied in [52] for the case of κ deformed spaces). It may also be noted that the R_τ and R_c matrices would in general lead to different physics. The calculation of the covariant R matrix to all orders in the deformation parameter is presently under investigation. This covariant R matrix to first order in the deformation parameter given in Eq. (65) is a new result.

B. Twisted oscillator algebra

In this section, we derive a novel class of twisted products between the creation and annihilation operators appearing in the mode expansion of the scalar field theory in κ space.

Having defined the twisted flip operator τ_φ, we are now in a position to define (anti)symmetric states of a theory defined in the κ-Minkowski space. We start by defining the deformed bosonic state as

$$f \star_\varphi g = m_\varphi(f \otimes g) = m_\varphi\tau_\varphi(f \otimes g). \quad (66)$$

Using the definitions of m_φ and τ_φ [see Eqs. (43) and (60)] in the above, we get

$$f \otimes g = \tau_\varphi(f \otimes g) \quad (67)$$

or equivalently, we can write

$$\mathcal{F}_\varphi(f \otimes g) = \tilde{\mathcal{F}}_\varphi(f \otimes g), \quad (68)$$

where we have used the mirror twist operator $\tilde{\mathcal{F}}_\varphi = \tau_0\mathcal{F}_\varphi\tau_0$. Now defining the twisted tensor product $f \otimes_\varphi g$ as $\mathcal{F}_\varphi(f \otimes g)$, from the above, we get

$$f \otimes_\varphi g = \tau_0(f \otimes_\varphi g). \quad (69)$$

For the product of two bosonic fields $\phi(x)$ and $\phi(y)$ under interchange, now we pick up an additional factor compared to the commutative case. This can be calculated using Eq. (67) and one gets

$$\phi(x) \otimes \phi(y) - e^{-(A\otimes N - N\otimes A)}\phi(y) \otimes \phi(x) = 0. \quad (70)$$

Expressing ϕ in the above equation using Fourier transforms and using the twisted flip operator in momentum space, we are led to the deformed commutation relations between the annihilation operators as

$$\tilde{\phi}(k)\tilde{\phi}(p) = e^{-ia[k_0(\partial_{p_i}p_i) - p_0(\partial_{k_i}k_i)]}\tilde{\phi}(p)\tilde{\phi}(k). \quad (71)$$

The $\Phi(x)$ appearing in the generalized Klein-Gordon equation (22) can be expressed as

$$\Phi(x) = \int d^4p\,\delta(p_0^2 - \omega^2)\bar{A}(p)e^{-ip\cdot x}, \quad (72)$$

where $\omega = \sqrt{p_i^2 + m^2}$. Using the mode decomposition

$$\Phi(x) = \int \frac{d^3 p}{\sqrt{p_i^2 + m^2}} [A(\omega, \vec{p}) e^{-ip\cdot x} + A^\dagger(\omega, \vec{p}) e^{ip\cdot x}], \tag{73}$$

from Eq. (71) we get

$$A^\dagger(p_0, \vec{p}) A(q_0, \vec{q}) - e^{-a(q_0 \partial_{p_i} p_i + \partial_{q_i} q_i p_0)} A(q_0, \vec{q}) A^\dagger(p_0, \vec{p}) = -\delta^3(p - q), \tag{74}$$

$$A^\dagger(p_0, \vec{p}) A^\dagger(q_0, \vec{q}) - e^{-a(-q_0 \partial_{p_i} p_i + \partial_{q_i} q_i p_0)} A^\dagger(q_0, \vec{q}) \times A^\dagger(p_0, \vec{p}) = 0, \tag{75}$$

$$A(p_0, \vec{p}) A(q_0, \vec{q}) - e^{-a(q_0 \partial_{p_i} p_i - \partial_{q_i} q_i p_0)} A(q_0, \vec{q}) A(p_0, \vec{p}) = 0. \tag{76}$$

