
ELSEVIER
Volume 213
Issue 6
June 2009
ISSN 0022-4049
JOURNAL OF PURE AND APPLIED ALGEBRA

Journal of Pure and Applied Algebra 213 (2009) 913–926

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Recurrence relations for characters of affine Lie algebra $A_\ell^{(1)}$

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ARTICLE INFO

Article history:
Received 31 March 2008
Available online 7 November 2008
Communicated by C.A. Weibel

MSC:
Primary: 17B67
secondary: 17B69
05A19

ABSTRACT

By using the known description of combinatorial bases for Feigin–Stoyanovsky's type subspaces of standard modules for affine Lie algebra $\widetilde{\mathfrak{sl}(l+1,\mathbb{C})}$, as well as certain intertwining operators between standard modules, we obtain exact sequences of Feigin–Stoyanovsky's type subspaces at fixed level k. This directly leads to systems of recurrence relations for formal characters of those subspaces.

1. Introduction

In a series of papers Lepowsky and Wilson have obtained a Lie theoretic proof of famous Rogers–Ramanujan partition identities (cf. [1,2]). They used the fact that the product side of these identities arise in principally specialized characters for all level 3 standard representations of affine Lie algebra $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$, and to obtain the sum side of identities they constructed combinatorial bases for these $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$-modules. The key ingredient in their construction is a twisted vertex operator construction of fundamental $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$-modules and the corresponding vertex operator relations for higher level modules.

By using Lepowsky–Wilson's approach with untwisted vertex operators, Lepowsky and Primc obtained in [3] new character formulas for all standard $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$-modules. Feigin and Stoyanovsky in [4] gave another proof and combinatorial interpretation of these character formulas, one of several new ingredients in the proof being the so-called principal subspaces, later also named Feigin–Stoyanovsky's subspaces. By using intertwining operators Georgiev (cf. [5]) extended Feigin–Stoyanovsky's approach and obtained combinatorial bases of principal subspaces of standard $\widetilde{\mathfrak{sl}(\ell+1,\mathbb{C})}$-modules together with the corresponding character formulas. In a similar way Capparelli, Lepowsky and Milas in [6,7] use the theory of intertwining operators for vertex operator algebras in order to obtain the recurrence relations for characters of principal subspaces of all arbitrary fixed level standard $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$-modules. It turned out that these recurrence relations are precisely the known recursion formulas of Rogers and Selberg, already solved by G. Andrews while working on Gordon identities (cf. [8,9]). The Capparelli–Lepowsky–Milas approach was further investigated by Calinescu in [10] in order to construct the exact sequences of principal subspaces of basic $\widetilde{\mathfrak{sl}(l+1,\mathbb{C})}$-modules and thus acquire a recurrence system for characters of these subspaces. Furthermore, in [11] Calinescu used the exact sequence method to obtain systems of recurrences for characters of all principal subspaces of arbitrary level standard $\widetilde{\mathfrak{sl}(3,\mathbb{C})}$-modules and the corresponding characters for some classes of principal subspaces. Finally, this approach was further developed in [12,13] where presentations of principal subspaces of all standard $\widetilde{\mathfrak{sl}(l+1,\mathbb{C})}$-modules are given.

Another construction of combinatorial bases of standard modules of affine Lie algebras was given in [14,15]. In this construction a new and interesting class of subspaces of standard modules emerged—the so-called Feigin–Stoyanovsky's *type* subspaces, which coincide with principal subspaces only for affine Lie algebra $\widetilde{\mathfrak{sl}(2,\mathbb{C})}$. It turned out that in the case of affine Lie algebra $\widetilde{\mathfrak{sl}(\ell+1,\mathbb{C})}$ these combinatorial bases are parametrized by $(k,\ell+1)$-admissible configurations,

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doi:10.1016/j.jpaa.2008.10.001

combinatorial objects introduced and studied in [16] and [17]. In the case of other "classical" affine Lie algebras only bases of basic modules are constructed by using the crystal base [18] character formula.

Inspired by the use of intertwining operators in the work of Capparelli, Lepowsky and Milas, in [19] a simpler proof for the existence of combinatorial bases of Feigin–Stoyanovsky's type subspaces was given in the case of affine Lie algebra $\tilde{\mathfrak{sl}(2,\mathbb{C})}$. In this paper we extend this approach to obtain exact sequences of Feigin–Stoyanovsky's type subspaces at fixed level k for affine Lie algebra $\tilde{\mathfrak{sl}(l+1,\mathbb{C})}$.

In order to state the main result we need some notation. Denote $\mathfrak{g}=\mathfrak{sl}(\ell+1,\mathbb{C})$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ with the corresponding root system R. Let

$$\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in R}\mathfrak{g}_\alpha$$

be the root space decomposition of $\mathfrak{g}$ with fixed root vectors x_α. Let

$$\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$$

be a chosen $\mathbb{Z}$-grading such that $\mathfrak{h}\subset\mathfrak{g}_0$. Grading on $\mathfrak{g}$ induces $\mathbb{Z}$-grading on $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_{-1}\oplus\tilde{\mathfrak{g}}_0\oplus\tilde{\mathfrak{g}}_1,$$

where $\tilde{\mathfrak{g}}_1=\mathfrak{g}_1\otimes\mathbb{C}[t,t^{-1}]$ is a commutative Lie algebra with a basis given by

$$\{x_\gamma(j)\mid j\in\mathbb{Z},\gamma\in\Gamma\}.$$

Here Γ denotes the set of roots belonging to $\mathfrak{g}_1$.

