

Categorified symmetries *

Urs Schreiber [†]

Department of Mathematics, Utrecht University
P.O. Box 80010, 3508 TA Utrecht
Budapestlaan 6, 3584 CD Utrecht
The Netherlands

and

Department Mathematik, Universität Hamburg
Bundesstrasse 55, D-20146 GERMANY

Zoran Škoda [‡]

Division of Theoretical Physics, Institut Rudjer Bošković
P.O.Box 180, HR-10002 Zagreb, CROATIA

ABSTRACT

Quantum field theory allows more general symmetries than groups and Lie algebras. For instance quantum groups, that is Hopf algebras, have been familiar to theoretical physicists for a while now. Nowadays many examples of symmetries of categorical flavor – categorical groups, groupoids, Lie algebroids and their higher analogues – appear in physically motivated constructions and facilitate constructions of geometrically sound models and quantization of field theories. Here we consider two flavours of categorified symmetries: one coming from noncommutative algebraic geometry where varieties themselves are replaced by suitable categories of sheaves; another in which the gauge groups are categorified to higher groupoids. Together with their gauge groups, also the fiber bundles themselves become categorified, and their gluing (or descent data) is given by nonabelian cocycles, generalizing group cohomology, where ∞ -groupoids appear in the role both of the domain and the coefficient object. Such cocycles in particular represent higher principal bundles, gerbes, – possibly equivariant, possibly with connection – as well as the corresponding *associated* higher vector bundles. We show how the Hopf algebra known as the Drinfeld double arises in this context.

This article is an expansion of a talk that the second author gave at the *5th Summer School of Modern Mathematical Physics* in 2008.

* Work supported by DAAD Germany-MSES Croatia bilateral project.

[†] e-mail address: urs.schreiber@googlemail.com

[‡] e-mail address: zskoda@irb.hr

Contents

1. Introduction	3
1.1. Categories and generalizations	4
1.2. Basic idea of descent	5
2. From noncommutative spaces to categories	5
2.1. Idea of a space and of a noncommutative space	5
2.2. Gel'fand-Naimark	6
2.3. Nonaffine schemes and gluing of quasicoherent sheaves . . .	6
2.4. Noncommutative generalizations of $Qcoh_X$	7
2.5. Abelian versus ∞ -categories	7
3. Monoidal categories as symmetries of NC spaces	8
3.1. Basic appearances of Hopf algebras	8
3.2. A problem with tensor product	9
3.3. Replacing Hopf (co)actions with geometrically admissible ac- tions of monoidal categories	9
3.4. Principal bundles on noncommutative schemes	11
4. Application to Hopf algebraic coherent states	12
5. Higher gauge theories	13
6. ∞-Categories and homotopy theory	13
6.1. ∞ -Categories versus model categories	14
6.2. Generalized spaces, topoi and (higher) categories	15
6.3. Strict ω -Groupoid-valued ∞ -stacks	16
7. Nonabelian cohomology, higher vector bundles and back- ground fields	17
7.1. Principal ∞ -bundles	20
7.2. Associated ∞ -bundles	24
7.3. Sections of associated ∞ -bundles	26
7.4. Connections on ∞ -bundles	27
7.4.1. The homotopy ∞ -groupoid	27
7.4.2. The geometric path ∞ -groupoid	31
7.4.3. Differential cocycles and connections	33
8. Quantization and quantum symmetries	36
8.1. Background field and space of states	37
8.2. Transgression of cocycles to mapping spaces	37
8.3. Branes and bibranes	38
8.4. Quantum propagation	39

9. Examples and applications	40
9.1. Ordinary vector bundles	40
9.2. The charged quantum particle	41
9.3. Group algebras and category algebras from bibrane monoids	42
9.4. Monoidal categories of graded vector spaces from bibrane monoids	43
9.5. Twisted vector bundles	43
9.6. Dijkgraaf-Witten theory	44
9.6.1. The 3-cocycle	45
9.6.2. Transgression of DW theory to loop space: the twisted Drinfeld double	47
9.6.3. The Drinfeld double modular tensor category from DW bibranes	48
9.6.4. The fusion product	49
9.7. Outlook: Chern-Simons theory	50
10. Conclusion	51

1. Introduction

The first part of this article is an overview for a general audience of mathematical physicists of (some appearances of) categorified symmetries of geometrical spaces and symmetries of constructions related to physical theories on spaces. Our main emphasis is on geometric and physical motivation, and the kind of mathematical structures involved. Sections 2-4 treat examples in noncommutative geometry, while 5-6 introduce nonabelian cocycles motivated in physics.

In sections 6-9 we discuss some technical details concerning differential cocycles and their quantization; part of these sections can be understood as a research announcement.

WARNING on versions: The original version of this article has been submitted in December 2008, and appeared in *5th Summer School of Modern Mathematical Physics*, SFIN, XXII Series A: Conferences, No A1, (2009), 397-424 (Editors: Branko Dragovich, Zoran Rakić). In this arXiv version we have slightly updated some introductory points, and in particular the subsections 2.5. and 7.2. are entirely new. Section 7.1. on connections on principal ∞ -bundles is also new and serves to provide some more background for the examples in section 9, where for instance the discussion of the electromagnetically charged quantum particle from a categorical perspective is new and the whole subsection 9.7. on Chern-Simons theory. We have also appended the list of additional (mainly new) references alphabetically just below the original references [1-48]. One should especially mention the important reference [arXiv:0905.0731 \[FHLT\]](#) which touches on similar issues of categorical foundations of quantum physics as the the work sketched here and in the larger manuscript [36], also from December 2008. We should also note that since publication many aspects of this and

related work were discussed or presented in the online project *nlab* [66] in which we are participating.

1.1. Categories and generalizations

We assume the reader is familiar with basics of the theory of categories, functors and sheaves, as the mathematical physics community has adopted these by now. At a few places for instance we use (co)limits in categories. Readers familiar with enriched and higher category theory ([3, 24, 65, 72]) can skip this subsection.

The concept of a category $\mathcal{C}$ is often extended in several directions [2, 4, 24], leading to the internal categories, internal groupoids, monoidal categories, enriched categories, strict n -categories, and various flavours of weak higher categories. We will just sketch the terminology for orientation.

Instead of a *set* $\mathcal{C}_1 = \text{Ob}\mathcal{C}$ of objects and set $\mathcal{C}_0 = \text{Mor}\mathcal{C}$ of morphisms, with the usual operations (assignment of identity $i : X \mapsto \text{id}_X$ to X ; domain (source) and codomain (target) maps $s, t : \mathcal{C}_1 \rightarrow \mathcal{C}_0$; composition of composable pairs of morphism $\circ : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$) one defines an **internal category** in some ambient category $\mathcal{A}$ by specifying *object of objects* $\mathcal{C}_0$ and *object of morphism* $\mathcal{C}_1$ which are both objects in $\mathcal{A}$, together with morphisms $i, s, t, \circ$ as above, and satisfying analogous relations. An **internal groupoid** is an internal category equipped with an inverse-assigning morphism $(\cdot)^{-1} : \mathcal{C}_1 \rightarrow \mathcal{C}_1$ satisfying the usual properties. For instance smooth groupoids (Lie groupoids) are internal groupoids in the category of manifolds [2, 14, 23, 40]). A category may be given additional structure, e.g. a **monoidal category** is equipped with tensor (monoidal) products and tensor unit object (cf. [4, 24, 29] and section 3.). Given a monoidal category $\mathcal{D}$, a **$\mathcal{D}$ -enriched category** $\mathcal{C}$ has a set of objects, but each set of morphisms $\text{hom}_{\mathcal{C}}(A, B)$ is replaced by an object D in $\mathcal{D}$; it is required that the composition be a monoidal functor. In particular $\mathcal{D}$ may be the category of small categories, in which case a $\mathcal{D}$ -enriched category is precisely a 2-category: it has morphisms between morphisms. This process may be iterated and leads to n -categories of various flavour, with n -morphisms or n -cells as morphisms between $(n-1)$ -morphisms. A strict $(n+1)$ -category is the same as $n\text{Cat}$ -enriched category where $n\text{Cat}$ is the category of strict n -categories and strict n -functors. If the cells for all $n \geq 0$ are allowed we are dealing with ω -categories.

It is natural to weaken the associativity conditions for compositions of k -cells for $0 < k < n$. This weakening is difficult to deal with, and there are multiple definitions, but this weakening is often naturally arising in applications and is more natural from the point of view of category theory itself. Thus one can talk about weak n -categories [3, 24, 65, 72].

The weakening is much easier if the higher cells are invertible – these are by definition the $(n, 1)$ -categories in the sense of Baez and Dolan, including the case of $(\infty, 1)$ -categories, which are of central importance in applications. More generally, we may talk on (n, k) -categories, of (in general, weak) n -categories only r -cells for $r > k$ are invertible, and in particular of (∞, k) -categories.

According to Grothendieck's **homotopy hypothesis** (from [58], ex-

plained also in [3, 65, 24, 72]), $(\infty, 0)$ -categories, i.e. weak ∞ -groupoids are equivalent to topological spaces; in particular $(\infty, 1)$ -categories can be modelled as categories enriched over a convenient category Top of topological spaces. Alternatively, instead of describing combinatorially n -cells and the algebra of various compositions among them, one can model $(\infty, 1)$ -categories as simplicial sets satisfying the inner Kan conditions which are certain existence properties which together replace the algebraic structure of higher compositions. This model is also known under the name of **quasi-categories**. Most recently, THOMAS NIKOLAUS ([67]) has found a mixture of algebraic and simplicial definition, in which the existence is accompanied with additional choices making the comparison between the topological and algebraic models of higher categories more transparent.

The language of Quillen model categories helps to compare various models for ∞ -categories, see 6.1.

1.2. Basic idea of descent

Suppose we are given a geometric space and its decomposition in pieces with some intersections, e.g. an open cover of a manifold. The manifold can be reconstructed as the disjoint union modulo the identification of points in the pairwise intersections. For this we need to specify the identifications explicitly, and they may be considered as additional data. Suppose we now want to glue not the underlying sets, but some structures above, e.g. vector bundles. Bundles on each open set U_α of the cover form a category Vec_α and there are restriction functors from Vec_α to the 'localized' category of bundles on $U_{\alpha\beta} := U_\alpha \cap U_\beta$. A global bundle F is determined by its restrictions F_α to each open set U_α of the cover, together with identifications

$(F_\alpha)|_{U_{\alpha\beta}} \xrightarrow{f_{\alpha\beta}} (F_\beta)|_{U_{\alpha\beta}}$ via some isomorphisms $f_{\alpha\beta}$. These isomorphisms satisfy the *cocycle condition* $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ and $f_{\alpha\alpha} = \text{id}$. The data $\{F_\alpha, f_{\alpha\beta}\}$ are called descent data ([19, 9, 47]). Equivalence classes of descent data are cohomology classes (with values in the automorphism group of the typical fiber) and they correspond to isomorphism classes bundles over the base space. There are vast generalizations of this theory, cf. [19, 9, 46, 36, 40, 41]. Gluing categories of quasicoherent sheaves/modules (see section 2. on their role) over noncommutative (NC) localizations which replace open sets ([34, 33, 44]) is a standard tool in NC geometry. Localization functors Q_α for different α , usually do not commute, what may be pictured as a noncommutativity of intersections of 'open sets'. In order to reconstruct the module from its localizations (restrictions to localized "regions") we need match at *both* consecutive localizations, $Q_\alpha Q_\beta$ and $Q_\beta Q_\alpha$.

2. From noncommutative spaces to categories

2.1. Idea of a space and of a noncommutative space

By a noncommutative (NC) space ([14, 34, 44]) we mean any object for which geometrical intuition is available and whose description is given by the data pertaining to some geometrical objects living on the 'space'. Suppose we measure observable corresponding to some property depending on

a local position in space. If the position changes from one part to another part of a space, we get different measurements, thus the measurements are expected to be functions of the local position. If the space is made out of points and we can make measurements closely about each point, then we get a function on the underlying set of points. This corresponds to the observables on phase space in classical physics; the quantum physics and noncommutative geometry mean that we can not decompose some 'spaces' to points, hence we can not really construct set-theoretic functions. Still, one can often localize observables to some geometrical 'parts' if not points.

2.2. Gel'fand-Naimark

The most standard case is when the 'space' is represented by a $\mathbb{C}$ -valued algebra A . If A is commutative then the points of the space correspond to the characters (nonzero homomorphisms $\chi : A \rightarrow \mathbb{C}$), or equivalently, to maximal ideals $I = \text{Ker} \chi$ in A . Knowing all functions at all points, physically means being able to measure all local quantities, and mathematically expresses the GEL'FAND-NAIMARK theorem: from the C^* -algebra of continuous $\mathbb{C}$ -valued functions on a compact Hausdorff space we can reconstruct back the space as the Gel'fand spectrum of A ([14, 23]). The Gel'fand spectrum can be constructed for NC algebras as well, but in that process we lose information and get smaller commutative algebras – the spectrum is roughly extracting the points, together with some topology, and there are not sufficiently many points to determine the NC space. Instead one is trying to express the geometrical and physical constructions we need in terms of algebra A , at least for good A -s of physical interest. This strategy usually works e.g. for small NC deformations of commutative algebras. Thus such a *quantum algebra* is by physicists usually called a NC space. We emphasise that there are more general NC spaces and more general types of their description.

2.3. Nonaffine schemes and gluing of quasicohherent sheaves

We often know how the local coordinate charts look like and glue them together. The global ring is in principle sufficient information in C^* -algebraic framework, but many constructions are difficult as one has to make correct choices in operator analysis. Thus sometimes one resorts to algebraic geometry, that is algebras of regular (polynomial) functions; but even commutative algebraic variety/scheme X is not always determined by its ring of global regular functions $\mathcal{O}(X)$. Even if it is, we may find convenient to glue together more complicated objects (say fiber bundles) over the space from pieces. One way or another, we need to glue the spaces represented by algebras of functions to an object which will not lose information as the global coordinate ring sometimes does. If we have some sort of a cover of the space by collection of open sets U_i where on each U_i the algebra of functions determines the space, then having all of them together conserves all local information; moreover we should be able to pass to other open sets. Thus one needs a correspondence which to every open set gives an algebra of observables, that is some **sheaf** of functions in the case of commutative space; to do the same for fibre bundles means that we need to do the same

for sheaves of sections of other bundles. It seems reasonable to take a **category of all sheaves of suitable kind on the space as a replacement of space**. This point of view in geometry was advocated by A. GROTHENDIECK in 1960-s (geometry of toposes). GABRIEL-ROSENBERG's theorem states that every algebraic scheme X (typical geometrical space in algebraic geometry) can be reconstructed, up to an isomorphism of schemes, from the abelian category $\mathbf{Qcoh}_X$ of quasicoherent sheaves on X ([34]). For noetherian schemes a smaller (tensor abelian) subcategory $\mathbf{Coh}_X$ of coherent sheaves is enough, and in some cases even its derived category $D^b(\mathbf{Coh}_X)$ in the sense of homological algebra ([31]).

2.4. Noncommutative generalizations of $Qcoh_X$

Examples suggest that instead of small deformations of commutative algebras of functions, we may consider deformations and similarly behaved analogues of categories $\mathbf{Qcoh}_X$. Principal examples appeared related to mirror symmetry. Mirror symmetry is a duality involving two Calabi-Yau 3-folds X and Y , saying that $N = 2$ SCFT-s A-model on X and B-model on Y and viceversa, A-model on Y and B-model on X are (nontrivially) equivalent as $N = 1$ SCFT-s (the difference at $N = 2$ level is in a ± 1 eigenvalue of an additional $U(1)$ -symmetry operator, what is physically not distinguishable). In 1994, MAXIM KONTSEVICH proposed the homological mirror symmetry conjecture [27], which is an equivalence of A_∞ -categories related to topological A- and B-models. In A-model, the A_∞ -category involved is the *Fukaya category* defined in terms of symplectic geometry on X (Y), and B-model is the A_∞ -enhancement of the derived category of coherent sheaves on Y (X). Kontsevich also suggested a definition of a category of B-branes in $N=2$ *Landau-Ginzburg models* ([21, 26]) which have very similar structure to, but are *different* from, the derived categories of coherent sheaves on quasiprojective varieties. There are known relations between Hochschild cohomology (expressed in terms of $D^b(\mathbf{Coh}_X)$) and n -point correlation functions in the corresponding SCFT. Around 2003, Kontsevich and, independently K. COSTELLO, found a way to go back and reconstruct SCFT from sufficiently good, but abstract A_∞ -categories [15, 26], where 'good' involves generalizations of certain properties of (A_∞ enhancement of) $D^b(\mathbf{Coh}_X)$ where X is a Calabi-Yau variety. This shows that indeed physically relevant generalizations and deformations of varieties of complex algebraic geometry may come out of generalizations of algebraic geometry in terms of categories of sheaves and their abstract generalizations.

2.5. Abelian versus ∞ -categories

While Abelian categories of quasicoherent sheaves ([68, 34]) contain all the information on a variety, this is not always true for their derived categories, or enhanced versions of those ([54, 62, 63]). Recent years have witnessed a subject of **derived algebraic geometry** (DAG, cf. [63, 26, 74, 75]), which helped redefine many constructions in the usual geometry in a fruitful and natural manner. The geometric spaces are there understood as higher categorical entities; in the functor of points view on schemes, the commutative

rings as (opposite to) local models are replaced by commutative dg- or simplicial rings, and presheaves of sets by presheaves of simplicial sets. This allows for notions of constructions up to coherent homotopy, and in particular of derived version of many natural constructions and functors. In particular many new moduli spaces are constructed in this framework.

While the framework of DAG has extended the *commutative* algebraic geometry and can treat the usual schemes as special case of derived schemes, the passage from the representation of the category of quasicoherent sheaves to its derived version loses some information in general. There are several ways to do a derived category; it is well known that the usual triangulated version is bad for many reasons (including the nonfunctoriality of cones) and that one can replace it by an enhanced version. The enhanced versions are very natural from the categorical point of view but even they do not contain full information which the abelian category of quasicoherent sheaves have; all of them are just a derived version of a usual scheme. While the usual schemes can be embedded into the bigger world of derived schemes, the replacement of the abelian category of quasicoherent sheaves by the enhanced triangulated category is not a faithful functor.

There are several versions of enhancements of a triangulated category; that is their replacements by a pretriangulated differential graded category, by pretriangulated A_∞ -category or by a stable ∞ -category ([54, 61, 53]). In characteristics zero all of the three approaches are equivalent. Being pretriangulated, or stable are properties rather than structures.

Now, coming back to noncommutative geometry. The replacement of an algebraic variety by a stable ∞ -category or say A_∞ version of it, can also be fitted into the functor of points point of view. Then there is no difference in formalism, between the derived algebraic geometry based on simplicial presheaves on derived affine schemes as with the derived geometry based on representing spaces by stable infinity categories. Thus there is no significant difference between derived commutative and derived noncommutative geometry; while there is a more serious difference between the commutative and noncommutative geometry at the level of abelian categories of sheaves, seen for example in localization theory of noncommutative algebras which are large in the sense "close to free algebras".

3. Monoidal categories as symmetries of NC spaces

3.1. Basic appearances of Hopf algebras

The role of **symmetry** objects extends to the NC world: they help us singling out good candidates for the underlying space-time of a theory, and one employs the covariance properties of the tensors built out of field variables when constructing model Lagrangeans. In QFT, one wants not only that the fields form a representation of a symmetry algebra, but also to describe the second quantized systems, where the Hilbert space $\mathcal{H}$ is replaced by its exponent – the direct sum of n -particle Hilbert spaces, for all n , which are (in bosonic case, for simplicity) the symmetric powers of 1-particle Hilbert space $\mathcal{H}$. Thus the symmetry has to be defined on tensor products of representation spaces (classical example: addition of angular momenta of sub-systems). **Hopf algebras** have the structure sufficient to define the tensor

product of representations and the dual representations ([28, 29]), and each finite group gives rise to a Hopf algebra (“group algebra”) with the same representation theory. Locally compact groups considered in axiomatic QFT, may also be generalized to Hopf algebra-like structures called locally compact quantum groups. If the underlying space is undeformed and in 4D, the axiomatic QFT actually proves that the full symmetry is described by a locally compact group, but in dimension 2, exotic braiding symmetries and quantum groups are allowed; in NC case a generic model will have a nonclassical symmetry. The symmetry algebra is here understood in the usual sense – consisting of all observables commuting with the Hamiltonian. The natural Hamiltonians preserve the symmetries of the underlying space geometry, but there are often other symmetries which are not the symmetries of the underlying space; there are also hidden symmetries not seen at Hamiltonian level, but only in solutions.