For the choice $\varphi = e^{-(A/2)} = e^{-((ia\partial_0)/2)}$, the generalized Klein-Gordon equation (37) is

$$\left[\partial_i^2 + \frac{4}{a^2} \sinh^2\left(\frac{ia\partial_0}{2}\right) - m^2\right] \Phi = 0. \tag{77}$$

We can decompose this field in positive and negative frequency modes and thus,

$$\Phi(x) = \int \frac{d^4 p}{2\Omega_k(p)} [A(\omega_k, \vec{p}) e^{-ip\cdot x} + A^\dagger(\omega_k, \vec{p}) e^{ip\cdot x}], \tag{78}$$

where $A^\dagger(\pm\omega_k, \vec{p}) = A^\dagger(\mp\omega_k, \vec{p})$. In the above, we have used

$$p_0^\pm = \pm\omega_k(p) = \pm\frac{2}{a} \sinh^{-1}\left(\frac{a}{2}\sqrt{p_i^2 + m^2}\right), \tag{79}$$

$$\Omega_k(p) = \frac{1}{a} \sinh(a\omega_k(p)). \tag{80}$$

Using this in Eq. (71), for the field satisfying the deformed generalized Klein-Gordon equation given above [Eq. (77)], we obtain various twisted commutation relations between creation and annihilation operators. They are

$$A^\dagger(p_0, \vec{p}) A(q_0, \vec{q}) - e^{-a(q_0 \partial_{p_i} p_i + \partial_{q_i} q_i p_0)} A(q_0, \vec{q}) A^\dagger(p_0, \vec{p}) = -\delta^3(p - q), \tag{81}$$

$$A^\dagger(p_0, \vec{p}) A^\dagger(q_0, \vec{q}) - e^{-a(-q_0 \partial_{p_i} p_i + \partial_{q_i} q_i p_0)} A^\dagger(q_0, \vec{q}) \times A^\dagger(p_0, \vec{p}) = 0, \tag{82}$$

$$A(p_0, \vec{p}) A(q_0, \vec{q}) - e^{-a(q_0 \partial_{p_i} p_i - \partial_{q_i} q_i p_0)} A(q_0, \vec{q}) A(p_0, \vec{p}) = 0. \tag{83}$$

Note that p_0 and q_0 are as given in Eq. (79). From this, one can easily derive the following relations

$$A^\dagger(p_0, e^{-(aq_0)/2}\vec{q}) A^\dagger(q_0, e^{(ap_0)/2}\vec{q}) - \mathcal{F}(q, p) A^\dagger(q_0, e^{-(ap_0)/2}\vec{q}) A^\dagger(p_0, e^{(aq_0)/2}\vec{q}) = 0, \tag{84}$$

$$A(p_0, e^{(aq_0)/2}\vec{p}) A(q_0, e^{-(ap_0)/2}\vec{q}) - \mathcal{F}(-q, -p) A(q_0, e^{(ap_0)/2}\vec{q}) A(p_0, e^{-(aq_0)/2}\vec{p}) = 0, \tag{85}$$

$$A^\dagger(p_0, e^{(aq_0)/2}\vec{p}) A(q_0, e^{(ap_0)/2}\vec{q}) - \mathcal{F}(-q, p) A(q_0, e^{-(ap_0)/2}\vec{q}) A^\dagger(p_0, e^{-(aq_0)/2}\vec{p}) = -\delta^3(p - q), \tag{86}$$

where $\mathcal{F}(q, p) = e^{3a(q_0 - p_0)}$. These relations were obtained in [37,48] using a different approach. Using these relations, a new product (the $\circ$ product) between the creation and annihilation operators is defined as follows:

$$A(p) \circ A(q) = e^{-((3a)/2)(p_0 - q_0)} A(p_0, e^{(aq_0)/2}\vec{p}) \times A(q_0, e^{-(ap_0)/2}\vec{q}) \tag{87}$$