For a standard $\tilde{\mathfrak{g}}$-module of level $k=\Lambda(c)$ with a highest weight vector v_Λ define Feigin–Stoyanovsky's type subspace $W(\Lambda)$ as the following subspace of $L(\Lambda)$:

$$W(\Lambda)=U(\tilde{\mathfrak{g}}_1)\cdot v_\Lambda.$$

By using a description of combinatorial bases for Feigin–Stoyanovsky's type subspaces for affine Lie algebra $\tilde{\mathfrak{sl}(l+1,\mathbb{C})}$ in terms of $(k,\ell+1)$-admissible vectors, as well as operators $\varphi_0,\dots,\varphi_{m-1}$ (constructed with certain intertwining operators between standard modules) and a simple current operator $[\omega]$, we obtain exact sequences of these subspaces at fixed level k (cf. Theorem 5.1):

$$0\to W_{k_\ell,k_0,k_1,\dots,k_{\ell-1}}\xrightarrow{[\omega]^{\otimes k}}W\xrightarrow{\varphi_0}\sum_{I_1\in D_1(K)}W_{I_1}\xrightarrow{\varphi_1}\cdots\xrightarrow{\varphi_{m-1}}W_{I_m}\to 0.$$

Here $W=W(k_0\Lambda_0+\cdots+k_\ell\Lambda_\ell)$ is a Feigin–Stoyanovsky's type subspace for level $k=k_0+\cdots+k_\ell$, $W_{k_\ell,k_0,k_1,\dots,k_{\ell-1}}=W(k_\ell\Lambda_0+k_0\Lambda_1+\cdots+k_{\ell-1}\Lambda_\ell)$, and $W_{I_1},\dots,W_{I_m}$ Feigin–Stoyanovsky's type subspaces derived from W by a procedure described in Section 5.1.

Exact sequences described above directly lead to systems of relations among formal characters of those subspaces (cf. Eq. (6.2)):

$$\sum_{I\in D(K)}(-1)^{|I|}\chi(W_I)(z_1,\dots,z_\ell;q)=(z_1q)^{k_0}\dots(z_\ell q)^{k_{\ell-1}}\chi(W_{k_\ell,k_0,\dots,k_{\ell-1}})(z_1q,\dots,z_\ell q;q).$$

The paper is organized as follows. Section 2 gives the setting. In Section 3 we define Feigin–Stoyanovsky's type subspaces and present the result on combinatorial bases. Section 4 gives the vertex operator construction for fundamental modules, and introduces the intertwining operators and a simple current operator. The last two sections contain the main results of the paper: Section 5 states the result on exactness (cf. Theorem 5.1), while in Section 6 we obtain the corresponding system of relations among characters of Feigin–Stoyanovsky's type subspaces at fixed level (cf. Eqs. (6.2) and (6.4)), and present the proof that such system has a unique solution.

I sincerely thank Mirko Primc for his valuable suggestions.

2. Affine Lie algebra $\tilde{\mathfrak{sl}(l+1,\mathbb{C})}$ and standard modules

Let $\mathfrak{g}=\mathfrak{sl}(\ell+1,\mathbb{C})$, $\ell\in\mathbb{N}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Denote by R the corresponding root system (identified in the usual way as a subset of $\mathbb{R}^{\ell+1}$):

$$R=\{\pm(\epsilon_i-\epsilon_j)\mid 1\le i<j\le\ell+1\}.$$

As usual, we fix simple roots $\alpha_i=\epsilon_i-\epsilon_{i+1}$, $i=1,\dots,\ell$, and have triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$. Denote by $Q=Q(R)$ the root lattice and by $P=P(R)$ the weight lattice of R. Let ω_i, $i=1,\dots,\ell$, denote the corresponding fundamental weights. Define also $\omega_0:=0$ for later purposes. Let $\langle\cdot,\cdot\rangle$ be the Killing form on $\mathfrak{g}$. Identify $\mathfrak{h}$ and $\mathfrak{h}^*$ using this form, denote by x_α fixed root vectors, and normalize the form in such a way that for the maximal root θ holds $\langle\theta,\theta\rangle=2$.

Associate to $\mathfrak{g}$ the affine Lie algebra $\tilde{\mathfrak{g}}$ (cf. [20])

$$\tilde{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d,$$

with Lie product given by

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\langle x, y\rangle \delta_{m+n,0} c,$$

for $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$. Here c denotes the canonical central element, and d the degree operator: $[d, x \otimes t^n] = nx \otimes t^n$.

We denote $x(n) = x \otimes t^n$ for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$, and define the following formal Laurent series in formal variable z:

$$x(z) = \sum_{n\in\mathbb{Z}} x(n) z^{-n-1}.$$

Furthermore, having denoted $\mathfrak{h}^e = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, $\tilde{\mathfrak{n}}_\pm = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}] \oplus \mathfrak{n}_\pm$, we have the triangular decomposition for $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h}^e \oplus \tilde{\mathfrak{n}}_+.$$

As usual, denote by $\{\alpha_0, \alpha_1, \dots, \alpha_\ell\} \subset (\mathfrak{h}^e)^*$ the corresponding set of simple roots, and by $\Lambda_0, \Lambda_1, \dots, \Lambda_\ell$ fundamental weights.

Let $L(\Lambda)$ denote standard (i.e., integrable highest weight) $\tilde{\mathfrak{g}}$-module with dominant integral highest weight

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell, \quad k_0, k_1, \dots, k_\ell \in \mathbb{Z}_+.$$

Define the level of $L(\Lambda)$ as $k = \Lambda(c) = k_0 + k_1 + \cdots + k_\ell$.

3. Feigin–Stoyanovsky's type subspaces

3.1. Definition

Fix the minuscule weight $\omega = \omega_\ell$ and define the following alternative basis for $\mathfrak{h}^*$:

$$\Gamma = \{\alpha \in R \mid \omega(\alpha) = 1\} = \{\gamma_1, \gamma_2, \dots, \gamma_\ell \mid \gamma_i = \epsilon_i - \epsilon_{\ell+1} = \alpha_i + \cdots + \alpha_\ell\}.$$

Consequently we obtain $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1,$$

where $\mathfrak{g}_0 = \mathfrak{h} + \sum_{\omega(\alpha)=0} \mathfrak{g}_\alpha$, $\mathfrak{g}_{\pm 1} = \sum_{\alpha\in\pm\Gamma} \mathfrak{g}_\alpha$. The corresponding $\mathbb{Z}$-grading on affine Lie algebra $\tilde{\mathfrak{g}}$ is

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1,$$

with $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, $\tilde{\mathfrak{g}}_{\pm 1} = \mathfrak{g}_{\pm 1} \otimes \mathbb{C}[t, t^{-1}]$. Note that

$$\tilde{\mathfrak{g}}_1 = \operatorname{span}\{x_\gamma(n) \mid \gamma \in \Gamma, n \in \mathbb{Z}\}$$

is a commutative subalgebra and a $\tilde{\mathfrak{g}}_0$-module.