Now we want to discuss the geometrical symmetries of “bare” underlying noncommutative space. For this we need to discuss more carefully a role of Hopf algebras. Recall ([29]) that a Hopf algebra H is an associative unital algebra equipped with a counital coassociative coproduct $\Delta : H \rightarrow H \otimes H$ which is a morphism of algebras. Starting with any (say finite) group one may form its group algebra $\mathbb{C}G$, which is a Hopf algebra whose representations coincide with the representations of the group. On the other hand, the group itself may be replaced by a suitable algebra of functions $\mathcal{O}(G)$ on it, and then the corepresentations of $\mathcal{O}(G)$ (linear maps $\rho : V \rightarrow V \otimes \mathcal{O}(G)$ with $(\rho \otimes \text{id})\rho = (\text{id} \otimes \Delta)\rho$) will correspond to the representations of G . $\mathcal{O}(G)$ is commutative, and one may consider noncommutative Hopf algebras instead; and they are in abundance ([29]), since the discovery of quantum groups.

3.2. A problem with tensor product

Mathematically, replacing the commutative algebras, by the noncommutative, one should change the tensor product of commutative algebras (a *categorical* coproduct in the category of commutative algebras) by the so-called **free product** of noncommutative algebras, in all considerations. This would yield straightforward transfer of many constructions and their properties. However, examples of Hopf algebras with respect to the categorical coproduct are just few, while usual NC Hopf algebras with respect to $\otimes$ (e.g. quantum groups) are abundant in physical applications. Many practitioners ignore these facts and simply work with the usual tensor product; however we suggest a better understanding of this situation ([45]) from the perspective of categories of modules (quasicoherent sheaves) rather than algebras.

3.3. Replacing Hopf (co)actions with geometrically admissible actions of monoidal categories

Recall that in the commutative case, $A\text{-Mod}$ is equivalent to $\mathbf{Qcoh}_{\text{Spec } A}$, and that we took a viewpoint that the categories like $\mathbf{Qcoh}_X$ are representing spaces. $H\text{-Mod}$ and $H\text{-Comod}$ where H is a Hopf algebra are rigid

monoidal categories. A **monoidal category** is a category equipped with a bifunctor $\otimes$ (monoidal or tensor product), which is associative up to coherent isomorphisms $M \otimes (N \otimes P) \cong (M \otimes N) \otimes P$, this category has a unit object $\mathbf{1}$ (satisfying $\mathbf{1} \otimes M \cong M \cong M \otimes \mathbf{1}$) and this category has dual objects M^* with usual properties (rigidity/autonomous category). Not only $\mathbf{Qcoh}_X$ remembers scheme X (Gabriel-Rosenberg theorem), also in favorite cases $H\text{-Mod}$ as a rigid monoidal category remembers the underlying Hopf algebra (or Hopf algebroids, appearing as symmetries of inclusions of factors [8], relevant to CFT): this is an aspect of so-called Tannakian duality used widely in physics, e.g. the Doplicher-Roberts duality dealing with reconstruction of a QFT in 4d out of knowledge of full symmetry algebra is also a form of Tannaka reconstruction theorem; some reconstructions in CFT are as well ([29], Ch.9).

The reason why the usual Hopf algebras geometrically still fit into NC world is that the Hopf actions $H \otimes A \rightarrow A$ (i.e. when A is H -module algebra [29]) or Hopf coaction $A \rightarrow A \otimes H$ (A is H -comodule algebra), induce an action of the **monoidal category** of H -comodules (1st case) or H -modules (2nd case) on $A\text{-Mod}$ (cf. [45] for recipes how to induce the categorical actions in these cases). Hopf coaction is hence replaced by an action bifunctor $\diamond : A\text{-Mod} \times H\text{-Mod} \rightarrow A\text{-Mod}$. The action axiom is the mixed associativity with product $\otimes$ in $H\text{-Mod}$, namely $M \diamond (N \diamond P) \cong (M \otimes N) \diamond P$. Replacing the Hopf algebra H by its monoidal category of left modules $H\text{-Mod}$, we can as well, replace the Hopf coaction of H on A by the corresponding action $\diamond$ of $H\text{-Mod}$.

While in these affine cases (co)actions of Hopf algebras, induce for noncommutative geometry more fundamental categorical actions, in some more general nonaffine situations the usual (co)actions of Hopf algebras do not make sense and need to be replaced by a categorical device anyway. For example, if we want to globalize the action of Hopf algebra to nonaffine noncommutative varieties, then the latter may not be represented by a single algebra, but rather by the gluing data for several algebras. As the (co)action usually does not make the affine pieces invariant, one really needs to talk about a coaction of Hopf algebra on the entire category of sheaves glued from pieces. But such coaction has no sense literally, unless we replace the Hopf algebra by its monoidal category of modules as well which can act in the categorical sense.

One should emphasise that not all actions of monoidal categories are good in this framework. A monoidal category usually has some origin, which is understood over the ground scheme. For example, the category of modules over a Hopf algebra H over a commutative ground ring k , knows that its Hopf algebra structure is made sense of by the tensor product in the category of k -modules. In particular, the tensor product on the category of H -modules exists because the coproduct of H is defined using the tensor product of k -modules. The category of H -modules naturally acts on the category of k -modules just by forgetting the H -module structure and tensoring over k . Now any action on a category of quasicoherent sheaves over some noncommutative k -scheme must be compatible with that original "defining" action in the sense that the direct image functor to the ground scheme (represented by the category of k -modules) intertwines the actual

action and the canonical action on the ground scheme. Abstract actions of monoidal categories, satisfying this property were defined in our earlier paper [45] as **geometrically admissible actions**.

3.4. Principal bundles on noncommutative schemes

We now have to answer which geometrically admissible actions are principal in the noncommutative setup in which the space is replaced by a category of would-be quasicoherent sheaves; what are locally trivial bundles and how can they be expressed by the cocycle like data. This is partly understood in the categorical framework, but not widely known.

Geometrical admissibility of actions of monoidal category in the case when both the total space and the base "quotient" space of the action are affine can be understood as a construction of the action by **lifting** the canonical action on the base to the total space. In category theory, such lifts were studied for (co)monads and lead to the concept of a distributive law. One of the authors has worked out a slightly more general case when the monad or comonad is replaced by a monoidal category ([73]). It is interesting that in the construction of *examples* of noncommutative ("quantum") principal bundles T. BRZEZIŃSKI and S. MAJID rediscovered the notion of mixed distributive laws under the name of **entwining structures** ([55]).

For principality, a minimal reasonable requirement is that the category of equivariant sheaves on the total space is canonically equivalent (via a descent theorem) to the category of usual sheaves on the base space. It has been shown by V. LUNTS and the second author that in the case of coaction of Hopf algebras, the relative Hopf modules are in very literal sense (phrased in terms of sheaves on noncommutative analogue of a simplicial object, namely the Borel construction) equivariant sheaves (a summary of that interpretation is treated in [45]). And indeed, the Schneider's theorem for Hopf-Galois extensions and its various generalizations, including for the distributive laws are then the descent along torsor theorems. Surely the natural topology for such theorems in the commutative case is the flat topology, while Zariski principal bundles are very special.

Flat noncommutative localizations form an analogue of Zariski topology in the noncommutative case. There are some substantial differences in the formulation of the descent in that case, in respect to the case of the usual Grothendieck topologies (see our survey [44]) which we for simplicity ignore in this article. Still the descent for such a noncommutative topology can be formulated and effectively used in some favorable cases. But it appears that the local situation is in such cases usually neither the tensor product, nor the free product of the base algebra and the fiber. Instead, the most frequent case is a special case of Hopf-Galois extensions which is algebraically similar to semidirect products of groups, namely the Hopf **smash product** of noncommutative algebras, which indeed has many important features of a "trivial principal bundle", though there exist more than one trivial principal bundle in general. For example, one of the authors has shown ([42]) that the quantum group analogue of the Gauss decomposition induces a local trivialization in the sense of noncommutative Ore localizations and smash products of the q -deformation of the fibration $SL_n \rightarrow SL_n/B$ expressed in

terms of quantized algebras of functions.

More generally, one can consider morphism $E \rightarrow X$ of noncommutative schemes ([33]), represented by abelian categories $\mathcal{C}_E, \mathcal{C}_X$, with an action of a monoidal category $\mathcal{M}$ of modules over a Hopf algebra, together with an atlas of localizations on $\mathcal{C}_X$, which are compatible in the sense that there is an induced action of the monoidal category on the localizations ([45]); and then if the localizations are affine, then one tests locally if the action is Hopf-Galois or a distributive law analogue of Hopf-Galois. At least such cases deserve to be called principal actions, and some slight generalizations are not difficult to define.

Moreover one can define associated bundles in some of these situations. In the case of coactions of Hopf algebras, the sections of the associated bundles locally in affine charts boil down to the cotensor product of the type $\mathcal{E} \otimes V$ where $\mathcal{E}$ is the total algebra of the principal bundle and V is a left comodule over the symmetry Hopf algebra ([42, 43]). For some reason in the literature there is almost no study of the global algebras of associated bundles but rather most of the spaces of sections in affine case; we will address this question in detail in another publication.

4. Application to Hopf algebraic coherent states

In the classical case of compact Lie groups (and some other classical generalizations), there is a projective operator valued measure on the space of coherent states (CS) which integrates to a constant operator on the Hilbert space corresponding to a unitary representation of the compact Lie group or to a space of sections of certain line bundle over a generalization of the homogenous space (from the Lie group case): the CS are not mutually orthogonal but they still play role in a resolution of unity operator formula. If the Hilbert space is a playground of some quantum mechanical situation, the Schrödinger equation can thus be written in CS representation. In addition CS have a number of special properties. Perelomov CS minimize generalized (covariant) uncertainty relations and transform in an appropriate covariant manner. Tensor operators of various “spin” may be treated simultaneously by forming CS operators, what is useful for discussing QFT on homogeneous spaces. I. TODOROV with collaborators ([20, 37]) has been taking advantage of CS in formulating gauged WZNW models in Hamiltonian formalism; but their CS are attached to quantum groups (cf. [28] for variants of Hopf algebras in 2dCFT context) whose general and, particularly, geometric theory was lacking; the open problem was to extend the “projective CS measure” to the quantum group case (existing formulas in simple cases in literature were just formal identities and usually the claimed invariance is incorrect). But Todorov et al. did not develop an appropriate geometric theory of coherent states.

Motivated by ([20, 37]), one of us has shown in [43] that using non-commutative localization and gluing one can study the geometry of line bundles over the quantum group homogeneous spaces and express the correct algebraic conditions for the analogues of Perelomov CS and of the invariant “projective” CS measure. Local coordinates on quantum G/B are described as a noncommutative principal fibration using categorical picture

with action-compatible noncommutative localizations and a smash-product picture in charts. In practice the construction of coordinates on the quotient space, boils down to gluing of the algebras of localized coinvariants (under the localized coaction of quantum Borel subgroup B_q) in coaction-compatible localized charts on G_q ([42]).

5. Higher gauge theories

An n -groupoid is a n -category in which all k -cells for all $1 \leq k \leq n$ are invertible (depending on the choice of context, this means strictly invertible or weakly invertible, i.e. up to higher cells). An n -group is a one-object n -groupoid. Smooth n -group(oid)s appear as analogues of Lie gauge groups for parallel transport along higher dimensional surfaces. One can build a theory of bundles with total space (which is now replaced by smooth n -category), possessing local trivialization and differential forms which are analogues of connection forms; two-torsors of [30], principal bigroupoid 2-bundles of [5] and gerbes [9] are examples. The cocycle data of a gerbe may be used to twist usual bundles and constructions with bundles, e.g. to get twisted K-theory. Instead of looking at the total space and differential forms, one may instead consider the effect of parallel transport to the points in a typical fiber. Thus an n -bundle with connection gets replaced by transport n -functor from some groupoid corresponding to the path geometry of the underlying space (fundamental n -groupoid, path n -groupoid) to the symmetry object of the fiber. The same formalism may incorporate gluing of (hyper)covers, by replacing the path groupoids with Čech n -groupoids of hypercovers: corresponding n -functors into symmetry n -group(oid) are the appropriate cocycles. On the other hand, the fiber bundles may be pulled back from the universal bundle over the classifying space of the group; the classifying space $\mathcal{B}G$ of an n -group corresponds to regarding the n -group G as a one-object $(n+1)$ -groupoid $\mathbf{B}G$. Thus in the next few sections we view cocycles as some weak maps into $\mathbf{B}G$. In general we find it useful to employ some abstract homotopy theory of a certain model of ∞ -categories described in section 6. to express what sort of “weak maps” the cocycles really are (for more details see [36]). We present two collections of definitions, one encoding the theory of nonabelian cocycles and ∞ -bundles in section 7., the other describing aspects of the quantum theory of the corresponding σ -models in section 8. Some examples and applications are in section 9.

6. ∞ -Categories and homotopy theory

While for a long time definitions of ∞ -categories were notorious for not being ready for showtime, recently André Joyal’s several decades-old suggestion that there *is* good $(\infty, 1)$ -category theory with good explicit incarnations in terms of simplicial sets and simplicially enriched categories has been fully realized and now provides a fully-fledged context in which to do higher category theory. A good deal of the full picture was clarified in [65], building on previous and ongoing work by many authors. The reader is referred to this reference for a comprehensive account of most of the technical concepts that we will invoke in the following sections, notably to the

appendix for a survey of category theory, simplicially enriched category theory, model category theory and its relation to higher category theory. We now try to briefly survey some aspects, to provide some kind of indication of the background for the technical discussion to follow.

6.1. ∞ -Categories versus model categories

This subsection is about the technical point of modelling ∞ -categories, and may be skipped in the first reading.

A Kan complex is a simplicial set whose k -cells may be thought of as k -morphisms in an ∞ -groupoid: the Kan condition ensures that these k -morphisms may be composed and have inverses.

An $(\infty, 1)$ -category is like an ∞ -groupoid, only that the 1-morphisms are not required to be necessarily invertible. Accordingly there is a slightly weakened version of the Kan condition, and simplicial sets satisfying that condition were called *weak Kan complexes* by BOARDMAN and VOGT. A. JOYAL fully realized that this is a good model for $(\infty, 1)$ -categories and introduced the term *quasicategory* ([60]) for these simplicial sets. This is a powerful model in that it allows to use many tools from simplicial homotopy theory for the study of $(\infty, 1)$ -categories.

A closely related and equivalent incarnation of $(\infty, 1)$ -categories is given by ordinary enriched categories, enriched in Kan complexes. This is a powerful, too, because it allows to apply many tools from enriched category theory. There is an operation called the *homotopy coherent nerve*, which takes a Kan-complex enriched category to a quasi-category. In practice this is used to pass back and forth between the two incarnations at will. There are other models, such as Segal categories and complete Segal spaces, too.

A particularly powerful additional toolset for presenting and handling $(\infty, 1)$ -categories is the old language of Quillen model categories, which deals with ordinary categories some of whose morphisms are marked in a way that indicates their hidden $(\infty, 1)$ -categorical origin: notably a model category has a singled out class of morphisms called weak equivalences, which need not be invertible in the category, but are supposed to be equivalences in the $(\infty, 1)$ -category presented by the model category.

Not all $(\infty, 1)$ -categories arise from model categories, but all $(\infty, 1)$ -categories do arise from categories with just weak equivalences ([51]): every category with weak equivalences determines, by a procedure called Dwyer-Kan simplicial localization at these weak equivalences, an $(\infty, 1)$ -category, and all $(\infty, 1)$ -categories arise this way. But if the category with weak equivalences does carry in addition the structure of a simplicial Quillen model category, then the corresponding $(\infty, 1)$ -category may more easily be expressed as simply the full $\mathbf{sSet}$ -subcategory on all those objects that are marked in the model structure as being *fibrant* and *cofibrant*.

The hom-simplicial sets of a simplicial model category are necessarily Kan complexes, hence ∞ -groupoids. If moreover the model category is combinatorial (meaning that there is particularly good control over its cofibrations), then the $(\infty, 1)$ -category obtained this way is (locally) *presentable* [65]: it is a reflective sub- $(\infty, 1)$ -category (a localization) of an $(\infty, 1)$ -category of $(\infty, 1)$ -presheaves. That every presentable $(\infty, 1)$ -category does arise this way from a combinatorial simplicial model category is essentially

the Dugger's theorem, which says that every combinatorial model category arises as the left Bousfield localization of the projective model structure on the category of simplicial presheaves on some site. Precisely if the localization defining a locally presentable $(\infty, 1)$ -category is *exact* in that the left adjoint to the inclusion of the reflective subcategory preserves finite ∞ -limits is the presentable $(\infty, 1)$ -category an $(\infty, 1)$ -topos: in that case this left adjoint is ∞ -stackification and the reflective subcategory is that of ∞ -stacks/ $(\infty, 1)$ -sheaves. If the localization is what is called *topological*, then these ∞ -stacks are precisely those ∞ -presheaves that satisfy *descent* with respect to Čech-nerves given by some Grothendieck topology on the underlying category. This is the ∞ -categorical version of the ordinary sheaf condition.

The ∞ -topos of ∞ -stacks on some site C plays the role of the collection of ∞ -groupoids equipped with geometric structure modeled by C . This is discussed in the next section.

6.2. Generalized spaces, topoi and (higher) categories

If the object of a category C play the role of test spaces and their morphisms behave as geometric homomorphisms between these test spaces, then the topos $\mathrm{Sh}(C)$ - the category of sheaves on C - may be understood as the category of generalized spaces modeled on C . This is a rephrase of Grothendieck's *functor of points* point of view on geometric spaces, by now largely extended by Lurie ([64]) and others.

Some of these generalized spaces are very general: all they provide is a consistent rule for how to probe them by throwing test spaces in C into them. If C is a concrete site, the concrete sheaves on C model such spaces that at least have an underlying topological space of points. Among these concrete generalized spaces are the tame ones that are locally isomorphic to objects in C .

For instance for $C = \mathrm{CartesianSpaces}$ (CartSp for short), the category whose objects are the spaces $\mathbb{R}^n$ for all $n \in \mathbb{N}$ and whose morphisms are smooth (infinitely differentiable) maps between these, we have a sequence of inclusions

$$\begin{array}{ccccccc} \mathrm{CartesianSpaces} & \hookrightarrow & \mathrm{SmoothManifolds} & \hookrightarrow & \mathrm{DiffeologicalSpaces} & \hookrightarrow & \mathrm{Sh}(\mathrm{CartesianSpaces}) \\ \text{representable} & & \text{locally representable} & & \text{sheaves with underlying} & & \\ \text{sheaves} & \hookrightarrow & \text{sheaves} & \hookrightarrow & \text{topological space} & \hookrightarrow & \text{all sheaves} \end{array} ,$$

where the entire inclusion from left to right is the Yoneda embedding.

Here diffeological spaces are sets equipped with a consistent rule for which maps of sets from an $\mathbb{R}^n$ into them are regarded as being smooth. Originally defined this way by SOURIAU and CHEN, one sees that more abstractly speaking these are precisely the *concrete* sheaves on CartSp ([49]), those sheaves which have an underlying topological space of points. For instance for Σ and X two smooth manifolds, their mapping space $[\Sigma, X]$ is naturally a diffeological space, which as a sheaf is given by the assignment $[\Sigma, X] : U \mapsto \mathrm{Hom}_{\mathrm{SmoothManifolds}}(\Sigma \times U, X)$ that says that a smooth map from U into $[\Sigma, X]$ is a smooth U -parameterized family of smooth maps from Σ to X .