$$A^\dagger(p) \circ A^\dagger(q) = e^{((3a)/2)(p_0 - q_0)} A^\dagger(p_0, e^{-(aq_0)/2}\vec{p}) \times A(q_0, e^{(ap_0)/2}\vec{q}) \tag{88}$$

$$A^\dagger(p) \circ A(q) = e^{((3a)/2)(p_0 + q_0)} A^\dagger(p_0, e^{(aq_0)/2}\vec{p}) \circ A(q_0, e^{(ap_0)/2}\vec{q}) \tag{89}$$

$$A(p) \circ A^\dagger(q) = e^{-((3a)/2)(p_0 + q_0)} A(p_0, e^{-(aq_0)/2}\vec{p}) \circ A^\dagger(q_0, e^{-(ap_0)/2}\vec{q}). \tag{90}$$

Using this new product rule, we can reexpress Eqs. (81)–(83) as

$$[A(p_0, \vec{p}), A(q_0, \vec{q})]_\circ = 0, \quad [A^\dagger(p_0, \vec{p}), A^\dagger(q_0, \vec{q})]_\circ = 0, \tag{91}$$

$$[A(p_0, \vec{p}), A^\dagger(q_0, \vec{q})]_\circ = \delta^3(\vec{p} - \vec{q}). \tag{92}$$

Thus, with this modified product rule, the algebra of creation and annihilation operators can be recast in the same form as the corresponding commuting operators.

We note here that the creation and annihilation operators satisfying the specific deformed products given in Eqs. (87)–(90) [and hence the commutation relations in Eqs. (91) and (92)] are the ones appearing in the mode decomposition of the scalar field [see Eq. (78)] satisfying the generalized Klein-Gordon equation (37) with a particular choice $\varphi(A) = e^{-(A/2)}$ [see Eq. (77)]. Thus it is clear that even for the scalar field obeying the field equation (37), more general [i.e., for other choices of/arbitrary $\varphi(A)$] dispersion relations than those given in Eqs. (79) and (80) are possible. This will lead to more general twisted products than those given in Eqs. (74)–(90), leading to generalized commutation relations in place of those in

Eqs. (91) and (92). Thus the twisted products [see Eqs. (87)–(90)] obtained in [37,48] are only a *particular case* of more general products between $A^\dagger$ and A that are possible.

The creation and annihilation operators satisfying the above given deformed commutation relations are the ones appearing in the mode decomposition of the scalar field satisfying the generalized Klein-Gordon equation (37). This generalized Klein-Gordon equation is invariant under the κ-Poincaré algebra defined in Eqs. (25) and (26), in addition to the usual $so(n-1,1)$ commutation relations between $M_{\mu\nu}$. This should be contrasted with the approach taken in [37,48] where the generalized Klein-Gordon equation was invariant under the action of a *deformed* κ-Poincaré algebra. Irrespective of this, we have obtained the deformed commutation relations between $A^\dagger$ and A given in [37,48], as a special case.

V. CONCLUSION

In this paper, we have studied the construction of scalar theory on κ-Minkowski spacetime. It is known that this noncommutativity of the coordinates leads to twisted coproducts for the generators of the κ-Poincaré algebra. These twisted coproducts are necessary for the implementation of the symmetry algebra on multiparticle states. We have summarized briefly, the explicit form of the twisted coproducts for a class of realization of deformed κ-spacetime coordinates in terms of commuting coordinates and derivatives. Here, the momenta do not transform like a vector unlike in the case of undeformed κ-Poincaré algebra. This results in the noninvariance of the naive generalization of the generalized Klein-Gordon equation [see Eq. (22) under the action of the deformed κ-Poincaré algebra. We then introduced Dirac derivatives which transform as a vector under the deformed κ-Poincaré algebra. After obtaining the coproducts of the generators of this deformed Poincaré algebra and the Casimir, generalized Klein-Gordon equations which are invariant under this algebra are introduced. The requirement of invariance alone does not lead to a unique generalized Klein-Gordon equation. These generalized Klein-Gordon equations do have higher derivative terms with respect to time while one of them has quartic space derivatives [see Eq. (36)] while the second has quadratic space derivatives [see Eq. (37)]. We have then discussed the $*$ product naturally induced by the realization of κ-spacetime coordinates in terms of the commuting ones and derivatives. From this $*$ product, one can read off the twist element and we showed that it can be expressed in terms of a space dilation operator and time derivative. It is clear that the twisted coproducts obtained using this twist element are different from those of deformed κ-Poincaré algebra as well as those of the undeformed κ-Poincaré algebra defined using Dirac derivatives.