For an integral dominant weight Λ define Feigin–Stoyanovsky's type subspace of $L(\Lambda)$ as

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda,$$

$U(\tilde{\mathfrak{g}}_1)$ denoting the universal enveloping algebra for $\tilde{\mathfrak{g}}_1$, and v_Λ a fixed highest weight vector of $L(\Lambda)$.

We proceed by defining colored partitions as maps

$$\pi : \{x_\gamma(-j) \mid \gamma \in \Gamma, j \geq 1\} \to \mathbb{Z}_+$$

with finite support. For given π define the monomial $x(\pi)$ in $U(\tilde{\mathfrak{g}}_1)$ as

$$x(\pi) = \prod x_\gamma(-j)^{\pi(x_\gamma(-j))}.$$

Since every π can be identified with a sequence $(a_i)_{i=0}^\infty$ with finitely many nonzero elements via $a_{\ell(j-1)+r-1} = \pi(x_{\gamma_r}(-j))$, for corresponding $x(\pi)$ we will write

$$x(\pi) = \dots x_{\gamma_1}(-2)^{a_\ell} x_{\gamma_\ell}(-1)^{a_{\ell-1}} \dots x_{\gamma_1}(-1)^{a_0}. \tag{3.1}$$

From Poincaré–Birkhoff–Witt theorem it follows that monomial vectors $x(\pi)v_\Lambda$, with $x(\pi)$ as in (3.1), span $W(\Lambda)$.

3.2. Combinatorial basis for $W(\Lambda)$

For given $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell$ a monomial $x(\pi)$ given by (3.1), or a monomial vector $x(\pi)v_\Lambda$ for such $x(\pi)$, is called $(k, \ell+1)$-admissible for Λ (or $W(\Lambda)$) if the following inequalities are met:

$$\begin{aligned} &a_0 \le k_0\\ &a_0 + a_1 \le k_0 + k_1\\ &\dots\\ &a_0 + a_1 + \cdots + a_{\ell-1} \le k_0 + \cdots + k_{\ell-1}, \end{aligned} \tag{3.2}$$

and

$$a_i + \cdots a_{i+\ell} \le k, \quad i \in \mathbb{Z}_+. \tag{3.3}$$

We say that (3.2) are the initial conditions for given Λ (or $W(\Lambda)$), and that (3.3) are the difference conditions (note that the difference conditions (3.3) do not depend on the choice of Λ).

In [19] Primc demonstrated, by using certain coefficients of intertwining operators between fundamental modules as well as a simple current operator, that the spanning set for $W(\Lambda)$ given in 3.1 can be reduced to basis consisting of $(k, \ell+1)$-admissible vectors:

Theorem 3.1. *The set of $(k, \ell+1)$-admissible monomial vectors $x(\pi)v_\Lambda$ is a basis of $W(\Lambda)$.*

Note that the formulation of Theorem 3.1 in terms of $(k, \ell+1)$-admissible monomials was first given in [16]. We will not give details of the proof here, although some of the content needed for it is to be presented in the the next section.

4. Intertwining operators and simple current operator

4.1. Fundamental modules

In this section we state some facts needed in the following sections. For precise definitions and proofs see [21–26].

Let us state the basics of the well-known vertex operator construction for fundamental $\tilde{\mathfrak{g}}$-modules (cf. [27,28]). Define Fock space $M(1)$ as induced $\hat{\mathfrak{h}}$-module

$$M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}c)} \mathbb{C},$$

where $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, such that $\mathfrak{h} \otimes \mathbb{C}[t]$ acts trivially and c as identity on one-dimensional module $\mathbb{C}$. Denote by $\{e^\lambda \mid \lambda \in P\}$ a basis of the group algebra $\mathbb{C}[P]$ of P. Then $M(1) \otimes \mathbb{C}[P]$ is an $\hat{\mathfrak{h}}$-module: $\hat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{n\in\mathbb{Z}\setminus\{0\}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c$ acts as $\hat{\mathfrak{h}}_{\mathbb{Z}} \otimes 1$, and $\mathfrak{h} = \mathfrak{h} \otimes t^0$ as $1 \otimes \mathfrak{h}$, with $h(0)$ given by $h(0).e^\lambda = \langle h, \lambda\rangle e^\lambda$ for $h \in \mathfrak{h}$, $\lambda \in P$.

For $\lambda \in P$ we have vertex operators:

$$Y(e^\lambda, z) = E^-(-\lambda, z)E^+(-\lambda, z)\epsilon_\lambda e^\lambda z^\lambda, \tag{4.1}$$

with

$$E^\pm(\lambda, z) = \exp\left(\sum_{n\ge1} \lambda(\pm n)\frac{z^{\mp n}}{\pm n}\right), \quad \lambda \in P$$

$$\epsilon_\lambda e^\lambda(v \otimes e^\mu) = \epsilon(\lambda, \lambda+\mu)v \otimes e^{\lambda+\mu}$$

$$z^\lambda(v \otimes e^\mu) = z^{\langle\lambda,\mu\rangle} v \otimes e^\mu, \quad v \in M(1), \lambda, \mu \in P,$$

where ϵ is a 2-cocyle corresponding to a central extension of P by certain finite cyclic group (cf. [23,21]).