From this point of view smooth manifolds are precisely the concrete sheaves on $\mathbf{CartSp}$ that are also *locally representable*. But there are also useful generalized spaces modeled on $\mathbf{CartSp}$ that are not concrete: an example is the space given by the rule $U \mapsto \Omega_{\text{cl}}^n(U)$ that sends a Cartesian space U to the set of closed smooth n -forms on it. This may be thought of as a model for an Eilenberg-MacLane space $K(n, \mathbb{R})$ in a useful sense, but it is not a concrete space. In fact, this space only has a single point, a single curve, a single surface, and generally a single k -dimensional probe for $k < n$. But then it has infinitely many n -dimensional probes.

But the theory of sheaves is not enough for a good discussion of general geometric objects. The fully general geometric objects modeled on test objects in a site C have not just a *set* of ways of mapping a test object $U \in C$ into them, but an ∞ -groupoid of ways of doing this: there is an $(\infty, 1)$ -topos $\mathbf{H} := \mathbf{Sh}_{(\infty, 1)}(C)$ of ∞ -groupoid valued sheaves (∞ -stacks) on C . If again $C = \mathbf{CartSp}$, then an ∞ -groupoid valued sheaf on C is a generalized Lie ∞ -groupoid. A locally presentable Lie ∞ -groupoid is an orbifold, or a higher generalization of that.

A convenient model for presenting and manipulating the $(\infty, 1)$ -category $\mathbf{Sh}_{(\infty, 1)}(C)$ is as the full $\mathbf{sSet}$ -enriched subcategory $(\mathbf{sPSh}(C))_{\text{proj, loc}}^\circ$ of the category $\mathbf{sPSh}(C) := [C^{op}, \mathbf{sSet}]$ of simplicial-set valued ordinary presheaves on C on those objects which are fibrant-cofibrant in what is called the projective local model structure on simplicial presheaves, with respect to the given Grothendieck topology on C . This subcategory is Kan-complex-enriched, hence enriched in ∞ -groupoids, hence an $(\infty, 1)$ -category. Its objects can be thought of as rectified ∞ -groupoid valued presheaves that satisfy an ∞ -sheaf/ ∞ -stack descent condition.

The tools for handling ∞ -toposes this way go back to KENNETH BROWN's work from 1973 ([13]). They have later been promoted by A. JOYAL and developed further by JARDINE ([59]), C. SIMPSON, D. DUGGER and others. With the results of [74, 75, 65] this toolset has found its intrinsic interpretation in higher category theory.

The sequence of inclusions of tame generalized spaces into ever more general generalized spaces mentioned above has been realized in [64] for the context of $(\infty, 1)$ -toposes. In this context a concrete ∞ -stack in $\mathbf{H}$ is not one with an underlying topological space, but one, in turn, with an underlying *petit* $(\infty, 1)$ -topos.

Apart from a hierarchy of *geometrically* more or less tame sheaves of ∞ -groupoids, there is also a hierarchy of *categorically* more or less tame ∞ -groupoids. To this we turn now.

6.3. Strict ω -Groupoid-valued ∞ -stacks

The category of strict ∞ -groupoids is the limit obtained by recursively forming groupoids strictly enriched in strict n -groupoids, starting with 0-groupoids = sets:

$$\mathbf{Str}\omega\mathbf{Grpd} = \lim_{\rightarrow} (\mathbf{Grpd} \hookrightarrow \mathbf{Str}2\mathbf{Grpd} = \mathbf{Grpd} - \mathbf{Grpd} \hookrightarrow \mathbf{Str}3\mathbf{Grpd} = 2\mathbf{Grpd} - \mathbf{Grpd} \hookrightarrow \cdots) .$$

Ross Street's ω -nerve functor

$$N : \text{Str}\omega\text{Grpd} \hookrightarrow \text{KanCplx} \hookrightarrow \text{sSet}$$

injects strict ω -groupoids into all ∞ -groupoids. One useful aspect of strict ω -groupoids is that they are half-way in between homological algebra and topology/full ∞ -groupoid theory: there is an equivalence (going back to Whitehead and amplified by R. BROWN, HIGGINS and others) of strict ω -groupoids with *crossed complexes*: these are like complexes of abelian groups, but may have non-abelian groups in low degree and be groupoidal in the lowest degree. Accordingly, ordinary chain complexes of abelian groups in non-negative degree in turn sit inside all crossed complexes as the models for the strict and abelian ∞ -groupoids. Combined with the ω -nerve this factors the familiar Dold-Kan map

$$\text{Ch}_\bullet(\text{Ab})^+ \hookrightarrow \text{CrsCpl} \simeq \text{Str}\omega\text{Grpd} \xrightarrow{N} \text{KanCplx} \simeq \infty\text{Grpd}.$$

as a hierarchy of more or less tame ∞ -groupoids. Street had also proposed a notion of *descent* for strict ω -groupoid-valued presheaves on a site C in [46]. Following a conjecture by one of the authors, DOMINIC VERITY has shown ([76]) that under mild conditions this notion is compatible with the correct notion of descent in $[C^{\text{op}}, \text{sSet}]$ (induced by the intrinsic ∞ -categorical theory) under the embedding $[C^{\text{op}}, \text{Str}\omega\text{Grpd}] \xrightarrow{N} [C^{\text{op}}, \text{sSet}]$. This allows to handle strict ω -groupoid-valued ∞ -stacks on C by themselves as useful special cases of general ∞ -groupoid valued ∞ -stacks. For instance the higher Lie groups known as the String-2-group or the Fivebrane-6-group have convenient models as strict n -groupoid-valued presheaves on $C = \text{CartSp}$. Notice that this is no contradiction to the fact that under the nerve strict ω -groupoids represent only a very restrictive subclass of all homotopy n -types: as soon as we are speaking about ∞ -groupoid valued *presheaves* on some site, the geometric realization functor

$$\Pi : \mathbf{H} \rightarrow \infty\text{Grpd}$$

that we discuss in more detail in section 7.4. will send such an ∞ -groupoid-valued sheaf to an ∞ -groupoid that combines the *geometric* homotopy groups encoded in the sheaves in each categorical degree, with the *categorical* homotopy groups themselves. For instance for $C = \text{CartSp}$ clearly every homotopy type $X \in \infty\text{Grpd}$ is in the image of Π : simply take the categorically discrete (and hence strict) ∞ -groupoid valued presheaf whose presheaf of objects is that represented by $|X|$.

7. Nonabelian cohomology, higher vector bundles and background fields

Fix now some site C of test spaces, and take the ambient context of ∞ -groupoids modeled on C to be the $(\infty, 1)$ -sheaf $(\infty, 1)$ -topos $\mathbf{H} := \text{Sh}_{(\infty, 1)}(C)$. As mentioned above, this may be presented by the model

category structure $\mathbf{sPSh}(C)_{\text{proj,loc}}$ on the functor category $\text{Func}(C^{\text{op}}, \mathbf{sSet})$ defined to be the left Bousfield localization of the global projective model structure at the set of Čech nerve-projections $C(\{U_i\}) \rightarrow U$ for $\{U_i \rightarrow U\}_i$ a covering family in C .

We shall give several of the following definitions both in their intrinsic ∞ -category theoretic formulation in $\mathbf{H}$ and also in terms of the model given by the ordinary category $\mathbf{sPSh}(C)$. The latter we shall often refer to just as “the model”.

Notably for X, A two objects of $\mathbf{H}$, we may think of a morphism $g : X \rightarrow A$ as a cocycle on X with values in A – a nonabelian cocycle if A is not

an Eilenberg-MacLane object – ; think of a 2-morphism $X \begin{array}{c} \xrightarrow{g} \\ \Downarrow \eta \\ \xrightarrow{g'} \end{array} A$

(necessarily an equivalence in the $(\infty, 1)$ -category $\mathbf{H}$) as a coboundary, and think of the set of equivalence classes of morphisms

$$H(X, A) := \pi_0 \mathbf{H}(X, A)$$

as the cohomology set of X with coefficients in A . This is a group if A is a group object, as discussed further below.

Many notions of cohomology ever considered are special cases of this simple definition for suitable choices of C . Notably for $C = *$, in which case $\mathbf{H} = \infty\text{Grpd} \simeq \text{Top}$ is the archetypical $(\infty, 1)$ -topos, does the above notion reduce to the familiar definition of (nonabelian) cohomology of topological spaces in terms of homotopy classes of maps into suitable coefficient objects. It is useful to think of all constructions here as refinements of this case, where continuous maps between topological spaces are replaced with richer structure preserving maps, such as smooth maps between ∞ -Lie groupoids.

In terms of the model, choosing a fibrant representative for A , a cocycle $X \rightarrow A$ is represented by an ∞ -anafunctor (this suggestive terminology for what is of course an old and basic concept in homotopy theory, we find useful to adopt from [25, 6]) from X to A : a span

$$(g : X \multimap A) := \begin{array}{ccc} & \hat{X} & \xrightarrow{g} A \\ & \downarrow \simeq & \\ & X & \end{array}$$

whose left leg is an acyclic fibration, which exhibits $\hat{X}$ as a *cover* of X (or rather as something akin to the Čech ∞ -groupoid of a cover). Cocycles are regarded as distinct only up to refinements of their covers. This makes

their composition by pullbacks

$$(X \xrightarrow{g} A \xrightarrow{r} A') := \begin{array}{ccc} g^* \hat{A} & \xrightarrow{\quad} & \hat{A} \xrightarrow{r} A' \\ \downarrow \simeq & & \downarrow \simeq \\ \hat{X} & \xrightarrow{g} & A \\ \downarrow \simeq & & \\ X & & \end{array}$$

well defined (noticing that acyclic fibrations are stable under pullback) and associative.

Several other simple notions for cohomology in an $(\infty, 1)$ -topos are useful:

- for $c : A \rightarrow B$ a morphism in $\mathbf{H}$, we can think of it both as a B -cocycle on A and as a characteristic class on A -cohomology, inducing a morphism of cohomologies $H(X, A) \rightarrow H(X, B)$ natural in X . We will later on notably be interested in the *curvature characteristic classes* of certain coefficient objects.
- given morphisms $i : X_0 \rightarrow X$ and $k : A_0 \rightarrow A$ we may define the relative cohomology of X with values in A and with respect to i and k as the corresponding hom-object in the arrow- $(\infty, 1)$ -category $\mathbf{H}^I$ of our $(\infty, 1)$ -topos

$$H(X; X_0, A; A_0) := \pi_0 \mathbf{H}^I \left(\begin{bmatrix} X_0 \\ \downarrow \\ X \end{bmatrix}, \begin{bmatrix} A_0 \\ \downarrow \\ A \end{bmatrix} \right).$$

A cocycle in this cohomology is a square

$$\begin{array}{ccc} X_0 & \longrightarrow & A_0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & A \end{array}$$

in $\mathbf{H}$, commuting up to a 2-morphism there, and can always be modeled by a strictly commuting square in the model.

This is notably of interest when A is pointed and $A_0 = *$ is that point. Then we write just $H(X; X_0, A)$ for the corresponding relative cohomology. Cocycles in here are A -cocycles on X that trivialize when pulled back to X_0 .

The curvature characteristic class mentioned above arises from cocycles in such relative cohomology in section 7.4.3..

- given a morphism $f : B \rightarrow C$ thought of as a characteristic class, let $A \rightarrow B$ denote its homotopy fiber. For a given object X , choose a representative cocycle for each C -cohomology class $H(X, C) \rightarrow \mathbf{H}(X, C)$. Then we may call the connected components of the homotopy pullback

$$\begin{array}{ccc} \mathbf{H}_f(X, A) & \longrightarrow & H(X, C) \\ \downarrow & & \downarrow \\ \mathbf{H}(X, B) & \longrightarrow & \mathbf{H}(X, C) \end{array}$$

the f -twisted cohomology on X with coefficients in A .

This we use for defining differential cohomology as curv-twisted flat differential cohomology.

Our goal now is to exhibit the following concepts internal to $\mathbf{H}$. For $X \in \mathbf{H}$ an ∞ -groupoid – thought of as target space (a generalized orbifold) – and for G an ∞ -group – the gauge ∞ -group or structure ∞ -group G – and given an $(\infty, 1)$ -category F – the ∞ -category of typical fibers – together with a morphism $\rho : \mathbf{B}G \longrightarrow F$ into a pointed codomain, $\mathrm{pt}_F : \mathrm{pt} \rightarrow F$ – which we think of as a representation – of G , we can speak of

- G -cocycles g on G ;
- the G -principal ∞ -bundle $P := g^* \mathbf{E}G$ on X classified by these;
- the ρ -associated ∞ -bundles $V := g^* \rho^* \mathbf{E}F$
- the collection $\Gamma(V)$ of sections of V ;
- connections ∇ on the G -principal ∞ -bundle P .

Except for the last one, the definition of these notions follows pretty much classical lore in homotopy theory, only that we work not necessarily in the traditional archetypical $(\infty, 1)$ -topos $\infty\mathrm{Grpd} \simeq \mathrm{Top}$ but in $\mathbf{H}$. This allows us to speak with ease for instance about the differential geometry of smooth $\mathbf{B}^n U(1)$ -principal bundles (otherwise known as $(n - 1)$ -bundle gerbes) or smooth nonabelian structures such as String-principal 2-bundles. But all constructions here work for arbitrary sites C , up to section 7.4., where connections on ∞ -bundles are introduced and special properties in the site are required.

7.1. Principal ∞ -bundles

Definition 7.1 (∞ -group) *Given a one-object ∞ -groupoid $\mathbf{B}G \in \mathbf{H}$ the ∞ -pullback*

$$\begin{array}{ccc} G & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathbf{B}G \end{array}$$

is the corresponding ∞ -group. In terms of the model, for $\mathbf{B}G$ a fibrant representative, this may be identified with the ordinary pullback

$$\begin{array}{ccc} G & \longrightarrow & (\mathbf{B}G)^I \\ \downarrow & & \downarrow (d_0 \times d_1) \\ \mathrm{pt} & \longrightarrow & \mathbf{B}G \times \mathbf{B}G \end{array}$$

where $I := \{0 \rightarrow 1\}$ is the categorical interval.

For G an ∞ -group, a G -principal ∞ -bundle $P \rightarrow X$ can be defined intrinsically in $\mathbf{H}$ simply as the ∞ -categorical fiber of a morphism $X \rightarrow \mathbf{B}G$, as we shall do shortly. In terms of the model for $\mathbf{H}$, this simple statement requires introducing the universal G -principal bundle, which we now do first.

Definition 7..2 (universal G -principal ∞ -bundle) For $\mathbf{B}G$ a fibrant representative in the model, the universal G -principal ∞ -bundle $\mathbf{E}G \twoheadrightarrow \mathbf{B}G$ is given by the ordinary pullback

$$\begin{array}{ccc} \mathbf{E}G & \xrightarrow{\simeq} & \mathrm{pt} \\ \downarrow & & \downarrow \\ (\mathbf{B}G)^I & \xrightarrow[d_0]{\simeq} & \mathbf{B}G \\ \downarrow d_1 \simeq & & \\ \mathbf{B}G & & \end{array} .$$

Lemma 7..3 The morphism $\mathbf{E}G \twoheadrightarrow \mathbf{B}G$ defined this way is indeed a fibration and its kernel is G : we have a short exact sequence

$$G \xrightarrow{i} \mathbf{E}G \xrightarrow{p} \mathbf{B}G .$$

Proof. This is a standard fact in homotopy theory, but maybe deserves to be highlighted here in the context of principal ∞ -bundles in $\mathbf{H}$.

That p is a fibration is the factorization lemma [13]. To see that G is indeed the kernel of this fibration, consider the diagram

$$\begin{array}{ccccc} G & \longrightarrow & \mathbf{E}_{\mathrm{op}} G & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{E}G & \longrightarrow & \mathbf{B}G^I & \xrightarrow{d_1} & \mathbf{B}G \\ \downarrow & & \downarrow d_0 & & \\ \mathrm{pt} & \longrightarrow & \mathbf{B}G & & \end{array} .$$

The right and bottom squares are pullback squares by definition. Moreover, G is by definition 7.1 the total pullback

$$\begin{array}{ccc}
 G & \xrightarrow{\quad} & \text{pt} \\
 \downarrow & \searrow & \downarrow \\
 & \mathbf{B}G^I & \xrightarrow{d_1} \mathbf{B}G \\
 & \downarrow d_0 & \\
 \text{pt} & \xrightarrow{\quad} & \mathbf{B}G
 \end{array}$$

Therefore also the top left square exists and is a pullback itself and hence so is the pasting composite of the two top squares. This says that i is the kernel of p .

Definition 7.4 (G -principal ∞ -bundles) *For $X \in \mathbf{H}$ and $G \in \mathbf{H}$ an ∞ -group, and for $g : X \rightarrow \mathbf{B}G$ a G -cocycle on X , the corresponding homotopy fiber $P \rightarrow X$, i.e. the ∞ -pullback*

$$\begin{array}{ccc}
 P & \xrightarrow{\quad} & * \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{g} & \mathbf{B}G
 \end{array}$$

is the G -principal ∞ -bundle classified by g . In terms of the model for $\mathbf{H}$, the cocycle is given by an ∞ -anafunctor $X \xleftarrow{\simeq} \hat{X} \xrightarrow{g} \mathbf{B}G$ and the corresponding G -principal ∞ -bundle $\pi_g : P \twoheadrightarrow X$ classified by g is given by the ordinary pullback diagram

$$\begin{array}{ccc}
 g^*\mathbf{E}G & \longrightarrow & \mathbf{E}G \\
 \downarrow & & \downarrow \\
 \hat{X} & \xrightarrow{g} & \mathbf{B}G \\
 \downarrow \simeq & & \\
 X & &
 \end{array}$$

For $n \leq 2$ this way of describing (universal) principal n -bundles was described in [32].

If G is a group or strict 2-group, this definition of G -principal bundles is equivalent to the definitions in [6, 5, 48].

Of course this statement involves higher categorical equivalences: for G a 2-group and $g : X \twoheadrightarrow \mathbf{B}G$ a cocycle, the pullback $g^*\mathbf{E}G$ is a priori a 2-groupoid, whereas in the literature on 2-bundles one expects this total space

to be a 1-groupoid. But this desired 1-groupoid is obtained by dividing out 2-isomorphisms in $g^*\mathbf{E}G$ and the result is weakly equivalent to the original 2-groupoid $g^*\mathbf{E}G \xrightarrow{\sim} (g^*\mathbf{E}G)_\sim$.

Principal ∞ -bundles and line bundle gerbes. For every ordinary (Lie) group G , i.e. a one object Lie groupoid $\mathbf{B}G$ in $\mathbf{H} = \mathrm{Sh}_{(\infty,1)}(\mathrm{CartSp})$, there is a 2-group $\mathrm{AUT}(G)$, i.e. a one-object Lie 2-groupoid $\mathbf{BAUT}(G)$ in $\mathbf{H}$ defined as the internal automorphism 2-group. The notion of G -gerbe introduced by Giraud corresponds to the notion of $\mathrm{AUT}(G)$ -principal 2-bundle as described here. For $G = U(1)$ we have that $\mathbf{BAUT}(U(1))$ is the 2-groupoid given by the crossed complex $U(1) \xrightarrow{0} \mathbb{Z}_2 \rightrightarrows *$, where $\mathbb{Z}_2$ acts on $U(1)$ by automorphisms. A $\mathrm{AUT}(U(1))$ -principal 2-bundle is what in the literature has been called a Jandl-gerbe. If one assumes that the $\mathbb{Z}_2$ -part of a $\mathrm{AUT}(U(1))$ -cocycle is trivial, one arrives at a plain $\mathbf{B}U(1)$ -principal 2-bundle in $\mathbf{H}$. If a cocycle for these is written down in a certain form, this is what in the literature is called a bundle gerbe. Similarly a cocycle for a $\mathbf{B}^2U(1)$ -principal 3-bundle in our sense, written down in a certain way, is called a bundle 2-gerbe. Generally therefore, for $n \in \mathbb{N}$, we may think of (certain representative cocycles for) $\mathbf{B}^nU(1)$ -principal ∞ -bundles as bundle n -gerbes.