Our main results are discussed in Sec. IV. Here we have derived the flip operators compatible with the algebraic structure of the system. First, we have obtained the twisted flip operator, which is compatible with the twisted coproducts of the deformed κ-Poincaré algebra and then we derive the covariant flip operator, which is compatible with the coproducts of the undeformed κ-Poincaré algebra defined using Dirac derivatives. In both cases, we have obtained the R matrices corresponding to the deformed flip operators (up to first order in the deformation parameter). It is shown that in the first case, the twisted flip operator contains elements that do not belong to the set of the generators of the symmetry algebra. In contrast, the covariant flip operator up to the first order in the deformation parameter involves only the generators of the symmetry algebra. The calculation of the covariant R matrix to all orders in the deformation parameter is being investigated now. Whether the two different R matrices we obtained for the κ-Minkoswksi space are equivalent or not is under investigation.

We have then studied implications of the twisted flip operator on the statistics of the scalar field quanta, satisfying the generalized Klein-Gordon equation defined in the κ spacetime. We have shown that the algebra of creation and annihilation operators is deformed and we obtain this deformed algebra explicitly. We have also shown that this deformed algebra reproduces a known result for a specific choice of the realization of the κ-spacetime coordinates [37]. Our analysis however leads to a much wider and novel class of deformed oscillator algebras.

Finally, in the Appendix, we have discussed the $*$ product for the κ spacetime defined using vector fields and obtain the twisted coproducts of the symmetry algebra induced by this $*$ product. We show that this twisted coproduct is the same as that of the undeformed κ-Poincaré algebra generated by ∂_μ, and $\tilde{M}_{\mu\nu}$.

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APPENDIX: THE $*$ PRODUCT AND COPRODUCTS FROM VECTOR FIELDS

Here we discuss the $*$ product and the twist element for the κ-spacetime using the commuting vector field. Using this twist element, we then derive the twisted coproducts and show that they are the same as the ones we obtained in Eqs. (13), (14), (17), and (18), with a specific choice $\varphi(a)$.

In Sec. III, we have obtained the $*$ product induced on the commutative space by the mapping of the function from deformed k spacetime. The explicit form of this $*$ product is given in Eq. (47). The $*$ product was first introduced in quantum mechanics as a way to handle the ordering problems encountered in the passage from classical phase space to the quantum one (or in obtaining a map between functions on classical phase space to the corresponding quantum mechanical operators). The approach later lead to the development of deformation quantization. In the studies of noncommutative field theories on the Moyal plane, it was realized that the product of functions on the Moyal plane can be replaced with the $*$ product of function on commutative space. Later, it was shown that the compatibility requirement of the $*$ product and the action of symmetry generators introduces the twisted coproducts into the discussions of symmetries [15].