The action of $\hat{\mathfrak{h}}$ extends to the action of $\tilde{\mathfrak{g}}$ via (4.1) in the following manner: $x_\alpha(n)$ acts on $M(1) \otimes \mathbb{C}[P]$ as the coefficient of x^{-n-1} in $Y(e^\alpha, z)$:

$$x_\alpha(z) = Y(e^\alpha, z),$$

and d as degree operator. Furthermore, the direct summands $M(1) \otimes e^{\omega_i}\mathbb{C}[Q]$ of $M(1) \otimes \mathbb{C}[P]$ are exactly standard $\tilde{\mathfrak{g}}$-modules $L(\Lambda_i)$, and we can identify highest weight vectors $v_{\Lambda_i} = 1 \otimes e^{\omega_i}$, $i = 0, 1, \dots, \ell$.

For $\lambda \in P$ we also have Dong–Lepowsky's intertwining operators:

$$\mathcal{Y}(e^\lambda, z) := E^-(-\lambda, z)E^+(-\lambda, z)e_\lambda z^\lambda e^{i\pi\lambda}c(\cdot, \lambda),$$

where $e_\lambda = e^\lambda\epsilon(\lambda, \cdot)$, $c(\alpha, \beta)$ defined in [21], Eq. (12.52).

For $\lambda_i := \omega_i - \omega_{i-1}$, $i = 1, \dots, \ell$, we have:

$$[Y(e^\gamma, z_1), \mathcal{Y}(e^{\lambda_i}, z_2)] = 0, \quad \gamma \in \Gamma,$$

which implies that every coefficient of all $\mathcal{Y}(e^{\lambda_i}, z)$ commutes with $x_\gamma(n)$, $\gamma \in \Gamma$, $n \in \mathbb{Z}$. In particular, this holds for coefficients

$$[i] = \mathrm{Res} z^{-1-\langle \lambda_i, \omega_{i-1}\rangle} c_i \mathcal{Y}(e^{\lambda_i}, z), \quad i = 1, \ldots, \ell,$$

which we also call intertwining operators.

We have

$$L(\Lambda_0) \xrightarrow{[1]} L(\Lambda_1) \xrightarrow{[2]} L(\Lambda_2) \xrightarrow{[3]} \cdots \xrightarrow{[\ell-1]} L(\Lambda_{\ell-1}) \xrightarrow{[\ell]} L(\Lambda_\ell),$$

and for suitably chosen c_i in the definition of intertwining operators also

$$v_{\Lambda_0} \xrightarrow{[1]} v_{\Lambda_1} \xrightarrow{[2]} v_{\Lambda_2} \xrightarrow{[3]} \cdots \xrightarrow{[\ell-1]} v_{\Lambda_{\ell-1}} \xrightarrow{[\ell]} v_{\Lambda_\ell}.$$

Next, define linear bijection $[\omega]$ on $M(1) \otimes \mathbb{C}[P]$ by

$$[\omega] = e^{\omega_\ell} \epsilon(\cdot, \omega_\ell).$$

We call $[\omega]$ a simple current operator (cf. [29]). It can be shown that

$$L(\Lambda_0) \xrightarrow{[\omega]} L(\Lambda_\ell) \xrightarrow{[\omega]} L(\Lambda_{\ell-1}) \xrightarrow{[\omega]} \cdots \xrightarrow{[\omega]} L(\Lambda_1) \xrightarrow{[\omega]} L(\Lambda_0),$$

and from vertex operator formula (4.1) we have

$$[\omega] v_{\Lambda_0} = v_{\Lambda_\ell}, \quad [\omega] v_{\Lambda_i} = x_{\gamma_i}(-1) v_{\Lambda_{i-1}}, \quad i = 1, \ldots, \ell. \tag{4.2}$$

Also, from (4.1) it follows $x_\alpha(z)[\omega] = [\omega] z^{\langle \omega_\ell, \alpha\rangle} x_\alpha(z)$ for $\alpha \in R$ or, written in components: $x_\alpha(n)[\omega] = [\omega] x_\alpha(n + \langle \omega_\ell, \alpha\rangle)$, $\alpha \in R$, $n \in \mathbb{Z}$, which specially for $\gamma \in \Gamma$ gives

$$x_\gamma(n)[\omega] = [\omega] x_\gamma(n+1). \tag{4.3}$$

In general, for every $x(\pi) v_\Lambda$ we have

$$[\omega] x(\pi) = x(\pi^-)[\omega],$$

where $x(\pi^-)$ denotes monomial obtained from $x(\pi)$ by lowering the degree of every constituting factor in $x(\pi)$ by one.

4.2. Higher level standard modules

Because of complete reducibility of tensor product of standard modules we can embed level k standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ with $\Lambda = k_0\Lambda_0 + \cdots + k_\ell \Lambda_\ell$ in the appropriate k-fold tensor product of fundamental modules

$$L(\Lambda) \subset L(\Lambda_\ell)^{\otimes k_\ell} \otimes \cdots \otimes L(\Lambda_1)^{\otimes k_1} \otimes L(\Lambda_0)^{\otimes k_0},$$

and we can take the corresponding highest weight vector v_Λ to be

$$v_\Lambda = v_{\Lambda_\ell}^{\otimes k_\ell} \otimes \cdots \otimes v_{\Lambda_1}^{\otimes k_1} \otimes v_{\Lambda_0}^{\otimes k_0}.$$

Denote again $[i] := 1 \otimes \cdots 1 \otimes [i] \otimes 1 \cdots \otimes 1$, $i = 1, \ldots, \ell$, i.e. the k-fold tensor product of $[i]$ with identity maps:

$$\begin{aligned} [i] : {} & L(\Lambda_\ell)^{\otimes k_\ell} \otimes \cdots \otimes L(\Lambda_i)^{\otimes k_i} \otimes L(\Lambda_{i-1})^{\otimes k_{i-1}} \otimes \cdots \otimes L(\Lambda_0)^{\otimes k_0} \\ & \to L(\Lambda_\ell)^{\otimes k_\ell} \otimes \cdots \otimes L(\Lambda_i)^{\otimes k_i+1} \otimes L(\Lambda_{i-1})^{\otimes k_{i-1}-1} \otimes \cdots \otimes L(\Lambda_0)^{\otimes k_0} \end{aligned} \tag{4.4}$$

for $k_i \geq 1$. These are again linear maps between corresponding standard level $k\,\tilde{\mathfrak{g}}$-modules that map highest weight vector into highest weight vector. Also, these maps commute with action of $x_\gamma(n)$, $\gamma \in \Gamma$, $n \in \mathbb{Z}$.