Not all higher principal bundles that appear in practice are of this abelian form. But by local semi-trivialization many cocycles for nonabelian G -principal ∞ -bundles may be realized as abelian principal ∞ -bundles on total spaces of nonabelian principal n -bundles for lower n .

For let $\mathbf{B}A \rightarrow \mathbf{B}\hat{G} \rightarrow \mathbf{B}G$ be a fibration sequence in $\mathbf{H}$. Then consider the diagram in $\mathbf{H}$ of the form

$$\begin{array}{ccccccc} \hat{G} & \longrightarrow & \hat{P} & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \downarrow & & \\ G & \longrightarrow & P & \longrightarrow & \mathbf{B}A & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \xrightarrow{x} & X & \xrightarrow{\hat{g}} & \mathbf{B}\hat{G} & \longrightarrow & \mathbf{B}G \end{array},$$

where every single square and hence all rectangles are ∞ -pullback squares. This exhibits $\hat{P} \rightarrow X$ as the total space of the $\hat{G}$ -principal ∞ -bundle classified by $\hat{g}$. But the diagram shows that this is encoded in an A -principal ∞ -bundle on the total space P of the underlying G -principal ∞ -bundle, satisfying the special property that its restriction to any fiber presents the cocycle that exhibits the extension $\hat{G} \rightarrow G$.

If A is an abelian ∞ -group, then this construction allows to speak of the possibly nonabelian $\hat{G}$ -principal ∞ -bundle $\hat{P}$ only in terms of abelian cocycles on P . This is for instance the case for $\mathrm{String}(n)$ -principal 2-bundles.

The 2-group $\text{String}(n)$ is defined by the fibration sequence

$$\cdots \rightarrow \mathbf{B}^2U(1) \rightarrow \mathbf{B}\text{String}(n) \rightarrow \mathbf{B}\text{Spin}(n) \xrightarrow{\frac{1}{2}p_1} \mathbf{B}^3U(1).$$

Hence a String-principal 2-bundle may equivalently be encoded by a certain bundle gerbe on the total space of the underlying $\text{Spin}(n)$ -principal bundle. These structures appear in the background of the heterotic string. See [69] for a survey. Similarly, using now the fiber sequence

$$\cdots \rightarrow \mathbf{B}^2U(1) \rightarrow \mathbf{B}\text{AUT}(U(1)) \rightarrow \mathbf{B}\mathbb{Z}$$

we find that $\text{AUT}((1))$ -principal 2-bundles are the same as bundle gerbes on certain double covers. These structures model the Kalb-Ramond field on an orbifold in string theory.

7.2. Associated ∞ -bundles

For many aspects of quantum theory it is crucial to pass from principal bundles to associated vector bundles. For instance the electromagnetic field on a space X is entirely encoded in a $U(1)$ -principal bundle $P \rightarrow X$ with connection ∇ . But to form the spaces of quantum states of the quantum particle that is charged under this field, one passes to the associated line bundle $E := P \times_{U(1)} \mathbb{C}$ – a rank-1 vector bundle – of the principal bundle, and then forms the space of sections of that. This space of linear sections in turn, may be understood as the collection of morphisms $\Psi : X \times \mathbb{C} \rightarrow E$ from the trivial line bundle into E . Here it is important this morphism is not required to be an isomorphism, but a general morphism in the category $\text{Vect}(X)$ of vector bundles over X . This means that for the quantum theory it is crucial to generalize from groupoid-valued stacks to category-valued stacks, and hence from ∞ -groupoid valued ∞ -stacks to $(\infty, 1)$ -category valued ∞ -stacks, and to realize associated ∞ -bundles in terms of these.

This is what we formalize now. While there is no good general intrinsic theory of $(\infty, 2)$ -toposes of $(\infty, 1)$ -category valued ∞ -stacks available yet, it is pretty clear what the $(\infty, 1)$ -category of $(\infty, 1)$ -category-valued ∞ -stacks should be: let sSet^+ be the model structure on marked simplicial sets introduced in [65], which provides a *simplicial* model structure that models quasi-categories. Then define the $(\infty, 1)$ -category of $(\infty, 1)$ -category-valued ∞ -stacks on C to be, as before, the left Bousfield localization of the global projective model structure $\text{Func}(C^{op}, \text{sSet}^+)$ at Čech nerves of covering families. In the following by $(\infty, 1)$ -category over C , we shall mean an object in this left Bousfield localization.

So let F be some $(\infty, 1)$ -category over C in this sense (*not* necessarily an ∞ -groupoid), which will play the role of the stack Vect of ordinary vector bundles. An ∞ -anafunctor

$$\rho : \mathbf{B}G \dashrightarrow F$$

may be thought of as an ∞ -group cocycle with values in F . If F is equipped with a point, $\text{pt} \xrightarrow{\text{pt}_F} F$, we may think of such a morphism ρ also as a representation of G . In analogy with the universal G -principal ∞ -bundle from definition 7.2 we obtain the universal F -bundle (with respect to the chosen point pt_F) as a pullback from the point:

Definition 7..5 (universal F -bundle) *For F an $(\infty, 1)$ -category with chosen point $* \xrightarrow{\text{pt}_F} F$ the universal F -bundle $\mathbf{E}F \longrightarrow F$ is the pullback*

$$\begin{array}{ccc} \mathbf{E}F & \longrightarrow & * \\ \downarrow & & \downarrow \text{pt}_F \\ F^I & \xrightarrow{d_0} & F \\ \downarrow d_1 & & \\ F & & \end{array} .$$

Here $I = \{0 \rightarrow 1\}$ crucially still denotes the *category* free on a single nontrivial morphisms, not the groupoid. This means that an object in F^I is not in general an invertible morphism in F .

Definition 7..6 (associated F -bundle) *Given a representation morphism $\rho : \mathbf{B}G \rightarrow F$ we call the lax (“comma”-) ∞ -pullback*

$$\begin{array}{ccc} \rho^* \mathbf{E}F & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\rho} & F \end{array}$$

which for F a fibrant representative in the model is given as the ordinary pullback

$$\begin{array}{ccc} \rho^* \mathbf{E}F & \longrightarrow & \mathbf{E}F \\ \downarrow & & \downarrow \\ \widehat{\mathbf{B}G} & \xrightarrow{\rho} & F \\ \downarrow \simeq & & \\ \mathbf{B}G & & \end{array}$$

the F -bundle ρ -associated to the universal G -bundle. Correspondingly the further ∞ -pullback along a g -cocycle

$$\begin{array}{ccccc}
g^* \rho^* \mathbf{E}F & \longrightarrow & \rho^* \mathbf{E}F & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{g} & \mathbf{B}G & \xrightarrow{\rho} & F
\end{array}$$

which is modeled by the sequence of ordinary pullbacks

$$\begin{array}{ccccc}
g^* \rho^* \mathbf{E}F & \longrightarrow & \rho^* \mathbf{E}F & \longrightarrow & \mathbf{E}F \\
\downarrow & & \downarrow & & \downarrow \\
\hat{\hat{X}} & \xrightarrow{g} & \widehat{\mathbf{B}G} & \xrightarrow{\rho} & F \\
\downarrow \simeq & & & & \\
X & & & &
\end{array}$$

is the F -bundle ρ -associated to the specific G -principal bundle $g^* \mathbf{E}G$.

The pullback V in

$$\begin{array}{ccccc}
V & \longrightarrow & \rho^* \mathbf{E}F & \longrightarrow & \mathbf{E}F \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & \widehat{\mathbf{B}G} & \xrightarrow{\rho} & F
\end{array}$$

is the representation space itself, the typical fiber of the ρ -associated bundles.

7.3. Sections of associated ∞ -bundles

Definition 7..7 (section) A section σ of a ρ -associated ∞ -bundle $V := \rho^* g^* \mathbf{E}F$ coming from a cocycle $X \xrightarrow{g} \mathbf{B}G$ is a lift of the cocycle through $\rho^* \mathbf{E}F \longrightarrow \mathbf{B}G$ or equivalently a morphism from the trivial F -bundle with fiber pt_F to V

$$\Gamma(V) := \left\{ \begin{array}{ccc} & \rho^* \mathbf{E}F & \\ \nearrow \sigma & \downarrow & \\ X & \xrightarrow{g} & \mathbf{B}G \end{array} \right\} \simeq \left\{ \begin{array}{ccc} & X & \\ \swarrow & \mathrm{Id} & \searrow \\ \mathrm{pt} & \xRightarrow{\sigma} & X \\ \searrow g & & \swarrow \rho \circ g \\ \mathrm{pt}_F & & F \end{array} \right\}$$

Lemma 7..8 *These two characterizations of sections are indeed equivalent.*

Proof. First rewrite

$$\left\{ \begin{array}{ccc} & X & \\ \swarrow & \downarrow g & \searrow \\ \text{pt} & \xrightarrow{\sigma} \mathbf{B}G & \\ \searrow \text{pt}_F & \downarrow \rho & \\ & F & \end{array} \right\} \simeq \left\{ \begin{array}{ccc} & \text{pt} \xrightarrow{\text{pt}_F} F & \\ \nearrow & \downarrow d_0 & \nwarrow \\ X & \xrightarrow{\sigma} F^I & \\ \searrow g & \downarrow d_1 & \nearrow \\ & \mathbf{B}G \xrightarrow{\rho} F & \end{array} \right\}$$

using the characterization of right (directed) homotopies by the (directed) path object F^I . Using the universal property of $\mathbf{E}F$ as a pullback this yields

$$\dots \simeq \left\{ \begin{array}{ccc} & & \mathbf{E}F \\ & \nearrow \sigma & \downarrow \\ X & \xrightarrow{g} \mathbf{B}G \xrightarrow{\rho} F & \end{array} \right\} \simeq \left\{ \begin{array}{ccc} & & \rho^* \mathbf{E}F \\ & \nearrow \sigma & \downarrow \\ X & \xrightarrow{g} \mathbf{B}G & \end{array} \right\}.$$

7.4. Connections on ∞ -bundles

A gauge background field is crucially not just an ∞ -bundle, but an ∞ -bundle with connection: the connection encodes the forces acting on the objects that are charged under the background field, its parallel transport enters the action functional for these objects. The underlying ∞ -bundle only encodes the global nontriviality of this parallel transport, while the crucial local physical information is in the connection.

In low categorical dimension n , following an original suggestion by John Baez (see [50] for an exposition and [40], [41] for details) it is by now “well known” that the n -connection is in fact *equivalent* to the parallel transport n -functor on the path n -groupoid that it induces. In [70] the full ∞ -categorical formulation of this phenomenon was indicated, with emphasis on Lie differential geometric aspects. We now discuss this with more emphasis on some general abstract properties, that we need for our examples in section 9.. A comprehensive discussion is to appear elsewhere, see [71].

7.4.1. The homotopy ∞ -groupoid

To obtain the ∞ -functor that sends each object $X \in \mathbf{H}$ to its path ∞ -groupoid $\Pi(X)$, we shall now observe that in nice cases this is defined *canonically*.¹

¹ U.S. thanks Richard Williamson for useful discussion of this point. It turns out that, in some disguise and up to some issues, this is almost a classical fact. But

Theorem 7..9 *Let C be a site whose objects are geometrically contractible in that Kan-complex-valued presheaves on C satisfy descent for simplicial presheaves on objects of C . Then $\mathbf{H} = \mathrm{Sh}_{(\infty,1)}(C)$ is a locally contractible $(\infty, 1)$ -topos in that the terminal global sections geometric morphism $\Gamma : \mathbf{H} \rightarrow \infty\mathrm{Grpd}$ is essential, i.e. in that we have a triple of adjoint $(\infty, 1)$ -functors*

$$(\Pi \dashv \mathrm{LConst} \dashv \Gamma) : \mathbf{H} \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\mathrm{LConst}} \\ \xrightarrow{\Gamma} \end{array} \infty\mathrm{Grpd} .$$

This is the case notably for the site $C = \mathrm{CartSp}$ with covering families given by ordinary covers that are good covers (all intersections of patches are contractible).

In this case we have moreover $(\Pi \circ \mathrm{LConst} \dashv \Gamma \circ \mathrm{LConst}) \simeq (\mathrm{Id} \dashv \mathrm{Id}) : \infty\mathrm{Grpd} \rightarrow \infty\mathrm{Grpd}$.

Remark. The last statement means that the *shape* of the locally contractible $(\infty, 1)$ -topos $\mathrm{Sh}_{(\infty,1)}(\mathrm{CartSp})$, in the sense of shape theory of $(\infty, 1)$ -toposes [65], is that of the point. This highlights the fact that $\mathrm{Sh}_{(\infty,1)}(\mathrm{CartSp})$ is a *gros* $(\infty, 1)$ -topos of “all” spaces modeled on CartSp , rather than something that is to be thought of as a generalized topological space itself: we may indeed usefully think of the objects in CartSp as nothing but thickened points, n -dimensional disks that only serve to encode the notion of smooth families around a given point. Useful comments along these lines can be found in [56].

Proof. For the first statement it is sufficient to produce a Quillen adjunction

$$(\Pi \dashv \mathrm{LConst}) : \mathrm{sPSh}(C)_{\mathrm{proj}, \mathrm{loc}} \begin{array}{c} \xrightarrow{\Pi} \\ \xleftarrow{\mathrm{LConst}} \end{array} \mathrm{sSet}_{\mathrm{Quillen}}$$

with the underlying functor of LConst simply being the constant presheaf functor. Almost by definition that has an sSet -enriched left adjoint given by sending a presheaf to its colimit. Since LConst evidently sends (acyclic) fibrations in $\mathrm{sSet}_{\mathrm{Quillen}}$ to (acyclic) fibrations in the global model structure $\mathrm{sPSh}(C)_{\mathrm{proj}}$, it follows that $\lim_{\rightarrow} : \mathrm{sPSh}(C)_{\mathrm{proj}} \rightarrow \mathrm{sSet}_{\mathrm{Quillen}}$ preserves cofibrations. But the cofibrations do not change under left Bousfield localization, so that also $\Pi := \lim_{\rightarrow} : \mathrm{sPSh}(C)_{\mathrm{proj}, \mathrm{loc}} \rightarrow \mathrm{sSet}_{\mathrm{Quillen}}$ preserves cofibrations. Moreover, by assumption $\mathrm{LConst} : \mathrm{sSet}_{\mathrm{Quillen}} \rightarrow \mathrm{sPSh}(C)_{\mathrm{proj}, \mathrm{loc}}$ preserves fibrant objects. Noticing that $\mathrm{sSet}_{\mathrm{Quillen}}$ is a left proper model

lifting the disguise and making the abstract $(\infty, 1)$ -topos-theoretic structure manifest turns out to be very useful. For a detailed commented review and more literature see [http://ncatlab.org/nlab/show/homotopy+groups+in+an+\(infinity,1\)-topos](http://ncatlab.org/nlab/show/homotopy+groups+in+an+(infinity,1)-topos).

category, this means that the conditions of corollary A.3.7.2 in [65] are satisfied, which says that $(\Pi := \lim_{\rightarrow} \dashv \text{LConst})$ is indeed a Quillen adjunction for the local model structure on $\text{sPSh}(\mathcal{C})$, as stated.

To see that the site CartSp does satisfy the required assumptions, let $\{U_i \rightarrow U\}$ be a good cover of $U \in \text{CartSp}$ and write $C(U_i) := \int^{[n] \in \Delta} \Delta[n] \cdot \coprod_{i_0, \dots, i_n} U_{i_0, \dots, i_n}$ for the corresponding Čech nerve, regarded as simplicial presheaf. Then for S a Kan complex we have

$$\begin{aligned} \text{sPSh}(C(U_i), \text{LConst}(S)) &:= \text{sPSh}\left(\int^{[n] \in \Delta} \Delta[n] \cdot \coprod_{i_0, \dots, i_n} U_{i_0, \dots, i_n}, \text{LConst}(S)\right) \\ &= \int_{[n] \in \Delta} \coprod_{i_0, \dots, i_n} \text{sPSh}(U_{i_0, \dots, i_n}, \text{LConst}(S)) \\ &= \int_{[n] \in \Delta} \coprod_{i_0, \dots, i_n} \text{sSet}(*, \text{LConst}(S)) \\ &= \text{sSet}\left(\int^{[n] \in \Delta} \Delta[n] \cdot \coprod_{i_0, \dots, i_n} *, S\right) \end{aligned}$$

which is a Kan complex weakly equivalent to S , since the simplicial set coming from the cover is a contractible Kan complex, since $U \in \text{CartSp}$ is topologically contractible. So the morphism

$$S = \text{sPsh}(U, \text{LConst}(S)) \rightarrow \text{sPSh}(C(U_i), \text{LConst}(S))$$

is a weak equivalence, which means that $\text{LConst}(S)$ satisfies descent.

Similarly we have that the right adjoint to the constant simplicial presheaf functor, $\Gamma := \lim_{\leftarrow} : \text{sPSh}_{\text{proh,loc}} \rightarrow \text{sSet}_{\text{Quillen}}$ preserves fibrant objects, and that LConst also preserves cofibrations (since the point $\mathbb{R}^0$ is cofibrant and tensoring with a simplicial set sends cofibrant presheaves to cofibrant presheaves). Since also $\text{sPSh}(C)_{\text{proj,loc}}$ is left proper (being the left Bousfield localization of a functor category with values in a left proper model category), corollary A.3.7.2 in [65] applies again to show that we have a triple of Quillen adjoint functors

$$(\Pi \dashv \text{LConst} \dashv \Gamma) : \text{sPSh}(C)_{\text{proj,loc}} \rightarrow \text{sSet}_{\text{Quillen}}.$$

By the above discussion $\Pi \circ \text{LConst} = \text{Id}_{\text{sSet}}$ and $\Gamma \circ \text{LConst} = \text{Id}_{\text{sSet}}$ are evidently composites of derived functors, which proves the last claim.

Remark: topological geometric realization. The colimit over a representable presheaf is the singleton set $*$. By [56], every object $X \in$

$\text{sPSh}(C)_{\text{proj}}$ has a cofibrant replacement $\hat{X}$ that is degreewise a coproduct of representables $\{U_{i_n} \in C\}$: $\hat{X} = \int^{[n] \in \Delta} \Delta[n] \cdot (\coprod_{i_n} U_{i_n})$. This means that the $(\infty, 1)$ -functor modeled by Π sends such X to (the Kan fibrant replacement of) the simplicial set obtained by contracting in this expression each representable to a point: $\Pi(\hat{X}) = \int^{[n] \in \Delta} \Delta[n] \cdot (\coprod_{i_n} *)$.

In particular, for $C = \text{CartSp}$ and X a manifold, the simplicial set $\Pi(\hat{X})$ is under the Quillen equivalence $\text{sSet}_{\text{Quillen}} \simeq \text{Top}$ a topological space that is weakly homotopy equivalent to X . So we may think of $|\Pi(-)| : \mathbf{H} \rightarrow \infty\text{Grpd} \xrightarrow{\sim} \text{Top}$ as being a topological geometric realization of structured objects in $\mathbf{H}$ to plain topological spaces, up to weak homotopy equivalence.

Indeed, by proposition 2.8 in [56], the cofibrant replacement of a simplicial presheaf X may be taken to be of the form $\hat{X} = \int^{[n] \in \Delta} \Delta[n] \cdot \widehat{X}_n$, with $\widehat{X}_n$ a replacement of the simplicially discrete presheaf X_n . This is sent by $|\Pi(-)|$ to the topological space $\int^{[n] \in \Delta} \Delta^n \times |\Pi(\widehat{X}_n)|$, which is the geometric realization of the simplicial topological space $|\Pi(\widehat{X}_\bullet)|$ obtained by geometrically realizing X in each degree.

So again in our running example of $C = \text{CartSp}$, we find in particular that if X is a simplicial manifold or simplicial diffeological space, then $|\Pi(\hat{X})|$ is, up to weak homotopy equivalence, the familiar topological geometric realization of X .

The fact alone that the path ∞ -groupoid functor is part of an essential geometric morphism of $(\infty, 1)$ -toposes $(\Pi \dashv \text{LConst} \dashv \Gamma)$ leads to some useful general statements about the geometric homotopy groups of objects in $\mathbf{H}$.