The $*$ product in the Moyal plane is constructed using the commuting generators of the symmetry algebra (namely Poincaré algebra). Later $*$ products were constructed for quantum spaces like $M(so_q(3))$, $M(so_q(1,3))$, etc. using commutative vector fields (which are not the generators of the symmetry algebra of the underlying space) [53]. For a pair of vector fields $\mathbf{X}$ and $\mathbf{Y}$, this $*$ product is given a series. An asymmetric $*$ product is defined by

$$f * g = \sum_{n=0}^{\infty} \frac{h^n}{n!}(\mathbf{X}^n f)(\mathbf{Y}^n g) \tag{A1}$$

and a symmetric one is defined as

$$f * g = \sum_{n=0}^{\infty} \frac{h^n}{2^n n!} \sum_{i=0}^{n} (-1)^n nC_i(\mathbf{X}^{n-i}\mathbf{Y}^i f)(\mathbf{X}^i \mathbf{Y}^{n-i} g). \tag{A2}$$

Using this approach, one can define a $*$ product between the coordinates leading to the commutation relations defining the κ spacetime given in Eq. (2). This product is defined in terms of the commuting vector fields $x_i\partial_i$ and ∂_0 as

$$f * g = f e^{-i(a/2)(x_i\partial_i \otimes \partial_0 - \partial_0 \otimes x_i \partial_i)} g. \tag{A3}$$

Here we have summed the series in Eq. (A2) to get the phase factor in the above equation. It is easy to verify that the above $*$ product gives Eq. (2) for commuting coordinates x_0, x_i.

As in the Moyal case, here too we can define the twisted coproduct in terms of the twist element $\mathcal{F}$. Noting $m_*(fg) = f * g = m(\mathcal{F} fg)$ and using Eq. (A3), we find $\mathcal{F} = e^{-i(a/2)(x_i\partial_i \otimes \partial_0 - \partial_0 \otimes x_i\partial_i)}$. Using $\Delta_t(g) = \mathcal{F}^{-1}\Delta(g)\mathcal{F}$, we find the twisted coproducts of the generators of the undeformed Poincaré algebra ∂_0, ∂_i, $\tilde{M}_{0i}$, and $\tilde{M}_{ij}$. We find

$$\Delta_t(\partial_0) = \Delta(\partial_0); \qquad \Delta_t(\tilde{M}_{ij}) = \Delta(\tilde{M}_{ij}) \tag{A4}$$

$$\Delta_t(\partial_i) = \partial_i \otimes e^{-((ia\partial_0)/2)} + e^{(ia\partial_0)/2} \otimes \partial_i \tag{A5}$$

$$\begin{aligned} \Delta_t(\tilde{M}_{i0}) = &-i[x_i\partial_0 \otimes e^{(ia\partial_0)/2} + e^{-((ia\partial_0)/2)} \otimes x_i\partial_0] \\ &- i[x_0\partial_i \otimes e^{-((ia\partial_0)/2)} + e^{(ia\partial_0)/2} \otimes x_0\partial_i] \\ &+ \frac{a}{2}[\partial_i \otimes x_j\partial_j e^{-((ia\partial_0)/2)} - x_j\partial_j e^{(ia\partial_0)/2} \otimes \partial_i]. \end{aligned} \tag{A6}$$

Exactly the same twisted coproducts result from $\mathcal{F}_\varphi^{-1}\Delta_0(\partial_\mu)\mathcal{F}_\varphi$, $\mathcal{F}_\varphi^{-1}\Delta_0(M_{\mu\nu})\mathcal{F}_\varphi$, where Δ_0 is the coproduct of the undeformed Poincaré generators with the choice $\varphi(a) = e^{-((ia\partial_0)/2)}$ [for the derivative operators these are the same as the ones given in Eqs. (13) and (14)]. We note that the $*$ product defined using the commutating vector fields in Eq. (A3) is same as the one we obtained in Eq. (47) and the twist element is exactly the same as that in Eq. (48) with the choice $\varphi(a) = e^{-((ia\partial_0)/2)}$. Thus it should not be surprising that these two different approaches lead to the same twisted coproducts. But it is clear the $*$ product defined in Eq. (47) is more general as it reduces to Eq. (A3) only for the choice $\varphi(a) = e^{-((ia\partial_0)/2)}$. We also note that the twisted coproducts derived above are given in terms of operators that are not in κ-Poincaré algebra.

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