On k-fold tensor products of standard $\tilde{\mathfrak{g}}$-modules we use also $[\omega]^{\otimes k}$ (we will denote it again $[\omega]$), a linear bijection for which the commutation formula analogous to (4.3) holds.

5. Exact sequences of Feigin–Stoyanovsky's type subspaces

5.1. Exactness and supplementary results

For fixed level k and arbitrary nonnegative integers $k_0, \ldots, k_\ell$ such that $k_0 + \cdots + k_\ell = k$ denote

$$W_{k_0,k_1,\ldots,k_\ell} = W(k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell),$$

$$v_{k_0,k_1,\ldots,k_\ell} = v_{k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell} = v_{\Lambda_\ell}^{\otimes k_\ell} \otimes \cdots \otimes v_{\Lambda_1}^{\otimes k_1} \otimes v_{\Lambda_0}^{\otimes k_0}.$$

Also, let $\mathcal{B}_{k_0,k_1,\ldots,k_\ell}$ be the set of $(k, \ell+1)$-admissible monomials $x(\pi)$ for $k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell$. Then vectors $x(\pi) v_{k_0,k_1,\ldots,k_\ell}$ belong to basis of $W_{k_0,k_1,\ldots,k_\ell}$.

Let us now fix some $K = (k_0, \ldots, k_\ell)$ satisfying $k_0 + \cdots + k_\ell = k$ and define $m = \sharp\{i = 0, \ldots, \ell - 1 \mid k_i \neq 0\}$. Denote

$$W = W_{k_0,k_1,\ldots,k_\ell}, \qquad v = v_{k_0,k_1,\ldots,k_\ell}, \qquad \mathcal{B} = \mathcal{B}_{k_0,k_1,\ldots,k_\ell}.$$

For $t \in \{0, \ldots, m-1\}$ define

$$D_{t+1}(K) = \{\{i_0, \ldots, i_t\} \mid 0 \le i_0 < \cdots < i_t \le \ell - 1 \text{ such that } k_{i_j} \neq 0, j = 0, \ldots, t\}.$$

Let $D_0(K) := \{\emptyset\}$ and denote by $D(K)$ the union of all $D_0(K), D_1(K), \ldots, D_m(K)$ (note that $D_m(K) = \{I_m\}$ is monadic).

For $I_{t+1} = \{i_0, \ldots, i_t\} \in D_{t+1}(K)$, $t = 0, \ldots, m-1$, we introduce the following notation:

$$W_{I_{t+1}} = W_{\{i_0,\ldots,i_t\}} = W_{k_0,\ldots,k_{i_0}-1,k_{i_0+1}+1,\ldots,k_{i_t}-1,k_{i_t+1}+1,\ldots,k_\ell},$$

similarly also for highest weight vector $v_{\{i_0,\ldots,i_t\}}$ and $\mathcal{B}_{\{i_0,\ldots,i_t\}}$. Also, define $W_\emptyset = W$, $v_\emptyset = v$, $\mathcal{B}_\emptyset = \mathcal{B}$.

By using $[i]$ given by (4.4), we construct the following mappings that commute with the action of $x_\gamma(n)$, $\gamma \in \Gamma$, $n \in \mathbb{Z}$: for $t = 0, \ldots, m-1$, let φ_t be a $U(\tilde{\mathfrak{g}}_1)$-homogeneous mapping

$$\varphi_t : \sum_{I_t \in D_t(K)} W_{I_t} \to \sum_{I_{t+1} \in D_{t+1}(K)} W_{I_{t+1}}$$

given component-wise by

$$\varphi_t(v_{I_t}) = \sum_{\substack{\{i\} \in D_1(K)\\ i \notin I_t}} (-1)^{p_{I_t}(i)} v_{I_t \cup \{i\}},$$

with $p_{I_t}(i)$ such that $i = j_{p_{I_t}(i)}$ in $I_t \cup \{i\} = \{j_0, \ldots, j_t\}$ for some $\{j_0, \ldots, j_t\} \in D_{t+1}(K)$.

Observe that φ_t can be described also by providing the components of the image: for $I_{t+1} = \{i_0, \ldots, i_t\} \in D_{t+1}(K)$ we have

$$\varphi_t(w)_{I_{t+1}} = \sum_{0 \le s \le t} (-1)^s a(w)_{I_{t+1} \setminus \{i_s\}} v_{I_{t+1}},$$

where $a(w)_{I_t}$ stands for the monomial part of $w = \sum_{I_t \in D_t(K)} a(w)_{I_t} v_{I_t}$ in W_{I_t}.

We can now state the exactness result:

Theorem 5.1. *For every $K = (k_0, \ldots, k_\ell)$ such that $k_0 + \cdots + k_\ell = k$ the following sequence is exact:*

$$0 \to W_{k_\ell,k_0,k_1,\ldots,k_{\ell-1}} \xrightarrow{[\omega]^{\otimes k}} W \xrightarrow{\varphi_0} \sum_{I_1 \in D_1(K)} W_{I_1} \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{m-1}} W_{I_m} \to 0.$$

Note that in the case of $K = (0, 0, \ldots, 0, k)$ we have $m = 0$ which yields the short exact sequence

$$0 \to W_{k,0,0,\ldots,0} \xrightarrow{[\omega]^{\otimes k}} W_{0,0,\ldots,0,k} \to 0.$$

Example 5.2. *For $\ell = 2$, $k = 2$ the exact sequences are:*

$$\begin{aligned}
&0 \to W_{0,2,0} \to W_{2,0,0} \to W_{1,1,0} \to 0\\
&0 \to W_{0,1,1} \to W_{1,1,0} \to W_{0,2,0} \oplus W_{1,0,1} \to W_{0,1,1} \to 0\\
&0 \to W_{1,1,0} \to W_{1,0,1} \to W_{0,1,1} \to 0\\
&0 \to W_{0,0,2} \to W_{0,2,0} \to W_{0,1,1} \to 0\\
&0 \to W_{1,0,1} \to W_{0,1,1} \to W_{0,0,2} \to 0\\
&0 \to W_{2,0,0} \to W_{0,0,2} \to 0.
\end{aligned}$$

The following lemmas (outlining some technical facts related to sets $\mathcal{B}_A$, $A \in D(K)$) will be used in the proof of Theorem 5.1.