Definition 7..10 Write $\text{Core}(\infty\text{Grpd})$ for the ∞ -groupoid of small ∞ -groupoids. Define

$$\text{Cov} := \mathbf{H}(-, \text{LConst}(\text{Core}(\infty\text{Grpd}))) : \mathbf{H} \rightarrow \infty\text{Grpd}.$$

For $X \in \mathbf{H}$ we call $\text{Cov}(X)$ the ∞ -groupoid of ∞ -covering spaces over X .

Theorem 7..11 (∞ -Galois theory) Let $\mathbf{H}$ be a locally contractible $(\infty, 1)$ -topos. We have naturally in $X \in \mathbf{H}$ the following statements.

- covering ∞ -spaces correspond to ∞ -local systems:

$$\text{Cov}(X) \simeq \infty\text{Func}(\Pi(X), \infty\text{Grpd});$$

- for each point $x : * \rightarrow |\Pi(X)|$ in the geometric realization of X , the automorphism ∞ -group of the induced fiber-functor $F_x : \text{Cov}(X) \rightarrow \infty\text{Grpd}$ is equivalent to the geometric homotopy groups $\Omega_x |\Pi(X)| = \text{Aut}_{\Pi(X)^{\text{op}}}(x)$ of $|\Pi(X)|$ at X :

$$\text{Aut}(F_x) \simeq \Omega_x |\Pi(X)|.$$

Proof. The first statement is simply the hom-equivalence corresponding to the $(\infty, 1)$ -adjunction $(\Pi \dashv \text{LConst})$:

$$\begin{aligned} \text{Cov}(X) &:= \mathbf{H}(X, \text{LConst}(\text{Core}(\infty\text{Grpd}))) \\ &\simeq \infty\text{Grpd}(\Pi(X), \text{Core}(\infty\text{Grpd})) \ . \\ &= \infty\text{Func}(\Pi(X), \infty\text{Grpd}) \end{aligned}$$

The second fact is abstract Tannaka duality, a formal consequence of applying the $(\infty, 1)$ -Yoneda lemma four times in a row: the fiber functor $F_x := \infty\text{Func}(* \xrightarrow{x} \Pi(X)) : \infty\text{Func}(\Pi(X), \infty\text{Grpd}) \rightarrow \infty\text{Grpd}$ may itself be regarded as an $(\infty, 1)$ -presheaf. By the $(\infty, 1)$ -Yoneda lemma for the $(\infty, 1)$ -Yoneda embedding $j : \Pi(X)^{\text{op}} \rightarrow \text{PSh}_{(\infty, 1)}(\Pi(X)^{\text{op}})$, this is equivalently $F_x \simeq \text{Hom}_{\text{PSh}_{(\infty, 1)}(\Pi(X)^{\text{op}})}(j(x), -)$. But this means that $F_x \simeq j(j(x))$ is itself a representable $(\infty, 1)$ -presheaf, an object in $\text{PSh}_{(\infty, 1)}(\text{PSh}_{(\infty, 1)}(\Pi(X)^{\text{op}})^{\text{op}})$. The statement then follows from applying the $(\infty, 1)$ -Yoneda lemma two more times:

$$\begin{aligned} \text{Aut}(F_x) &\simeq \text{Aut}(j(j(x))) \\ &\simeq \text{Aut}(j(x)) \\ &\simeq \text{Aut}_{\Pi(X)^{\text{op}}}(x) \\ &\simeq \Omega_x |\Pi(X)| \ , \end{aligned}$$

where we suppressed some evident subscripts for readability.

7.4.2. The geometric path ∞ -groupoid

We now want to obtain a notion of path ∞ -groupoid *internal* to $\mathbf{H}$. For that we use the above adjunction to reflect the homotopy ∞ -groupoid Π back into $\mathbf{H}$.

Definition 7..12 For $\mathbf{H}$ a locally contractible $(\infty, 1)$ -topos, write

$$(\Pi \dashv \flat) := (\text{LConst} \circ \Pi \dashv \text{LConst} \circ \Gamma) : \mathbf{H} \rightarrow \mathbf{H}$$

for the composite adjunction. We call Π the path ∞ -groupoid functor.

While entirely abstractly defined, it turns out that the path ∞ -groupoid functor does induce an intrinsic notion of *geometric paths* in $\mathbf{H}$. We make this explicit for $C = \text{CartSp}$ with the following statement.

Theorem 7..13 For $C = \text{CartSp}$ in the model $[C^{\text{op}}, \text{sSet}]$ the functor Π is equivalently given by the left Quillen functor which is the left-derived Yoneda extension Π_R of the smooth singular simplicial complex functor $C \rightarrow [C^{\text{op}}, \text{sSet}] : U \mapsto U^{\Delta_R^\bullet}$, where Δ_R is the canonical cosimplicial object exhibiting the geometric smooth n -simplex.

Proof. Choose a functorial factorization

$$\begin{array}{ccc} U & \xrightarrow{\quad} & \Pi_R(U) \\ & \searrow & \downarrow \simeq \\ & & U^{\Delta_R^\bullet} \end{array}$$

in $\text{sPSh}(C)_{\text{proj}}$ of the evident inclusion $U \rightarrow U^{\Delta_R^\bullet}$. Notice that since the representable U is cofibrant in $\text{sPSh}(C)_{\text{proj,loc}}$, also $\Pi_R(U)$ is cofibrant. For general $X \in \text{sPSh}(C)$ we then set

$$\Pi_R(X) := \int^{U \in C} \Pi_R(U) \cdot X(U).$$

Here the coend over the tensoring of $\text{sPSh}(C)$ over sSet

$$\int (-) \cdot (-) : [C, \text{sPSh}(C)_{\text{proj}}]_{\text{inj}} \times [C^{\text{op}}, \text{sSet}]_{\text{proj}} \times \rightarrow \text{sPSh}(C)_{\text{proj}}$$

is a left Quillen bifunctor by proposition A.2.26 and remark A.2.27 of [65]. Since by construction $\Pi_R(-)$ regarded as an object in $[C, \text{sPSh}(C)_{\text{proj}}]_{\text{inj}}$ is cofibrant, this means that $\Pi_R(-) = \int^{U \in C} \Pi_R(U) \cdot (-)(U)$ preserves cofibrations and acyclic cofibrations. Moreover, this Π_R extends to an sSet -enriched functor and as such has an sSet -enriched right adjoint $\flat_R : X \mapsto \text{sPSh}(\Pi_R(-), X)$. Therefore

$$(\Pi_R \dashv \flat_R) : \text{sPSh}(C)_{\text{proj}} \xrightleftharpoons{\quad} \text{sPSh}(C)_{\text{proj}}$$

is a Quillen adjunction for the global model structure. It remains to show that this descends to a Quillen adjunction on the local model structure. For this notice that Π_R sends projections $C(\{U_i\}) \rightarrow U$ of good covers $\{U_i \rightarrow U\}$ out of Čech nerves of good covers to weak equivalences.

This is because using that $\Pi_R(U) \rightarrow *$ is a global weak equivalence for $U \in \text{CartSp}$, and that the Čech nerve is cofibrant, we have

$$\begin{array}{ccc} \int^{[n] \in \Delta} \coprod_{i_0, \dots, i_n} \Pi_R(U_{i_0, \dots, i_n}) \cdot \Delta[n] & \xrightarrow{\simeq} & \int^{[n] \in \Delta} \coprod_{i_0, \dots, i_n} * \cdot \Delta[n] \\ \downarrow \simeq & & \downarrow \simeq \\ \Pi_R(C(\{U_i\})) = \int^{[n] \in \Delta} \coprod_{i_0, \dots, i_n} \Pi_R(U_{i_0, \dots, i_n}) \cdot \Delta[n] & \longrightarrow & \int^{[n] \in \Delta} \coprod_{i_0, \dots, i_n} * \cdot \Delta[n] \xrightarrow{\simeq} * \end{array},$$

where $\Delta : \Delta \rightarrow \text{sSet} : [n] \mapsto N([n]/\Delta)^{\text{op}}$ is the Bousfield-Kan cofibrant replacement of Δ and of $*$ in $[\Delta, \text{sSet}]_{\text{proj}}$, and we use again that all coends over tensors here are Quillen bifunctors, and finally, on the right, that U is topologically contractible.

From this we can now conclude that $\flat_R$ preserves fibrant objects in $\mathbf{sPSh}(C)_{\text{proj,loc}}$. This is because the fibrant objects in the left Bousfield localization are the globally fibrant objects that satisfy descent on all Čech nerves of good covers as above. And since both $C(\{U_i\})$ as well as U and therefore also $\mathbf{\Pi}_R(C(\{U_i\}))$ and of course $\mathbf{\Pi}_R(U)$ are cofibrant in $\mathbf{sPSh}(C)_{\text{proj,loc}}$, we have by adjunction that

$$\mathbf{sPSh}(C(\{U_i\}) \rightarrow U, \flat_R(A)) \simeq \mathbf{sPSh}(\mathbf{\Pi}_R(C(\{U_i\})) \rightarrow \mathbf{\Pi}_R(U), A)$$

is the enriched hom of a weak equivalence between cofibrant objects into a fibrant object in the simplicially enriched model category $\mathbf{sPSh}(C)_{\text{proj}}$, and so is itself a weak equivalence (in $\mathbf{sSet}_{\text{Quillen}}$). But this says that $\flat_R(A)$ satisfies descent.

Again by appeal to corollary A.3.7.2 in [65] we therefore have the desired local Quillen adjunction

$$(\mathbf{\Pi}_R \dashv \flat_R) : \mathbf{sPSh}(C)_{\text{proj,loc}} \rightleftarrows \mathbf{sPSh}(C)_{\text{proj,loc}} .$$

It remains to show that the ∞ -functor modeled by $\mathbf{\Pi}_R$, i.e. its left derived functor, is indeed equivalent to the abstractly defined $\mathbf{\Pi}$. This follows again using Dugger's cofibrant replacement theorem and the remarks about geometric realization in section 7.4.1.: for X a simplicial presheaf and $\mathbf{\Pi}_R(\hat{X}) = \int^{[n] \in \Delta} \Delta[n] \coprod_{i_n} \mathbf{\Pi}_R(U_{i_n})$ the value of the left derived functor of $\mathbf{\Pi}_R$, this is related by a zig-zag of weak equivalences, as in the diagram above, to $\int^{[n] \in \Delta} \Delta[n] \cdot \coprod_{i_n} \mathbf{LConst}^* = \mathbf{LConst} \Pi(\hat{X}) =: \mathbf{\Pi}(\hat{X})$.

Remark. The ∞ -groupoid $\mathbf{\Pi}_R(X)$ may be thought of as generated in degree n from the $(n - k)$ -dimensional smooth paths in the smooth space of k -morphisms of X . The unit of the adjunction $X \rightarrow \mathbf{\Pi}(X)$ identifies X as the object of constant paths inside $\mathbf{\Pi}(X)$. In low categorical degree, a very explicit description of $\mathbf{\Pi}_R(X)$ for X a diffeological (Čech-)groupoid is given in [41]. There it is also discussed how morphism out of $\mathbf{\Pi}_R(X)$ encode connections on higher principal bundles and nonabelian gerbes on X . This aspect we describe now in the full abstract generality of a locally contractible $(\infty, 1)$ -topos $\mathbf{H}$.

7.4.3. Differential cocycles and connections

With the path ∞ -groupoid available, it is immediate to say what a *flat* connection on a principal ∞ -bundle is: a local system as seen by $\mathbf{\Pi}$. In cases where the obstruction to flatness is measured suitably by some characteristic class curv – the curvature –, we can define non-flat connections as cocycles in the **curv-twisted cohomology** of $\mathbf{\Pi}$.

Throughout now $\mathbf{H}$ is assumed to be a locally contractible $(\infty, 1)$ -topos. Notice that the units and counits of the adjunctions $(\mathbf{\Pi} \dashv \mathbf{LConst} \dashv \Gamma)$ induce canonical natural morphisms

$$X \rightarrow \mathbf{\Pi}(X)$$

and

$$\mathfrak{b}(A) \rightarrow A.$$

Definition 7..14 For $g : X \rightarrow \mathbf{B}G$ a cocycle with corresponding G -principal ∞ -bundle $P \rightarrow X$, we say that an extension $\nabla : \mathbf{\Pi}(X) \rightarrow \mathbf{B}G$ in

$$\begin{array}{ccc} X & \xrightarrow{g} & \mathbf{B}G \\ \downarrow & \nearrow \nabla & \\ \mathbf{\Pi}(X) & & \end{array}$$

is a flat connection on P with underlying cocycle g . If the underlying cocycle is trivial, then we call a corresponding flat connection a flat/closed G -valued differential forms datum.

$$\mathbf{B}G_{dR} : U \mapsto \mathbf{H}^I \left(\begin{bmatrix} U \\ \downarrow \\ \mathbf{\Pi}(U) \end{bmatrix}, \begin{bmatrix} * \\ \downarrow \\ \mathbf{B}G \end{bmatrix} \right)$$

for the sheaf of closed G -valued differential forms, where $\mathbf{H}^I$ is the arrow- $(\infty, 1)$ -category of $\mathbf{H}$.

We now turn to the discussion of general, not-necessarily flat connection connections on principal ∞ -bundles. Of the full theory we here just treat the special case where G is braided, meaning that $A := \mathbf{B}G$ is itself a group object with one further delooping $\mathbf{B}A$. The general theory is discussed elsewhere [71]. A simple but important example to keep in mind is $G = \mathbf{B}^n U(1)$, in which case $\mathbf{B}A = \mathbf{B}^{n+2}U(1)$.

Theorem 7..15 For $\mathbf{H}$ a locally contractible $(\infty, 1)$ -topos and $A \in \mathbf{H}$ a group object, we have a fiber sequence

$$\mathfrak{b}(A) \rightarrow A \xrightarrow{\text{curv}} \mathbf{B}A_{dR}.$$

This means that for $g : X \rightarrow A$ a given cocycle, the obstruction to lifting it to a flat differential cocycle $X \rightarrow \mathfrak{b}(A)$, which by adjunction corresponds to $\mathbf{\Pi}(X) \rightarrow A$, is precisely the nontriviality of its curv characteristic class $X \rightarrow A \rightarrow \mathbf{B}A_{dR}$.

Definition 7..16 A differential cocycle refining a cocycle $g : X \rightarrow \mathbf{B}G$ – or equivalently a connection on the G -principal ∞ -bundle $P \rightarrow X$ classified

by g – is a cocycle in curv -twisted A -cohomology, i.e. in the ∞ -pullback $\mathbf{H}_{\text{curv}}(X, \mathbf{B}G)$ in

$$\begin{array}{ccc} \mathbf{H}_{\text{curv}}(X, \mathbf{B}G) & \xrightarrow{F} & H(X, \mathbf{B}^2 G_{dR}) \\ \downarrow \eta & & \downarrow \\ \mathbf{H}(X, \mathbf{B}G) & \longrightarrow & \mathbf{H}(X, \mathbf{B}^2 G_{dR}) \end{array}$$

where $H(X, \mathbf{B}^2 G_{dR}) := \pi_0 \mathbf{H}(X, \mathbf{B}^2 G_{dR})$ is the set of G -valued de Rham cohomology classes and the right vertical morphism is a choice of cocycle representative for each class. For ∇ a differential cocycle we call $\eta(\nabla)$ the underlying cocycle and $F(\nabla)$ its curvature characteristic class.

Theorem 7..17 *In the model for $\mathbf{H}$ a differential cocycle/connection on X is given by a fixed cofibrant replacement $\emptyset \hookrightarrow Y \xrightarrow{\sim} X$ of X and a diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & \mathbf{B}G \\ \downarrow & & \downarrow \\ \Pi(Y) & \xrightarrow{\nabla} & \mathbf{E}BG \end{array}$$

such that the composite morphism in

$$F(\nabla) : \Pi(Y) \xrightarrow{\nabla} \mathbf{E}BG \rightarrow \mathbf{B}^2 G$$

equals the corresponding curvature de Rham cocycle. A morphism between such cocycles is a commuting diagram

$$\begin{array}{ccc} Y & \begin{array}{c} \xrightarrow{g_1} \\ \Downarrow \\ \xrightarrow{g_2} \end{array} & \mathbf{B}G \\ \downarrow & & \downarrow \\ \Pi(Y) & \begin{array}{c} \xrightarrow{\nabla_1} \\ \Downarrow \\ \xrightarrow{\nabla_2} \end{array} & \mathbf{E}BG \end{array}$$

in the $\mathbf{sSet}$ -enriched category $[C^{\text{op}}, \mathbf{sSet}]$, that keeps the curvature fixed, in that

$$\begin{array}{ccc} \Pi(Y) & \begin{array}{c} \xrightarrow{\nabla_1} \\ \Downarrow \\ \xrightarrow{\nabla_2} \end{array} & \mathbf{E}BG \longrightarrow \mathbf{B}^2 G \\ & & = F(\nabla_1) = F(\nabla_2) : \Pi(Y) \rightarrow \mathbf{B}^2 G. \end{array}$$

Given a differential cocycle (g, ∇) and a representation $\rho : \mathbf{B}G \rightarrow F$, the total space $E = g^* \rho^* F$ of the corresponding ρ -associated F -bundle is accompanied by its action ∞ -groupoid E_∇ with respect to the action of the paths in the base on the fibers, under the connection.

Definition 7..18 *The ∞ -groupoid E_∇ associated in the model to a given differential cocycle (g, ∇) is the pullback*

$$\begin{array}{ccc} E_\nabla & \xrightarrow{\quad} & \mathbf{E}F \\ \downarrow & & \downarrow \\ \Pi(Y) & \xrightarrow{\nabla} \mathbf{E}BG \longrightarrow & F \amalg_{\mathbf{B}G} \mathbf{E}BG \end{array} .$$

This fits canonically into a commuting diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad} & E_\nabla \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \Pi(Y) \end{array}$$

in the model.

We can also consider applying Π to objects that are not smooth ∞ -groupoids, but smooth $(\infty, 1)$ -categories. Notably if Σ is a causal *Lorentzian manifold* then this may naturally be regarded as a smooth poset, a smooth category with exactly none or one morphism between every ordered pair (σ_1, σ_2) of points: one if σ_2 is in the future of σ_1 , none otherwise. Then $\Pi(\Sigma)$ is a smooth $(\infty, 1)$ -category whose morphisms are generated from spacelike paths in Σ and timelike jumps, and whose 2-morphisms are generated from those of the form

$$\begin{array}{ccc} \sigma_1 & \xrightarrow{\quad} & \sigma'_1 \\ \downarrow & \Downarrow & \downarrow \\ \sigma_2 & \xrightarrow{\quad} & \sigma'_2 \end{array} ,$$

where horizontal morphism are given by spacelike paths and are invertible, while the vertical 1-morphisms are given by future-directed jumps and are non-invertible.

8. Quantization and quantum symmetries

We want to think of an associated ∞ -bundle $E \rightarrow X$ with connection ∇ as a *background field* (a generalization of an electromagnetic field) on X to which a higher dimensional *fundamental brane* – such as a *particle*, a *string* or a *membrane* – propagating on X may *couple*. If a piece of *worldvolume* of this fundamental brane is modeled by an $(\infty, 1)$ -category Σ then (following for instance [16]) we want to say that

- the space of fields over Σ is $C_\Sigma := \text{hom}(\Sigma, X)$, the object of maps from the worldvolume to target space X ;
- the space of states over Σ is the space of sections $\Gamma(\tau_\Sigma V)$ of the background field V *transgressed* to the space of fields.
- the quantum time propagation along a piece of worldvolume $\Sigma_{\text{in}} \rightarrow \Sigma \leftarrow \Sigma_{\text{out}}$ is given by pull-push of sections through the span $[\Sigma_{\text{in}}, X] \leftarrow [\Sigma, X] \rightarrow [\Sigma_{\text{out}}, X]$, weighted by the parallel transport of ∇ over Σ – the path integral.

We now try to give this a precise meaning.

8.1. Background field and space of states

Definition 8..1 A background structure for a σ -model is

- an ∞ -groupoid $X \in \mathbf{H}$ called target space;
- an ∞ -group G , called the gauge group;
- a G -principal ∞ -bundle $P \rightarrow X$ with a connection ∇ , called the background gauge field.
- a representation ρ called the background matter content.

Then for Σ a smooth $(\infty, 1)$ -category, to be called parameter space or worldvolume, we say

- $X \times \Sigma$ is the extended configuration space;
- an action functional is a connection on a ρ -associated ∞ -vector bundle on $X \times \Sigma$ whose restriction to X is (E, ∇) , called the gauge-interaction part of the action, whereas the part depending on Σ is called the kinetic action.

8.2. Transgression of cocycles to mapping spaces

Following [40], we identify transgression to mapping spaces with the internal hom applied to cocycles:

Definition 8..2 (transgression of cocycles) For $X \xrightarrow{\rho \circ g} F$ a cocycle classifying a ρ -associated ∞ -bundle on X and for Σ any other ∞ -groupoid, we say that the transgression $\tau_\Sigma(\rho \circ g)$ of $\rho \circ g$ to X^Σ is its value under the pointed internal hom in $\mathbf{H}$:

$$\tau_\Sigma(\rho \circ g) := \text{hom}(\Sigma, \rho \circ g) : \text{hom}(\Sigma, X) \rightarrow \text{hom}(\Sigma, F).$$

8.3. Branes and bibranes

From the second part of definition 7.7 one sees that spaces of states, being spaces of sections, are given by certain morphisms between background fields pulled back to spans/correspondences of target spaces. From the diagrammatics this has an immediate generalization, which leads to the notion of *branes* and *bibranes*.

Definition 8.3 (branes and bibranes) A brane for a background structure $(X, \rho \circ g)$ is a morphism $\iota : Q \rightarrow X$ equipped with a section of the

background field pulled back to Q , i.e. a transformation $\begin{array}{ccc} & Q & \\ \swarrow & & \searrow \iota \\ \text{pt} & \xRightarrow{V} & X \\ \downarrow \text{pt}_F & & \downarrow \rho \circ g \\ & F & \end{array} .$

More generally, given two background structures (X, g, ρ) and (X', g', ρ) , a

biblane between them is a span $\begin{array}{ccc} & Q & \\ \swarrow \iota & & \searrow \iota' \\ X & & X' \end{array}$ equipped with a transforma-

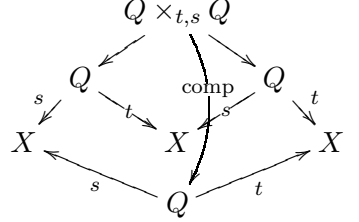
tion $\begin{array}{ccc} & Q & \\ \swarrow \iota & & \searrow \iota' \\ X & \xRightarrow{V} & X' \\ \downarrow \rho \circ g & & \downarrow \rho' \circ g' \\ & F & \end{array} .$

Bibranes may be composed –“fused” – along common background structures $(X, \rho \circ g)$: the composite or *fusion* of a bibrane V on Q with a bibrane V' on Q' is the bibrane $V \cdot V'$ given by the diagram

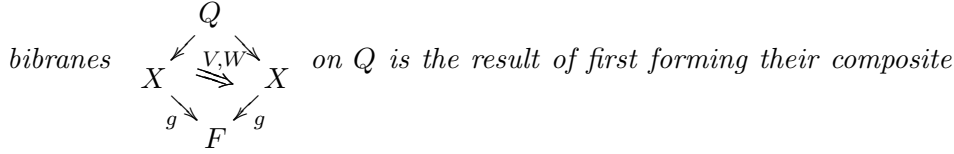
$$\begin{array}{ccc} & Q \times_{X'} Q & \\ & \swarrow \quad \searrow & \\ X & \xRightarrow{s^*V \cdot t^*V'} & X \\ & \swarrow \quad \searrow & \\ & F & \end{array} \quad \rho \circ g \quad \rho'' \circ g'' \quad := \quad \begin{array}{ccccc} & Q \times_{X'} Q' & & Q' & \\ & \swarrow s \quad \searrow t & & & \\ & Q & & Q' & \\ & \swarrow \quad \searrow & & \swarrow \quad \searrow & \\ X & \xRightarrow{V} & X' & \xRightarrow{V'} & X'' \\ & \swarrow \quad \searrow & \downarrow \rho' \circ g' & \swarrow \quad \searrow & \\ & F & & F & \end{array} \quad \rho \circ g \quad \rho'' \circ g''$$

If Q carries further structure, the fused bibrane on $Q \times_{t,s} Q$ may be pushed down again to Q , such as to produce a monoidal structure on bibranes on Q . Consider therefore a category $Q \xRightarrow[t]{s} X$ internal to ω -groupoids, equivalently a monad in the bicategory of spans internal to $\omega\mathbf{Groupoids}$,

with composition operation the morphism of spans



Definition 8..4 (monoidal structure on bibranes) *Given an internal category as above, and given an F -cocycle $g : X \rightarrow F$, the composite of two*



bibrane on $Q \times_{t,s} Q$ and then pushing that forward along comp :

$$V \star W := \int_{\text{comp}} (s^* V) \cdot (t^* W).$$

Here for finite cases, which we concentrate on, push-forward is taken to be the right adjoint to the pullback in a proper context.

Remarks. Notice that branes are special cases of bibranes and that bibrane composition restricts to an action of bibranes on branes. Also recall that the sections of a cocycle on X are the same as the branes of this cocycle for $\iota = \text{Id}_X$.

The idea of bibranes was first formulated in [18] in the language of modules for bundle gerbes. We show in section 9.5. how this is reproduced within the present formulation.

8.4. Quantum propagation

For $\rho : \mathbf{BG} \rightarrow F$ a representation, the corresponding representation space V is in applications typically equipped with a bimonoidal structure $(V, +, \cdot)$.

Given a sufficiently tame ∞ -groupoid $\Psi \rightarrow V$ over V , we may think of it under ∞ -groupoid cardinality as presenting a linear combination in V , where each element in V is weighted by the ∞ -groupoid cardinality of the fiber above it. In this way ∞ -groupoids over V are a way of presenting linear combinations in V without actually computing these. In particular, they may not converge in any sense.

Since the typical fiber of a ρ -associated ∞ -bundle is V , similarly an ∞ -groupoid $\Psi \rightarrow E$ over E may be thought of as representing a section of E , that may possibly be very singular.

For $\Sigma_{\text{in}} \rightarrow \Sigma \leftarrow \Sigma_{\text{out}}$ a piece of worldvolume with specified action functional $\exp(S)$

$$\begin{array}{ccccc}
 E & \xrightarrow{\quad} & \mathbf{E}F & \xrightarrow{\text{Id}} & \mathbf{E}F \\
 \downarrow & \searrow E_{\exp(S)} & \downarrow & \downarrow & \downarrow \\
 \Sigma \times X & \xrightarrow{\quad} & F & \xrightarrow{\quad} & F \amalg_{\mathbf{B}G} \mathbf{E}BG \\
 \downarrow & \searrow \Pi(\Sigma \times X) & \downarrow & \downarrow & \downarrow \\
 \Sigma & \xrightarrow{\quad} & \Pi(\Sigma) & \xrightarrow{\exp(S)} & F \amalg_{\mathbf{B}G} \mathbf{E}BG
 \end{array}$$

consider the corresponding span

$$\text{hom}_{\Sigma} \left(\left[\begin{array}{c} \Sigma_{\text{in}} \\ \downarrow \\ \Pi(\Sigma_{\text{in}}) \end{array} \right], \left[\begin{array}{c} E \\ \downarrow \\ E_{\exp(S)} \end{array} \right] \right) \leftarrow \text{hom}_{\Sigma} \left(\left[\begin{array}{c} \Sigma \\ \downarrow \\ \Pi(\Sigma) \end{array} \right], \left[\begin{array}{c} E \\ \downarrow \\ E_{\exp(S)} \end{array} \right] \right) \rightarrow \text{hom}_{\Sigma} \left(\left[\begin{array}{c} \Sigma_{\text{out}} \\ \downarrow \\ \Pi(\Sigma_{\text{out}}) \end{array} \right], \left[\begin{array}{c} E \\ \downarrow \\ E_{\exp(S)} \end{array} \right] \right).$$

Then the pull-push of ∞ -groupoids through the bottom part we may regard as modelling the quantum propagation along Σ .

9. Examples and applications

We start with some simple applications to illustrate the formalism and then exhibit some maybe interesting aspects in the context low dimensional or finite group QFT.

9.1. Ordinary vector bundles

Let G be an ordinary group, hence a 1-group, and denote by $F := \mathbf{Vect}$ the 1-category of vector spaces over some chosen ground field k . A linear representation ρ of G on a vector space V is indeed the same thing as a functor $\rho : \mathbf{B}G \rightarrow \mathbf{Vect}$ which sends the single object of $\mathbf{B}G$ to V .

The canonical choice of point $\text{pt}_F : \text{pt} \rightarrow \mathbf{Vect}$ is the ground field k , regarded as the canonical 1-dimensional vector space over itself. Using this we find from definition 7.5 that the *universal Vect-bundle* is $\mathbf{E}F = \mathbf{Vect}_*$, the category of *pointed* vector spaces with $\mathbf{Vect}_* \longrightarrow \mathbf{Vect}$ the canonical forgetful functor. Using this one finds from definition 7.6 that the ρ -associated vector bundle to the universal G -bundle is $V//G \longrightarrow \mathbf{B}G$,

where $V//G := (V \times G \xrightarrow[p]{p_1} V)$ is the action groupoid of G acting on V , the weak quotient of V by G .

For $g : X \xrightarrow{g} \mathbf{B}G$ a cocycle describing a G -principal bundle and for V the corresponding ρ -associated vector bundle according to definition 7.6, one sees that sections $\sigma \in \Gamma(V)$ in the sense of definition 7.7 are precisely sections of V in the ordinary sense.

9.2. The charged quantum particle

In this section we indicate how the familiar path integral quantization of the electromagnetically charged quantum particle arises from the general discussion. We will here fall short of attempting to discuss the measure on paths with respect to which the integral is done. While this is arguably the crucial technical point of making sense of the path integral, it may still be of interest to see here how just the underly structure of the path integral arises.

The background field for the charged particle that we consider is the electromagnetic field. The data involved is

- the target space X – a smooth manifold;
- the gauge group $G = U(1)$;
- a choice of representation

$$\rho : \mathbf{B}G \rightarrow \mathbf{Vect}_{\mathbb{C}},$$

taken to be the canonical representation on $V = \mathbb{C}$;

- the background field given by
 - a $U(1)$ -principal bundle $P \rightarrow X$ classified by a [[cocycle]] $g : X \rightarrow \mathbf{B}U(1)$ in $\mathbf{H}$ which in the model is given by an anafunctor $X \xleftarrow{\sim} Y \rightarrow \mathbf{B}U(1)$;
 - a connection ∇ on this bundle, which in the model is given by a diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & \mathbf{B}U(1) \\ \downarrow & & \downarrow \\ \Pi(Y) & \xrightarrow{\nabla} & \mathbf{E}BU(1) \end{array}$$

and whose field strength is given by the composite

$$F : \Pi(Y) \xrightarrow{\nabla} \mathbf{E}BU(1) \rightarrow \mathbf{B}^2U(1).$$

In [40] a realization of this setup on terms of smooth strict 2-groupoids is given. It is shown there in particular that such differential cocycles (g, ∇) correspond precisely to ordinary line bundles with connection: Y may be chosen to be the Čech 2-groupoid induced from a good cover $\{U_i \rightarrow X\}$, g is a transition function/ Čech cocycle $\{g_{ij} \in C^\infty(U_i \cap U_j), U(1)\}$ with respect to this cover, ∇ encodes the parallel transport of the correspondingly the local differential form data $A_i \in \Omega^1(U_i)$ and $F(\omega)$ is the parallel surface

transport of the corresponding curvature 2-form $F \in \Omega^2(X)$, which here in the physical model is the electromagnetic field strength tensor.

Inspection shows that the corresponding action groupoid $E_{\text{exp}(S)}$ can be characterized as

- objects are triples (x, σ, v) with $\sigma \in \Sigma$, $x \in X$ and v a vector in the fiber of E over X .
- morphisms $(x, \sigma, v) \rightarrow (x', \sigma', v')$ correspond to paths $[0, 1] \rightarrow X \times \Sigma$ from (σ, x) to (σ', x') , such that evaluating the action on this path takes v to v' .

Consider a “delta-section” of E , given by the terminal groupoid $\Pi : * \rightarrow E$ over E , that picks one vector v in the fiber E_x over a point x .

For $[t_1, t_2] \subset \Sigma$ an interval, the pull-push of this Ψ through the bottom part of the span in section 8.4. produces over the fiber $\simeq V$ of E over y the 0-truncated ∞ -groupoid which over v' is the set of those paths from x to y , whose action takes v to v' . If everything were suitably finite, the decategorification of this V -colored set would then indeed yield the familiar expression

$$\Psi'(y) = \int_{x \xrightarrow{\gamma} y} \exp(S_{\text{kin}}(\gamma)) \text{tra}_{\nabla}(\gamma) \Psi(x)$$

for the path integral of the charged particle.

9.3. Group algebras and category algebras from bibrane monoids

In its simplest version the notion of monoidal bibranes from section 8.3. reproduces the notion of *category algebra* $k[C]$ of a category C , hence also that of a *group algebra* $k[G]$ of a group G . Recall that the category algebra $k[C]$ of C is defined to have as underlying vector space the linear span of C_1 , $k[C] = \text{span}_k(C_1)$, where the product is given on generating elements $f, g \in C_1$ by

$$f \cdot g = \begin{cases} g \circ f & \text{if the composite exists} \\ 0 & \text{otherwise} \end{cases}$$

To reproduce this as a monoid of bibranes in the sense of section 8.3., take the category of fibers in the sense of section 7.2. to be $F = \text{Vect}$ as in section 9.1.. Consider on the space (set) of objects, C_0 , the trivial line

bundle given as an F -cocycle by $i : C_0 \longrightarrow \text{pt} \xrightarrow{\text{pt}_k} \text{Vect}$. An element in the monoid of bibranes for this trivial line bundle on the span given by the source and target map

$$\begin{array}{ccc} & C_1 & \\ s \swarrow & & \searrow t \\ C_0 & & C_0 \end{array} \quad \text{is a transformation of the form}$$

$$\begin{array}{ccc} & C_1 & \\ s \swarrow & & \searrow t \\ C_0 & \xrightarrow{V} & C_0 \\ i \swarrow & & \searrow i \\ & \text{Vect} & \end{array} .$$

In terms of its components this is canonically identified

with a function $V : C_1 \rightarrow k$ from the space (set) of morphisms to the ground

field and every such function gives such a transformation. This identifies the C -bibranes with functions on C_1 .

Given two such bibranes V, W , their product as bibranes is, according to definition 8.4, the push-forward along the composition map on C of the function on the space (set) of composable morphisms

$$C_1 \times_{t,s} C_1 \rightarrow k$$

$$(\xrightarrow{f} \xrightarrow{g}) \mapsto V(f) \cdot W(g).$$

This push-forward is indeed the product operation on the category algebra.

9.4. Monoidal categories of graded vector spaces from bibrane monoids

The straightforward categorification of the discussion of group algebras in section 9.3. leads to bibrane monoids equivalent to monoidal categories of graded vector spaces.

Let now $F := 2\mathbf{Vect}$ be a model for the 2-category of 2-vector spaces. For our purposes and for simplicity, it is sufficient to take $F := \mathbf{BVect} \hookrightarrow 2\mathbf{Vect}$, the 2-category with a single object, vector spaces as morphisms with composition being the tensor product, and linear maps as 2-morphisms. This can be regarded as the full sub-2-category of $2\mathbf{Vect}$ on 1-dimensional 2-vector spaces. And we can assume $\mathbf{BVect}$ to be strictified.

Then bibranes over G for the trivial 2-vector bundle on the point, i.e.

transformations of the form

$$\begin{array}{ccc} & G & \\ \swarrow & & \searrow \\ \text{pt} & \rightleftarrows & \text{pt} \\ \searrow & & \swarrow \\ & \mathbf{BVect} & \end{array}$$

canonically form the category

$\mathbf{Vect}^G$ of G -graded vector spaces. The fusion of such bibranes reproduces the standard monoidal structure on $\mathbf{Vect}^G$.

9.5. Twisted vector bundles

The ordinary notion of a brane in string theory is: for an abelian gerbe $\mathcal{G}$ on target space X a map $\iota : Q \rightarrow X$ and a $\mathrm{PU}(n)$ -principal bundle on Q whose lifting gerbe for a lift to a $U(n)$ -bundle is the pulled back gerbe $\iota^*\mathcal{G}$. Equivalently: a twisted $U(n)$ -bundle on Q whose twist is $\iota^*\mathcal{G}$. Equivalently: a gerbe module for $\iota^*\mathcal{G}$.

We show how this is reproduced as a special case of the general notion of branes from definition 8.3, see also [41].

The bundle gerbe on X is given by a cocycle $g : X \multimap \mathbf{BBU}(1)$. The co-efficient group has a canonical representation $\rho : \mathbf{B}^2U(1) \rightarrow F := \mathbf{BVect} \hookrightarrow$

2Vect on 2-vector spaces (as in section 9.4.) given by

$$\rho : \bullet \begin{array}{c} \text{Id} \\ \circlearrowleft \\ c \in \mathcal{U}(1) \\ \circlearrowright \\ \text{Id} \end{array} \bullet \mapsto \bullet \begin{array}{c} \mathbb{C} \\ \Downarrow \cdot c \\ \mathbb{C} \end{array} \bullet .$$

See also [41, 38].

By inspection one indeed finds that branes in the sense of diagrams

$$\begin{array}{ccc} & Q & \\ \swarrow \iota & & \searrow \\ \text{pt} & \xrightarrow{V} & X \\ \swarrow \text{pt}_F & & \searrow \rho \circ g \\ & \mathbf{BVect} & \end{array}$$

are canonically identified with twisted vector bundles on Q with twist given by the ι^*g : the naturality condition satisfied by the com-

ponents of V is

$$\begin{array}{ccc} & \mathbb{C} & \\ \swarrow \mathbb{C} & \Downarrow \text{Id} & \searrow \mathbb{C} \\ \mathbb{C} & \xrightarrow{\quad} & \mathbb{C} \\ \downarrow (\pi_1^* E)_y & \searrow \pi_{13}^* g_{\text{tw}}(y) & \downarrow \pi_3^* E_y \\ \mathbb{C} & \xrightarrow{\quad} & \mathbb{C} \end{array} = \begin{array}{ccc} & \mathbb{C} & \\ \swarrow \mathbb{C} & \downarrow (\pi_2^* E)_y & \searrow \mathbb{C} \\ \mathbb{C} & \xrightarrow{\pi_{12}^* g_{\text{tw}}(y)} & \mathbb{C} \xrightarrow{\pi_{23}^* g_{\text{tw}}(y)} \mathbb{C} \\ \downarrow \pi_1^* E_y & \searrow \mathbb{C} & \downarrow \pi_3^* E_y \\ \mathbb{C} & \xrightarrow{\quad} & \mathbb{C} \end{array} \quad , \text{ for all } y \in Y \times_X Y \times_X Y \times_X Y$$

$y \in Y \times_X Y \times_X Y \times_X Y$ in the triple fiber product of a local-sections admitting map $\pi : Y \rightarrow X$ whose homotopy coherent nerve $Y^\bullet$, regarded as an ∞ -category, provides the cover for the ∞ -anafunctor $X \xleftarrow{\simeq} Y^\bullet \xrightarrow{g} \mathbf{B}^2 U(1)$ representing the gerbe. See [41] for details. $E \rightarrow Y$ is the vector bundle on the cover encoded by the transformation V . The above naturality diagram says that its transition function g_{tw} satisfies the usual cocycle condition for a bundle only up to the twist given by the gerbe g : if $Y \rightarrow X$ is a cover by open subsets $Y = \sqcup_i U_i$, then the above diagram is equivalent to the familiar equation

$$(g_{\text{tw}})_{ij}(g_{\text{tw}})_{jk} = (g_{\text{tw}})_{ik} \cdot g_{ijk} .$$

In this functorial cocyclic form twisted bundles on branes were described in [39, 41].