Lemma 5.3. *For $A, B \in D(K)$ the following holds:*

$$A \subset B \Rightarrow \mathcal{B}_B \subset \mathcal{B}_A.$$

Proof. If $A = \{i_0, \ldots, i_t\}$, the initial conditions for W_A are:

$$\begin{aligned}
&a_0 \leq k_0\\
&a_0 + a_1 \leq k_0 + k_1\\
&\ldots\\
&a_0 + \cdots + a_{i_0} \leq k_0 + \cdots + k_{i_0} - 1\\
&a_0 + \cdots + a_{i_0+1} \leq k_0 + \cdots + k_{i_0+1}\\
&\ldots\\
&a_0 + \cdots + a_{i_t} \leq k_0 + \cdots + k_{i_t} - 1\\
&a_0 + \cdots + a_{i_t+1} \leq k_0 + \cdots + k_{i_t+1}\\
&\ldots\\
&a_0 + \cdots + a_{\ell-1} \leq k_0 + \cdots k_{\ell-1}
\end{aligned} \tag{5.1}$$

(in the case of $A = \emptyset$ these are just (3.2)). It is obvious that the initial conditions for W_B follow from those for W_A if we strengthen every $(j+1)$th inequality in (5.1) for each $j \in B \setminus A$:

$$a_0 + \cdots + a_j \leq k_0 + \cdots + k_j - 1.$$

Hence, for every $x(\pi) \in \mathcal{B}_B$ also $x(\pi) \in \mathcal{B}_A$ holds. $\square$

Lemma 5.4. *Let $B_1, B_2 \in D(K)$. Then*

$$\mathcal{B}_{B_1} \cap \mathcal{B}_{B_2} = \mathcal{B}_{B_1 \cup B_2}.$$

Proof. From Lemma 5.3 we have

$$\begin{aligned}
&\mathcal{B}_{B_1 \cup B_2} \subseteq \mathcal{B}_{B_1}\\
&\mathcal{B}_{B_1 \cup B_2} \subseteq \mathcal{B}_{B_2},
\end{aligned}$$

so $\mathcal{B}_{B_1 \cup B_2} \subseteq \mathcal{B}_{B_1} \cap \mathcal{B}_{B_2}$ holds. On the other hand, a monomial $x(\pi) \in \mathcal{B}_{B_1} \cap \mathcal{B}_{B_2}$ satisfies the initial conditions both for W_{B_1} and W_{B_2}, which means that it always satisfies the stronger inequality out of two equally indexed ones, which implies that $x(\pi) \in \mathcal{B}_{B_1 \cup B_2}$. $\square$

For $B \in D(K)$ let us define

$$\mathcal{B}^B = \mathcal{B}_B \setminus \bigcup_{\substack{C \in D(K)\\ C \supset B}} \mathcal{B}_C.$$

The next lemma describes the above defined set more explicitly:

Lemma 5.5. *For $B \in D(K)$ and $J_B = I_m \setminus B$, the set $\mathcal{B}^B$ consists of all $x(\pi) = \ldots x_{\gamma_\ell}(-1)^{a_{\ell-1}} \ldots x_{\gamma_2}(-1)^{a_1} x_{\gamma_1}(-1)^{a_0} \in \mathcal{B}_B$ such that*

$$a_0 + \cdots + a_j = k_0 + \cdots + k_j, \quad j \in J_B.$$

Proof. Note that I_m is the largest set in $D(K)$ and that for $B = I_m$ we have $J_B = \emptyset$. In this case the lemma trivially asserts

$$\mathcal{B}^{I_m} = \mathcal{B}_{I_m},$$

which is an obvious consequence of the definition of $\mathcal{B}^{I_m}$. Next, for $B \subsetneq I_m$ one directly checks that for every $C \supset B$, $C \in D(K)$, the set $\mathcal{B}_C$ consists of all $x(\pi) = \ldots x_{\gamma_\ell}(-1)^{a_{\ell-1}} \ldots x_{\gamma_2}(-1)^{a_1} x_{\gamma_1}(-1)^{a_0} \in \mathcal{B}_B$ that satisfy (besides the initial conditions imposed by B) also the following:

$$a_0 + \cdots + a_i \leq k_0 + \cdots + k_i - 1, \quad i \in C \setminus B.$$

Therefore, $\mathcal{B}^B$ consists of all $x(\pi) \in \mathcal{B}_B$ for which such inequalities do not hold (for every strict superset of B belonging to $D(K)$). This means that $\mathcal{B}^B$ consists of all those $x(\pi) \in \mathcal{B}_B$ for which the following equalities take place:

$$a_0 + \cdots + a_i = k_0 + \cdots + k_i, \quad i \in C \setminus B, C \supset B, C \in D(K).$$

All the indices from $C \in D(K)$, for all $C \supset B$, exactly comprise $J_B = I_m \setminus B$, since I_m is the largest set in $D(K)$. The assertion follows. $\square$

Lemma 5.6. *For $A \in D(K)$ the set $\mathcal{B}_A$ can be expressed as a disjoint union*

$$\mathcal{B}_A = \bigcup_{\substack{B \in D(K) \\ B \supseteq A}} \mathcal{B}^B. \tag{5.2}$$