9.6. Dijkgraaf-Witten theory

Dijkgraaf-Witten theory [17] is the σ -model which in our terms is specified by the following data:

- the target space $X = \mathbf{B}G$ is the one-object groupoid corresponding to a finite ordinary group G ;

- the background field $\alpha : \mathbf{B}G \rightarrow \mathbf{B}^3U(1)$ is a $\mathbf{B}^2U(1)$ -principal 3-bundle on $\mathbf{B}G$, classified by a group 3-cocycle on G .

More in detail, for G a finite group, let $\mathcal{B}G \in \infty\mathbf{Grpd}$ be the corresponding bare one-object groupoid. Then we may identify $\mathbf{B}G = \mathbf{LConst}\mathcal{B}G$. From purely formal manipulations with the adjunctions in our locally contractible $(\infty, 1)$ -topos of smooth ∞ -groupoids, using theorem 7..9 we find that

$$\begin{aligned} \Pi(\mathbf{B}G) &= \Pi(\mathbf{LConst}\mathcal{B}G) \\ &\simeq \mathbf{LConst} \circ \Pi \circ \mathbf{LConst}\mathcal{B}G \\ &= \mathbf{LConst}\mathcal{B}G \\ &= \mathbf{B}G \end{aligned},$$

which simply reflects the fact that there are no non-constant paths in the discrete $\mathbf{B}G$. Then from definition 7..14 it follows that every principal ∞ -bundle on such $\mathbf{B}G$ uniquely carries a flat connection. In this sense the cocycle $\alpha : \mathbf{B}G \rightarrow \mathbf{B}^3U(1)$ is indeed already the full background gauge field.

For Σ some manifold, a field configuration of the DW model is a morphism $\Sigma \rightarrow \mathbf{B}G$ in $\mathbf{H}$. Again just formally using the adjunction $(\Pi \dashv \mathbf{LConst})$ we find that this is equivalent to a morphism $\Pi(\Sigma) \rightarrow \mathcal{B}G$. By the remark below theorem 7..9 we learn that field configurations for the DW model on smooth manifolds correspond to topological G -principal bundles on the underlying topological space, i.e. simply to ordinary G -principal bundles on X . Of course the same can be seen also immediately in components by modelling $X \rightarrow \mathbf{B}G$ by an anafunctor out of the Čech nerve of a good cover.

9.6.1. The 3-cocycle

To understand the 3-cocycle and its transgression that we discuss later on, we make explicit what $\mathbf{B}G$ looks like:

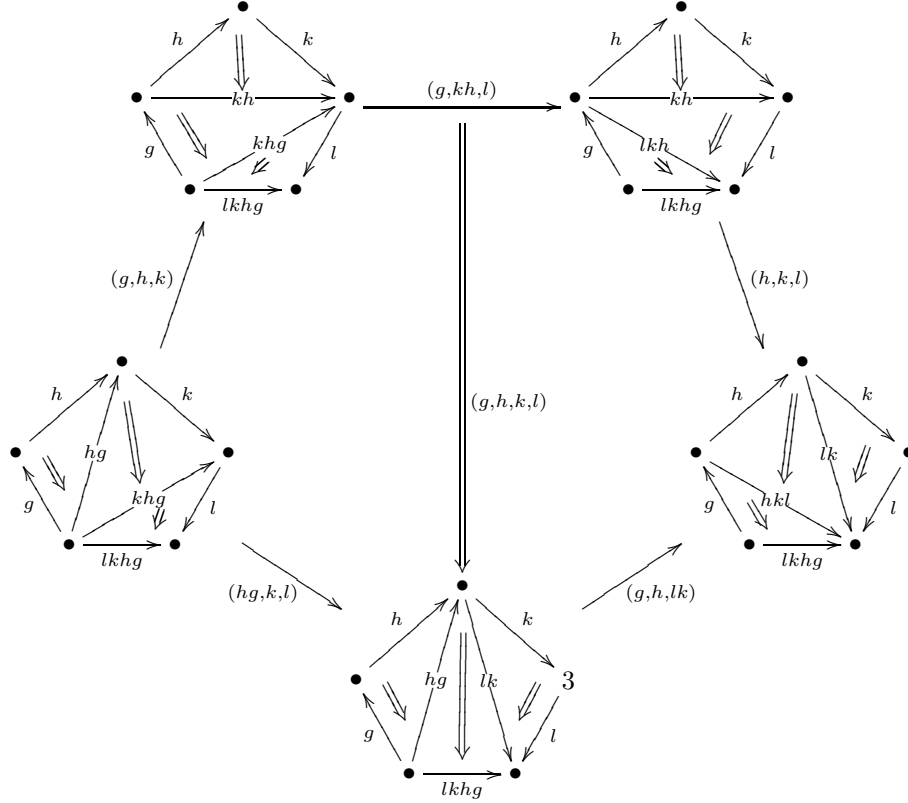
1-morphisms element elements of G ; 2-morphisms are triangles of the form

$$\left\{ \begin{array}{c} \bullet \\ \swarrow g \quad \searrow h \\ \bullet \quad \quad \bullet \\ \downarrow \Downarrow \\ \bullet \xrightarrow{hg} \bullet \end{array} \right\}, \text{ 3-morphisms are tetrahedra of the form}$$

$$\left\{ \begin{array}{ccc} \bullet & \xrightarrow{h} & \bullet \\ \uparrow g \quad \Downarrow & \searrow hg & \downarrow k \\ \bullet & \xrightarrow{khg} & \bullet \end{array} \quad \xrightarrow{(g,h,k)} \quad \begin{array}{ccc} \bullet & \xrightarrow{h} & \bullet \\ \uparrow g \quad \searrow kh & \Downarrow & \downarrow k \\ \bullet & \xrightarrow{khg} & \bullet \end{array} \right\}$$

together with their formal inverses. Finally 4-morphisms are 4-simplices of

the form



If we think of $\mathbf{BG}$ as modeled by a Kan complex, then this is precisely what it looks like in low degrees, if however we think of $\mathbf{BG}$ as being a strict ω -groupoid, then we need to consider the strict 4-groupoid which is generated from k -morphisms as indicated above, modulo the relation that every 5-simplex built from these 4-simplices commutes. This gives a strict 4-groupoid equivalent to the familiar one-object groupoid corresponding to G .

The ∞ -functor $\alpha : \mathbf{BG} \rightarrow \mathbf{B}^3U(1)$ has to send the generating 3-morphisms (g, h, k) to a 3-morphism in $\mathbf{B}^3U(1)$, which is an element $\alpha(g, h, k) \in U(1)$. In addition, it has to map the generating 4-morphisms between pasting diagrams of these 3-morphisms to 4-morphisms in $\mathbf{B}^3U(1)$. Since there are only identity 4-morphisms in $\mathbf{B}^3U(1)$ and since composition of 3-morphisms in $\mathbf{B}^3U(1)$ is just the product in $U(1)$, this says that α has to satisfy the equations

$$\forall g, h, k, l \in G : \alpha(g, h, k)\alpha(g, kh, l)\alpha(h, k, l) = \alpha(hg, k, l)\alpha(g, h, lk)$$

in $U(1)$. This identifies the ∞ -functor α with a group 3-cocycle on G . Conversely, every group 3-cocycle gives rise to such an ∞ -functor and one can

check that coboundaries of group cocycles correspond precisely to transformations between these ω -functors. For the strict ω -groupoid picture notice that α uniquely extends to the additional formal inverses of cells in Y which ensure that $Y \xrightarrow{\sim} \mathbf{BG}$ is indeed an acyclic fibration. For instance the 3-cell

$$\left\{ \begin{array}{ccc} \bullet & \xrightarrow{h} & \bullet \\ \uparrow g & \swarrow \nearrow hg & \downarrow k \\ \bullet & \xrightarrow{khg} & \bullet \end{array} \xrightarrow{(g,h,k)'} \begin{array}{ccc} \bullet & \xrightarrow{h} & \bullet \\ \uparrow g & \swarrow \nearrow kh & \downarrow k \\ \bullet & \xrightarrow{khg} & \bullet \end{array} \right\}$$

has to go to $\alpha(g, h, k)^{-1}$.

9.6.2. Transgression of DW theory to loop space: the twisted Drinfeld double

When we transgress DW theory, along the lines of section 8.2., to the free loop space $\Lambda G := \text{hom}(\mathbf{B}\mathbb{Z}, \mathbf{BG})$, the background gauge field $\mathbf{B}^2U(1)$ -3-bundle (a 2-gerbe or group 3-cocycle) reduced to just a $\mathbf{B}U(1)$ -2-bundle (a gerbe or group 2-cocycle).

Proposition 9.1 *The background field α of Dijkgraaf-Witten theory transgressed according to definition 8.2 to the mapping space of parameter space $\Sigma := \mathbf{B}\mathbb{Z}$ – a combinatorial model of the circle –*

$$\tau_{\mathbf{B}\mathbb{Z}}\alpha := \text{hom}(\mathbf{B}\mathbb{Z}, \alpha)_1 : \Lambda G \rightarrow \mathbf{B}^2U(1)$$

is the groupoid 2-cocycle known as the twist of the Drinfeld double ([10, 29]):

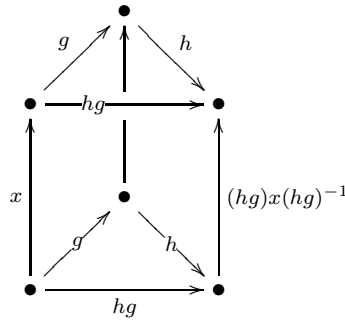
$$(\tau_{\mathbf{B}\mathbb{Z}}\alpha) : (x \xrightarrow{g} gxg^{-1} \xrightarrow{h} (hg)x(hg)^{-1}) \mapsto \frac{\alpha(x, g, h) \alpha(g, h, (hg)x(hg)^{-1})}{\alpha(h, gxg^{-1}, g)}.$$

Proof. A 2-cell (x, g, h) in ΛG

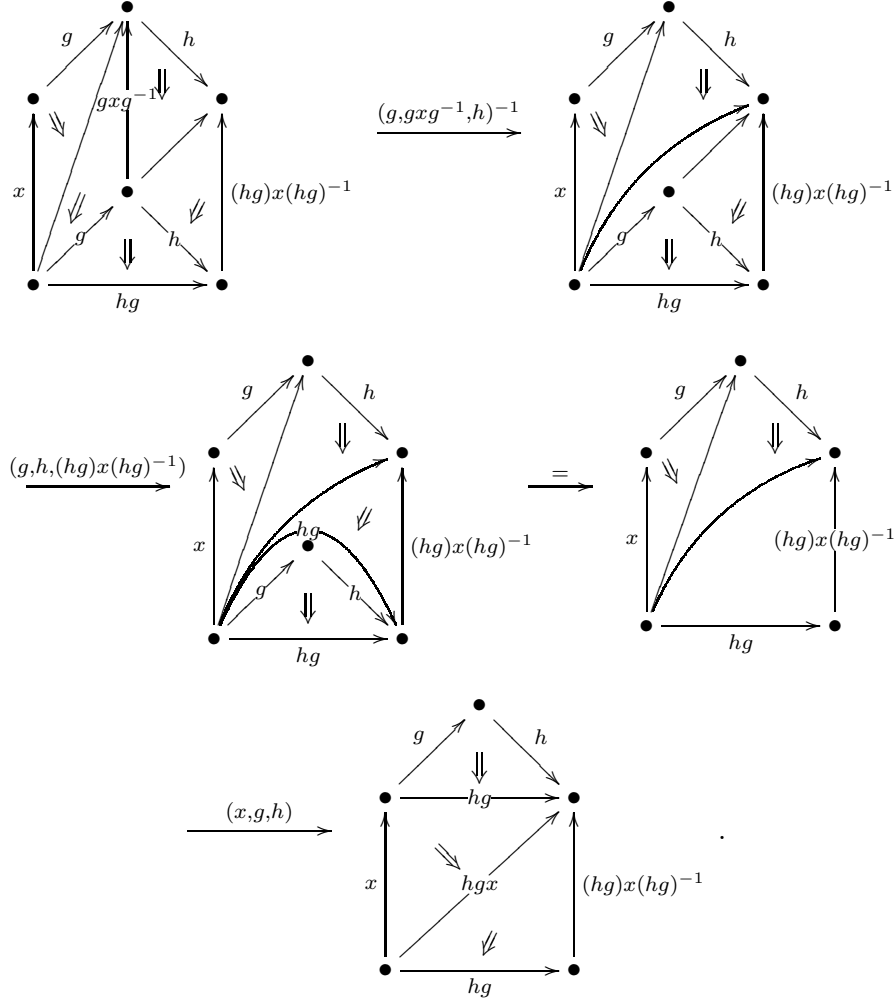
$$\begin{array}{ccc} & gxg^{-1} & \\ g \nearrow & & \searrow h \\ x & \xrightarrow{hg} & (hg)x(hg)^{-1} \end{array}$$

corresponds to a

closed prism



in $\mathbf{BG}$. The 2-cocycle $\tau_{\mathbf{B}\mathbb{Z}}\alpha$ sends the 2-cell in ΛG to the evaluation of α on this prism. One representative of such a 3-morphism, going from the back and rear to the top and front of this prism, is



This manifestly yields the cocycle as claimed.

9.6.3. The Drinfeld double modular tensor category from DW bibranes

Let again $\rho : \mathbf{B}^2U(1) \rightarrow 2\mathbf{Vect}$ be the representation of $\mathbf{BU}(1)$ from section 9.4. and let $\tau_{\mathbf{B}\mathbb{Z}}\alpha : \Lambda G \rightarrow \mathbf{B}^2U(1)$ be the 2-cocycle obtained in section 9.6.2. from transgression of a Dijkgraaf-Witten line 3-bundle on $\mathbf{BG}$ and consider the ρ -associated 2-vector bundle $\rho \circ \tau_{\mathbf{B}\mathbb{Z}}\alpha$ corresponding to that. Its sections according to definition 7.7 form a category $\Gamma(\tau_{\mathbf{B}\mathbb{Z}}\alpha)$.

Corollary 9..2 *The category $\Gamma(\tau_{\mathbf{B}\mathbb{Z}}\alpha)$ is canonically isomorphic to the representation category of the α -twisted Drinfeld double of G .*

Proof. Follows by inspection of our definition of sections applied to this case and using the relation established in 9.6.2. between nonabelian cocycles and the ordinary appearance of the Drinfeld double in the literature:

a section is a natural transformation $\sigma : \text{const}_k \rightarrow \tau_{\mathbf{B}\mathbb{Z}}\alpha : \Lambda G \rightarrow 2\text{Vect}$. Its components are therefore an assignment $\sigma : G \rightarrow \text{Vect}$ such that over each

$$\begin{array}{ccc} & g x g^{-1} & \\ g \nearrow & & \searrow h \\ x & \xrightarrow{\quad} & (h g) x (h g)^{-1} \end{array}$$

the naturality prism equations

$$\begin{array}{ccc} & \mathbb{C} & \\ & \swarrow \quad \searrow & \\ \mathbb{C} & & \mathbb{C} \\ \downarrow \sigma(x) & \xrightarrow{\quad} & \downarrow \sigma((gh)x(gh)^{-1}) \\ \mathbb{C} & & \mathbb{C} \end{array} \quad = \quad \begin{array}{ccc} & \mathbb{C} & \\ & \swarrow \quad \searrow & \\ \mathbb{C} & & \mathbb{C} \\ \downarrow \sigma(x) & \xrightarrow{\quad} & \downarrow \sigma((gh)x(gh)^{-1}) \\ \mathbb{C} & & \mathbb{C} \end{array} \quad .$$

(The left diagram has a diagonal arrow from the bottom-left $\mathbb{C}$ to the top-right $\mathbb{C}$ labeled $\sigma_{(gh)}^{\sigma((gh)x(gh)^{-1})}$. The right diagram has a diagonal arrow from the top-left $\mathbb{C}$ to the bottom-right $\mathbb{C}$ labeled $\sigma_{(gh)}^{\sigma((gh)x(gh)^{-1})}$ and a central square with arrows labeled $\sigma(g)$, $\sigma(h)$, and $(\tau_{\mathbf{B}\mathbb{Z}}\alpha)(g,h)$.)

This defines an $\tau_{\mathbf{B}\mathbb{Z}}\alpha$ -twisted equivariant vector bundle over ΛG . As in the discussion there, this exhibits σ as a twisted representation of ΛG . This establishes the claim [10].

9.6.4. The fusion product

In the case that α is trivial, the representation category of the twisted Drinfeld double is well known to be a modular tensor category. The fusion tensor product on this category is reproduced from a monoid of bibranes on ΛG .

We may think of $* \rightarrow \mathbf{B}\mathbb{Z} \leftarrow *$ as the cobordism cospan of a closed string. Homming this cospan into the target space $\mathbf{B}G$ produces the span of groupoids. A bibrane on this is, by

$$\begin{array}{ccc} & \Lambda G & \\ s \swarrow & & \searrow t \\ \mathbf{B}G & & \mathbf{B}G \end{array} \quad \text{of groupoids. A bibrane} \quad \begin{array}{ccc} & \Lambda G & \\ s \swarrow & & \searrow t \\ \mathbf{B}G & \xrightarrow{\quad \sigma \quad} & \mathbf{B}G \\ \text{const}_* \downarrow & & \downarrow \text{const}_* \\ & 2\text{Vect} & \end{array}$$

the above, an untwisted representation of ΛG . $\sigma : x \mapsto \sigma(x)$. Analogous to section 9.4., we find that the bibrane fusion of σ with some other σ' is the representation $\sigma \star \sigma' : x \mapsto \bigoplus_{y \in G} \sigma(xy^{-1}) \otimes \sigma'(y)$. This is indeed the fusion product on these representations.

9.7. Outlook: Chern-Simons theory

Dijkgraaf-Witten theory for a finite group G and a group 3-cocycle α is supposed to be a finite analog of the richer Chern-Simons theory which is defined for a Lie group G and a certain bundle gerbe on G . In $(\infty, 1)$ -topos language this can be made precise in that both theories (at least as far as their classical formulation goes, which is fully understood) are literally defined on the same type of data, only that the extra structure on G differs, which is a difference that the abstract structure of $\mathbf{H}$ takes care of automatically:

in both cases the target space object is $X = \mathbf{B}G$ in $\mathbf{H}$ and the background gauge field is a $\mathbf{B}^2U(1)$ -principal 3-bundle with connection ∇ , given by a differential cocycle (α, ∇)

$$\begin{array}{ccc} \mathbf{B}G & \xrightarrow{\alpha} & \mathbf{B}^3U(1) \\ \downarrow & & \downarrow \\ \Pi(\mathbf{B}G) & \xrightarrow{\nabla} & \mathbf{E}\mathbf{B}^3U(1) \end{array}$$

as in section 7.4.. We had seen that in the Dijkgraaf-Witten case of finite G , this general statement reduces to the much simpler statement that the background field is already determined by the morphism $\alpha : \mathbf{B}G \rightarrow \mathbf{B}^3U(1)$, which moreover in this case is nothing but a bare group 3-cocycle on G with coefficients in $U(1)$.

But the very same morphism $\mathbf{B}G \rightarrow \mathbf{B}^3U(1)$ is something much richer in the case that G is a genuine Lie group. There are various ways to characterize this morphism in terms of a concrete model. One way to think of it is as a G -equivariant bundle 2-gerbe on the point, a bundle gerbe on G with some extra structure and properties.

Similarly the differential cocycle: by unwinding what a morphism $\Pi_R(\mathbf{B}G) \rightarrow \mathbf{B}^3U(1)$ is in the model, one finds that it can for instance be given by a degree 4-class in the complex of differential forms on the simplicial manifold $\cdots G \times G \rightrightarrows G \rightrightarrows *$. That is given by a certain 3-form on G and a 2-form on $G \times G$, satisfying some relation. Details of this using explicit models in terms of bundle gerbes have been worked out by various authors, see for instance [77] for a good account.