Proof. One inclusion is clear: for $B \in D(K)$, $B \supseteq A$, we have $\mathcal{B}^B \subseteq \mathcal{B}_B \subseteq \mathcal{B}_A$. On the other hand, take $x(\pi) = \ldots x_{\gamma_\ell}(-1)^{a_{\ell-1}} \ldots x_{\gamma_2}(-1)^{a_1} x_{\gamma_1}(-1)^{a_0}$ in $\mathcal{B}_A$ and define $J = \{j \mid a_0 + \cdots + a_j = k_0 + \cdots + k_j\}$. By Lemma 5.5 it is obvious that $x(\pi) \in \mathcal{B}^B$ for $B = I_m \setminus J$ and that $B \supseteq A$ (of course, in the case of $J = \emptyset$ we have $B = I_m$) and the equality stated in the lemma holds. Furthermore, suppose that for some $B_1, B_2 \in D(K)$, $B_1, B_2 \supseteq A$, there exists $x(\pi) \in \mathcal{B}^{B_1} \cap \mathcal{B}^{B_2}$. But, then we have $x(\pi) \in \mathcal{B}_{B_1}$ and $x(\pi) \in \mathcal{B}_{B_2}$, so $x(\pi) \in \mathcal{B}_{B_1} \cap \mathcal{B}_{B_2} = \mathcal{B}_{B_1 \cup B_2}$, which is, due to $B_1 \cup B_2 \supset B_1, B_2$, in opposition with $x(\pi) \in \mathcal{B}_{B_1}$ and $x(\pi) \in \mathcal{B}_{B_2}$. We conclude that the union on the right-hand side of (5.2) must be disjoint. □

5.2. Proof of exactness

We prove Theorem 5.1. It is clear from the definition that $[\omega] = [\omega]^{\otimes k}$ is injective. Therefore, as the first nontrivial step we prove that $Im([\omega]^{\otimes k}) = Ker(\varphi_0)$. Recall that $\mathcal{B} = \mathcal{B}_\emptyset$ and let $x(\pi)v \in Ker(\varphi_0)$ for $x(\pi) = \ldots x_{\gamma_\ell}(-1)^{a_{\ell-1}} \ldots x_{\gamma_2}(-1)^{a_1} x_{\gamma_1}(-1)^{a_0} \in \mathcal{B}_\emptyset$. From

$$\varphi_0(x(\pi)v) = \sum_{I_1 \in D_1(K)} x(\pi)v_{I_1} = 0,$$

by using the fact that for $x(\pi)$ which satisfies the difference conditions and does not satisfy the initial conditions for $W(\Lambda)$ the assertion $x(\pi)v_\Lambda = 0$ holds, one gets

$$x(\pi) \in \mathcal{B}_\emptyset \setminus \bigcup_{I_1 \in D_1(K)} \mathcal{B}_{I_1} = \mathcal{B}_\emptyset \setminus \bigcup_{C \in D(K)} \mathcal{B}_C = \mathcal{B}^\emptyset.$$

By using Lemma 5.5 we conclude that for $x(\pi)$

$$a_0 + \cdots + a_j = k_0 + \cdots + k_j, \quad j \in J_\emptyset = I_m$$

holds. Since every k_i, $i \in \{0, \ldots, \ell - 1\} \setminus J_\emptyset$, equals zero, the above equalities imply the equality between *all* corresponding left-hand and right-hand partial sums:

$$a_0 + \cdots + a_i = k_0 + \cdots + k_i, \quad i \in \{0, \ldots, \ell - 1\}.$$

We now have $a_i = k_i$, $i \in \{0, \ldots, \ell - 1\}$, and

$$Ker(\varphi_0) = \mathrm{span}\{x(\pi)v \in W \mid x(\pi) = \ldots x_{\gamma_\ell}(-1)^{k_{\ell-1}} \ldots x_{\gamma_1}(-1)^{k_0}\}.$$

On the other hand, take $x(\pi_1)v_{k_\ell, k_0, \ldots, k_{\ell-1}}$ where

$$x(\pi_1) = \ldots x_{\gamma_\ell}(-1)^{b_{\ell-1}} \ldots x_{\gamma_2}(-1)^{b_1} x_{\gamma_1}(-1)^{b_0} \in \mathcal{B}_{k_\ell, k_0, \ldots, k_{\ell-1}}$$

and calculate using (4.2) and (4.3):

$$\begin{aligned}[\omega]^{\otimes k}(x(\pi_1)v_{k_\ell, k_0, \ldots, k_{\ell-1}}) &= x(\pi_1^-)x_{\gamma_\ell}(-1)^{k_{\ell-1}} \ldots x_{\gamma_2}(-1)^{k_1} x_{\gamma_1}(-1)^{k_0} [w]^{\otimes k}(v_{k_\ell, k_0, \ldots, k_{\ell-1}}) \\ &= x(\pi_1^-)x_{\gamma_\ell}(-1)^{k_{\ell-1}} \ldots x_{\gamma_2}(-1)^{k_1} x_{\gamma_1}(-1)^{k_0} v.\end{aligned}$$

If vector $x(\pi_1)v_{k_\ell, k_0, \ldots, k_{\ell-1}}$ satisfies the initial conditions for $W_{k_\ell, k_0, \ldots, k_{\ell-1}}$, then

$$x(\pi_1^-)x_{\gamma_\ell}(-1)^{k_{\ell-1}} \ldots x_{\gamma_2}(-1)^{k_1} x_{\gamma_1}(-1)^{k_0} v$$

satisfies the difference conditions. It is therefore enough to check:

$$\begin{aligned}&b_0 \le k_\ell \Rightarrow k_0 + \cdots + k_{\ell-1} + b_0 \le k \\ &b_0 + b_1 \le k_\ell + k_0 \Rightarrow k_1 + \cdots + k_{\ell-1} + b_0 + b_1 \le k \\ &\ldots \\ &b_0 + \cdots + b_{\ell-1} \le k_\ell + k_0 + \cdots + k_{\ell-2} \Rightarrow k_{\ell-1} + b_0 + \cdots + b_{\ell-1} \le k.\end{aligned}$$

Also, the image of $x(\pi_1)v_{k_\ell, k_0, \ldots, k_{\ell-1}}$ obviously satisfies the initial conditions for W and thus we obtain

$$Im([\omega]^{\otimes k}) = \mathrm{span}\{x(\pi)v \in W \mid x(\pi) = \ldots x_{\gamma_\ell}(-1)^{k_{\ell-1}} \ldots x_{\gamma_1}(-1)^{k_0}\}.$$

The assertion that $Im([\omega]^{\otimes k}) = Ker(\varphi_0)$ now immediately follows.