Then for Σ a piece of cobordism, a field configuration ϕ for the Chern-Simons quantum field theory is a morphism in $\mathbf{H}^I$ from $\Sigma \rightarrow \Pi(\Sigma)$ to $\mathbf{B}G \rightarrow \Pi(\mathbf{B}G)$. This defines on Σ the differential cocycle

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\phi} & \mathbf{B}G & \xrightarrow{\alpha} & \mathbf{B}^3U(1) \\ \downarrow & & \downarrow & & \downarrow \\ \Pi(\Sigma) & \xrightarrow{\Pi(\phi)} & \Pi(\mathbf{B}G) & \xrightarrow{\nabla} & \mathbf{E}\mathbf{B}^3U(1) \end{array} .$$

For 3-dimensional Σ , its volume-holonomy is the familiar Chern-Simons action. One way to see this is by differentially approximating the ∞ -Lie

groups involved here by their corresponding ∞ -Lie algebras (L_∞ -algebras) one passes from this diagram to a corresponding diagram of dg-algebras: if $P \rightarrow \Sigma$ is the (ordinary) G -principal bundle classified by α , this is

$$\begin{array}{ccccc} \Omega_{\text{vert}}^\bullet(P) & \xleftarrow{A_{\text{vert}}} & \text{CE}(\mathfrak{g}) & \xleftarrow{\mu} & \text{CE}(b^2\mathfrak{u}(1)) , \\ \uparrow & & \uparrow & & \uparrow \\ \Omega^\bullet(P) & \xleftarrow{(A, F_A)} & W(\mathfrak{g}) & \xleftarrow{(\text{cs}_\mu, P_\mu)} & W(b^2\mathfrak{u}(1)) \end{array}$$

where $\mathfrak{g}$ is the Lie algebra of G , $\text{CE}(\mathfrak{g})$ its Chevalley-Eilenberg algebra, $W(\mathfrak{g})$ its Weil algebra, $\text{CE}(b^2\mathfrak{u}(1))$ the dg-algebra on a single degree-3 generator with trivial differential and $W(b^2\mathfrak{u}(1))$ the one with free differential, accordingly $\mu \in \text{CE}(\mathfrak{g})$ a Lie algebra 3-cocycle and $P_\mu = d_{W(b^2\mathfrak{u}(1))}\text{cs}_\mu$ the invariant polynomial in transgression with it. The image of the degree 3 generator under the total horizontal bottom morphism is the Chern-Simons form $\text{cs}_\mu(A, F_A)$ of the $\mathfrak{g}$ -valued connection 1-form A on P . This differential approximation to the differential Chern-Simons cocycle in $\mathbf{H}$ is discussed in [35]. A full account shall be given elsewhere.

10. Conclusion

We discussed that symmetries assembled into categories and higher analogues allow for a systematic and uniform treatment of many phenomena in noncommutative geometry, geometry and physics. The emphasis has been on monoidal categories acting on categories of sheaves in NC geometry; and on higher cocycles for smooth ∞ -groupoids. We sketched generalized notions of background fields and aspects of their induced σ -models.

Let us list some related topics not touched on here. Some σ -models and couplings can be defined using infinitesimal versions of gauge n -groupoids. E.g. a remarkable AKSZ construction [1] utilizes essentially Lie algebroids as gauge “Lie algebras”. The relation between higher groupoids and L_∞ -algebroids (particularly “integration”) is an active area of research (cf. its role in our context in [36]).

With actions of higher groups, notions of equivariance for categorified objects (e.g. gerbes) under usual or higher groups need some treatment. The first author has studied Z_2 -equivariant gerbes as an expression of so-called Jandl structures in CFT; and the second author studied 2-equivariant object in 2-fibered categories (presented at WAGP06, Vienna 2006; the basic definition is sketched in [45]).

Acknowledgements. Large part of this article was written when Z.Š. was at MPI, Bonn, whom he thanks for hospitality. A bilateral project Germany (DAAD)-Croatia (MZOS) on homological algebra and applications has supported meetings between the two authors. Z.Š. thanks organizers of the 5th Mathematical Physics Meeting in Belgrade, particularly Prof.

B. DRAGOVIĆ, for the opportunity to give a presentation. The updates for the arXiv version were done in Utrecht (Urs) and at l’IHES (Zoran). U.S. thanks DOMENICO FIORENZA and RICHARD WILLIAMSON for very useful discussion of matters related to section 7, 8, 9.

References

- [1] M. Alexandrov, A. Schwarz, O. Zaboronsky, M. Kontsevich, *The geometry of the master equation and TQFT*, Int. J. Mod Phys. A **12**, 7 (1997), 1405–1429.
- [2] J. Baez, A. Lauda, *Higher-dimensional algebra V: 2-groups*, Theory Appl. Categ. **12** (2004), 423–491, [arXiv:math.QA/0307200].
- [3] J. Baez, M. Shulman, *Lectures on n -categories and cohomology*, [arXiv:math.CT/0608420].
- [4] J. Baez, A. Crans, *Higher-dimensional algebra VI: Lie 2-algebras*, Theory Appl. Categ. **12** (2004), 492–528, [arXiv:math.QA/0307263].
- [5] I. Baković, *Bigroupoid 2-bundles*, LMU thesis, 2008; *The simplicial interpretation of bigroupoid 2-torsors*, [arXiv:0902.3436].
- [6] T. Bartels, *2-Bundles*, [arXiv:math/0410328] [math.CT].
- [7] P. Bantay, *Characters of crossed modules and premodular categories* [arXiv:math.QA/0512542].
- [8] G. Böhm, K. Szlachányi, *Hopf algebroid symmetry of abstract Frobenius extensions of depth 2*, Comm. Alg. **32**, 11, 4433–4464 (2004)
- [9] L. Breen, *On the classification of 2-gerbes and 2-stacks*, Astérisque **225** (1994); J. Giraud, *Cohomologie non abélienne*, Grundle Math. Wiss. 179, Springer 1971; T. Beke, *Higher Čech theory*, K-Theory **32** 4 (2004); 293–322; I. Moerdijk, *Introduction to the language of stacks and gerbes*, [arXiv:math.AT/0212266].
- [10] R. Dijkgraaf, V. Pasquier, P. Roche, *Quasi Hopf algebras, group cohomology and orbifold models*, Nucl. Phys. B Proc. Suppl. **18B** 60–72 (1991); S. Majid, *Quantum double for quasiHopf algebras*, Lett. Math. Phys. 45 (1998) 1–9; P. Bressler, *Levels and characters*, lecture at NCG III, Mittag-Leffler May 19, 2004; and subsequent notes on cyclic objects, inertia groupoids, gerbes (2002–2004); S. Willerton, *The twisted Drinfeld double of a finite group via gerbes and finite groupoids* [arXiv:math/0503266]; V. Hinich, *Drinfeld double for orbifolds*, [arXiv:math.QA/0511476].
- [11] R. Brown, P. Higgins, R. Sivera, *Nonabelian algebraic topology*, book draft.
- [12] R. Brown, M. Golasiński, *A model structure on the homotopy theory of crossed complexes*, Cahier Top. Géom. Diff. Catég. **30** (1) (1989) 61–82
- [13] R. Brown, *Abstract homotopy Theory and generalized sheaf cohomology*, Trans. Am. Math. Soc. **186** (1973), 419–458
- [14] A. Connes, *Noncommutative geometry*, Academic Press, San Diego 1994.
- [15] K. Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214, [arXiv:math/0412149] [math.QA].
- [16] D. Freed, *Higher Algebraic Structures and Quantization*, Comm. Math. Phys. **159** (1994) 343–398 [arXiv:hep-th/9212115].
- [17] D. Freed, *Chern-Simons theory with finite gauge group*, Comm. Math. Phys. **156** (1993) 435–472 [arXiv:hep-th/9111004]
- [18] J. Fuchs, C. Schweigert, K. Waldorf, *Bi-branes: Target space geometry for world sheet topological defects*, J. Geom. Phys. **58** (2008) 576–598 [arXiv:hep-th/0703145]

- [19] A. Grothendieck et al., *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), LNM **224**, Springer (1971), retyped as: [arXiv:math.AG/0206203].
- [20] L. K. Hadjiivanov, R. R. Paunov, I. T. Todorov, $\mathcal{U}_q$ -covariant oscillators and vertex operators, J. Math. Phys. **33** (4), pp. 1379–1394, April 1992.
- [21] A. M. Kapustin, D. O. Orlov, *Lectures on mirror symmetry, derived categories and D-branes*, Uspehi Mat. Nauk **59**, 5(2004), 101–134 (Rus. Math. Surveys **59**, 5(2004), 907–940), [arXiv:math.AG/0308173]; *Remarks on A-branes, mirror symmetry, and the Fukaya category*, J. Geom. Phys. **48**:84 (2003) [arXiv:hep-th/0109098].
- [22] Y. Lafont, F. Metayer, and K. Worytkiewicz, *A folk model structure on omega-cat*, [arXiv:math.CT/0712.0617]; F. Metayer, *Cofibrant complexes are free*, [arXiv:math.CT/0701746].
- [23] N. P. Landsman, *Mathematical topics between classical and quantum mechanics*, Springer, New York 1998. xx+529 pp.
- [24] T. Leinster, *Higher categories, higher operads*, London Math. Soc. LNS **298**, Cambridge Univ. Press 2004, [arXiv:math.CT/0305049].
- [25] M. Makkai, *Avoiding the axiom of choice in general category theory*, J. Pure Appl. Algebra **108** (1996), no. 2, 109–173, [http://www.math.mcgill.ca/makkai/anafun].
- [26] L. Katzarkov, M. Kontsevich, T. Pantev, *Hodge theoretic aspects of mirror symmetry*, [arXiv:0806.0107].
- [27] M. Kontsevich, *Homological algebra of mirror symmetry*, Proc. ICM Zurich 1994, Vol. 1, 2, 120–139, Birkhäuser, Basel, 1995.
- [28] G. Mack, V. Schomerus, *Quasihopf quantum symmetry in quantum theory*, Nucl. Phys. **B370** (1992) 185–230.
- [29] S. Majid, *Foundations of quantum group theory*, CUP 1995, 2000.
- [30] L. Mauri, M. Tierney, *Two descent, two-torsors and local equivalence*, J. Pure Appl. Algebra **143**, 1-3 (1999), 313–327.
- [31] D. O. Orlov, *Derived categories of coherent sheaves, and motives*. Uspehi M. N. **60**, 6 (2005), no. 6(366), 231–232 (Russ. Math. Surveys **60**, 6 (2005), 1242–1244).
- [32] D. Roberts, U. Schreiber, *The inner automorphism 3-group of a strict 2-group*, J. Homotopy Relat. Struct. **3** (2008) no. 1, 193–244, [arXiv:0708.1741] [math.CT].
- [33] A. L. Rosenberg, *Noncommutative schemes*, Comp. Math. **112** (1998), 93–125;
- [34] A. L. Rosenberg, *Topics in noncommutative algebraic geometry, homological algebra and K-theory*, preprint MPIM2008-57, at <http://www.mpim-bonn.mpg.de>
- [35] H. Sati, U. Schreiber, J. Stasheff, L_∞ -connections and applications to String- and Chern-Simons n -transport, in *Recent Developments in QFT*, eds. B. Fauser et al., Birkhäuser, Basel (2008), [arXiv:0801.3480] [math.DG].
- [36] H. Sati, U. Schreiber, Z. Škoda, D. Stevenson, *Twisted differential nonabelian cohomology: Twisted $(n-1)$ -brane n -bundles and their Chern-Simons $(n+1)$ -bundles with characteristic $(n+2)$ -classes*, [http://www.math.uni-hamburg.de/home/schreiber/nactwist.pdf]
- [37] H. Sazdjian, Y. S. Stanev, I. T. Todorov, *$SU(3)$ -coherent state operators and invariant correlation functions and their quantum group counterparts*, J. Math. Phys., **36** (1995), 2030–2052.
- [38] U. Schreiber, *AQFT from n -functorial QFT*, Comm. Math. Phys. **291**, 2, 2009 357–401, [arXiv:0806.1079]

- [39] U. Schreiber *Quantum 2-states: Sections of 2-vector bundles*, talk at *Higher categories and their applications*, Fields Institute, Jan. 2007, [<http://www.math.uni-hamburg.de/home/schreiber/atd.pdf>]
- [40] U. Schreiber, K. Waldorf, *Parallel transport and functors*, [[arXiv:0705.0452](https://arxiv.org/abs/0705.0452)]; sequel *Smooth functors vs. differential forms*, [[arXiv:0802.0663](https://arxiv.org/abs/0802.0663)].
- [41] U. Schreiber, K. Waldorf, *Connections on nonabelian gerbes and their holonomy*, [[arXiv:math.DG/0808.1923](https://arxiv.org/abs/math/0808.1923)].
- [42] Z. Škoda, *Localizations for construction of quantum coset spaces*, Banach Center Publications **61** (2003), 265–298 [[math.QA/0301090](https://arxiv.org/abs/math/0301090)].
- [43] Z. Škoda, *Coherent states for Hopf algebras*, Letters in Mathematical Physics **81**, 1, (2007) 1–17 [[math.QA/0303357](https://arxiv.org/abs/math/0303357)].
- [44] Z. Škoda, *Noncommutative localization in noncommutative geometry*, London Math. Soc. Lecture Note Series 330, ed. A. Ranicki; pp. 220–313, Cambridge Univ. Press, Cambridge, 2006, [[math.QA/0403276](https://arxiv.org/abs/math/0403276)].
- [45] Z. Škoda, *Some equivariant constructions in noncommutative algebraic geometry*, Georgian Math. J. **16** (2009), n. 1, 183–202, [[arXiv:0811.4770](https://arxiv.org/abs/0811.4770)].
- [46] R. Street, *The algebra of oriented simplexes*, J. Pure Appl. Alg. **49**, 3 (1987) 283–335; R. Street, *Categorical and combinatorial aspects of descent theory*, Appl. Categ. Structures **12** (2004), no. 5–6, 537–576, [[arXiv:math.CT/0303175](https://arxiv.org/abs/math/0303175)].
- [47] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry, 1–104, Math. Surveys Monogr. **123**, Amer. Math. Soc. Providence, RI, 2005, [[arXiv:math.AG/0412512](https://arxiv.org/abs/math/0412512)].
- [48] C. Wockel, *A global perspective to gerbes and their gauge stacks*, [[arXiv:0803.3692](https://arxiv.org/abs/0803.3692)], to appear in Forum Math.

NEW REFERENCES ADDED AFTER PUBLICATION

- [49] John C. Baez, Alexander E. Hoffnung, *Convenient categories of smooth spaces*, [arxiv/0807.1704](https://arxiv.org/abs/0807.1704)
- [50] J. Baez, J. Huerta, *An invitation to higher gauge theory*, [[arXiv:1003.4485](https://arxiv.org/abs/1003.4485)]
- [51] C. Barwick, D. Kan, *Relative categories; another model for the homotopy theory of homotopy theories I,II*, preprints at <http://www.math.harvard.edu/~clarkbar>
- [52] D. Ben-Zvi, J. Francis, D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, [arXiv:0805.0157](https://arxiv.org/abs/0805.0157), J. Amer. Math. Soc. 2010.
- [53] D. Ben-Zvi, D. Nadler, *Loop spaces and connections*, [arxiv/1002.3636](https://arxiv.org/abs/1002.3636)
- [54] A. I. Bondal, M. M. Kapranov, *Enhanced triangulated categories* (Russian) Mat. Sbornik **181** (1990), No.5, 669–683; Engl. transl. USSR Math. Sbornik, **70** (1991), No. 1, 93–107
- [55] T. Brzeziński, S. Majid, *Coalgebra bundles*, Comm. Math. Phys. **191**(1998), n. 2, 467–492.
- [56] D. Dugger, *Universal homotopy theories*, Advances in Mathematics **164** (2001), see also the unpublished notes *Sheaves and homotopy theory* <http://www.uoregon.edu/~ddugger/cech.html>
- [57] D. Freed, M. Hopkins, J. Lurie, C. Teleman, *Topological quantum field theories from compact Lie groups*, [arxiv/0905.0713](https://arxiv.org/abs/0905.0713)
- [58] A. Grothendieck, *Pursuing stacks*, 1983 manuscript.
- [59] R. Jardine, *Simplicial objects in a Grothendieck topos*, Contemporary Mathematics **55**, I (1986), 193–239; Cocycle categories, [arXiv:math/0605198](https://arxiv.org/abs/math/0605198), *Simplicial presheaves*, Fields Institute lectures (2007) <http://www.math.uwo.ca/~jardine/papers/Fields-01.pdf>

-
- [60] A. Joyal, Quasi-categories and Kan complexes, (in Special volume celebrating the 70th birthday of Prof. Max Kelly) J. Pure Appl. Algebra 175 (2002), no. 1-3, 207222
 - [61] D. Kaledin, *Tokyo 2007/2008 lectures "Homological methods in non-commutative geometry"*, <http://imperium.lenin.ru/~kaledin/tokyo/final.pdf>
A slightly different version of a course in Seoul:
<http://imperium.lenin.ru/~kaledin/seoul>
 - [62] V. A. Lunts, D. O. Orlov, *Uniqueness of enhancement for triangulated categories*, J. Amer. Math. Soc. **23** (2010), 853–908, [arXiv/0908.4187](https://arxiv.org/abs/0908.4187)
 - [63] Jacob Lurie, *Derived algebraic geometry I: stable ∞ -categories*, <http://arxiv.org/abs/math/0608228>
 - [64] Jacob Lurie, *Derived algebraic geometry V: structured spaces*, <http://arxiv.org/abs/math/0905.0459>
 - [65] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies **170**, Princeton University Press 2009, [math.CT/0608040](https://arxiv.org/abs/math/0608040); published version <http://www.math.harvard.edu/~lurie/papers/highertopoi.pdf>
 - [66] *nLab*, a collaborative online wiki project for mathematics, physics and philosophy, from the perspective of higher category theory, <http://ncatlab.org/nlab>
 - [67] Thomas Nikolaus, *Algebraic models for higher categories*, [arXiv/1003.1342](https://arxiv.org/abs/1003.1342)
 - [68] Dmitri Orlov, *Quasi-coherent sheaves in commutative and non-commutative geometry*, Izv. RAN. Ser. Matem. **67**, 3 (2003), 119–138, <http://www.mpim-bonn.mpg.de/preprints/send?bid=56>
 - [69] Hisham Sati, *Geometric and topological structures related to M-branes*, [arXiv:1001.5020](https://arxiv.org/abs/1001.5020)
 - [70] Hisham Sati, Urs Schreiber, Jim Stasheff, *Twisted differential String- and Five-brane structures*, [arXiv/0910.4001](https://arxiv.org/abs/0910.4001)
 - [71] Urs Schreiber *Differential cohomology in an $(\infty, 1)$ -topos*, <http://ncatlab.org/schreiber/>
 - [72] Carlos Simpson, *Homotopy theory of higher categories*, draft of a book, [arxiv/1001.4071](https://arxiv.org/abs/1001.4071)
 - [73] Z. Škoda, *Distributive laws for actions of monoidal categories*, [arXiv:math.CT/0406310](https://arxiv.org/abs/math/0406310)
 - [74] B. Toën, G. Vezzosi, *Homotopical algebraic geometry I: Topos theory*, [math.AG/0207028](https://arxiv.org/abs/math/0207028), II: geometric stacks and applications [math.AG/0404373](https://arxiv.org/abs/math/0404373)
 - [75] B. Toën, G. Vezzosi, *From HAG to DAG: derived moduli stacks in Axiomatic, enriched and motivic homotopy theory*, 173–216, NATO Sci. Ser. II Math. Phys. Chem., 131, Kluwer Acad. Publ., Dordrecht, 2004.
 - [76] Dominic Verity, *Relating descent notions*, <http://ncatlab.org/nlab/files/VerityDescent.pdf>
 - [77] K. Waldorf, *Multiplicative bundle gerbes with connection*, [arxiv/0804.4835](https://arxiv.org/abs/0804.4835)