We continue the proof by showing that $Im(\varphi_t) = Ker(\varphi_{t+1})$, $t = 0, \dots, m-2$. First we show that $Im(\varphi_t) \subseteq Ker(\varphi_{t+1})$ by proving that

$$\varphi_{t+1}(\varphi_t(w)) = 0$$

for every

$$w = \sum_{I_t \in D_t(K)} w_{I_t} \in \sum_{I_t \in D_t(K)} W_{I_t},$$

$t = 0, \dots, m-2$. For all $I_{t+2} = \{i_0, \dots, i_{t+1}\} \in D_{t+2}(K)$ the following holds:

$$\begin{aligned}
\varphi_{t+1}(\varphi_t(w))_{I_{t+2}} &= \sum_{0 \le s_1 \le t+1} (-1)^{s_1} a(\varphi_t(w))_{I_{t+2} \setminus \{i_{s_1}\}} v_{I_{t+2}} \\
&= \sum_{0 \le s_2 < s_1 \le t+1} (-1)^{s_1}(-1)^{s_2} a(w)_{I_{t+2} \setminus \{i_{s_1}, i_{s_2}\}} v_{I_{t+2}} + \sum_{0 \le s_1 < s_2 \le t+1} (-1)^{s_1}(-1)^{s_2 - 1} a(w)_{I_{t+2} \setminus \{i_{s_1}, i_{s_2}\}} v_{I_{t+2}} \\
&= \sum_{0 \le s_2 < s_1 \le t+1} (-1)^{s_1}(-1)^{s_2} a(w)_{I_{t+2} \setminus \{i_{s_1}, i_{s_2}\}} v_{I_{t+2}} - \sum_{0 \le s_1 < s_2 \le t+1} (-1)^{s_1}(-1)^{s_2} a(w)_{I_{t+2} \setminus \{i_{s_1}, i_{s_2}\}} v_{I_{t+2}} = 0,
\end{aligned}$$

hence $\varphi_{t+1}(\varphi_t(w)) = 0$ follows.

Next, we prove that $Ker(\varphi_{t+1}) \subseteq Im(\varphi_t)$. Take

$$w = \sum_{I_{t+1} \in D_{t+1}(K)} w_{I_{t+1}} \in \sum_{I_{t+1} \in D_{t+1}(K)} W_{I_{t+1}}$$

such that $\varphi_{t+1}(w) = 0$. This means that for every $I_{t+1} \in D_{t+1}(K)$ and $\{i\} \in D_1(K)$ such that $i \notin I_{t+1}$ one has

$$\varphi_{t+1}(w)_{I_{t+1} \cup \{i\}} = 0.$$

Fix $A = \{i_0, \dots, i_t\} \in D_{t+1}(K)$ and arbitrary $\{i\} \in D_1(K)$, $i \notin A$. We have $A \cup \{i\} = \{j_0, \dots, j_{t+1}\}$ for some $\{j_0, \dots, j_{t+1}\} \in D_{t+2}(K)$, so $i = j_{p_A(i)}$ for some $p_A(i) \in \{0, \dots, t+1\}$.

From $\varphi_{t+1}(w)_{A \cup \{i\}} = 0$ and

$$\begin{aligned}
\varphi_{t+1}(w)_{A \cup \{i\}} &= \sum_{0 \le s \le t+1} (-1)^s a(w)_{(A \cup \{i\}) \setminus \{j_s\}} v_{A \cup \{i\}} \\
&= (-1)^{p_A(i)} a(w)_A v_{A \cup \{i\}} + \sum_{\substack{0 \le s \le t+1 \\ s \ne p_A(i)}} (-1)^s a(w)_{(A \cup \{i\}) \setminus \{j_s\}} v_{A \cup \{i\}} = 0
\end{aligned}$$

the following equation holds:

$$a(w)_A v_{A \cup \{i\}} = \sum_{\substack{0 \le s \le t+1 \\ s \ne p_A(i)}} (-1)^{s-1+p_A(i)} a(w)_{(A \cup \{i\}) \setminus \{j_s\}} v_{A \cup \{i\}}. \tag{5.3}$$

By Lemma 5.6 we can write

$$a(w)_A = \sum_{\substack{B \in D(K) \\ B \supseteq A}} a(w)_A^B,$$

where $a(w)_A^B$ denotes the part of $a(w)_A$ in $\mathcal{B}^B$. Analogous decomposition holds also for summands on the right-hand side of (5.3). Introducing these expressions into (5.3) gives

$$\sum_{\substack{B \in D(K) \\ B \supseteq A}} a(w)_A^B v_{A \cup \{i\}} = \sum_{\substack{0 \le s \le t+1 \\ s \ne p_A(i)}} (-1)^{s-1+p_A(i)} \sum_{\substack{B \in D(K) \\ B \supseteq (A \cup \{i\}) \setminus \{j_s\}}} a(w)^B_{(A \cup \{i\}) \setminus \{j_s\}} v_{A \cup \{i\}}. \tag{5.4}$$

Some of the summands in (5.4) are trivial. We want to see for which $x(\pi) \in \mathcal{B}^B$ ($B \in D(K)$ such that $B \supseteq A$) also $x(\pi) \in \mathcal{B}_{A \cup \{i\}}$ holds. There are two cases:

1. $A \cup \{i\} \subseteq B$: because of Lemma 5.3 the statement obviously holds.
2. $A \cup \{i\} \nsubseteq B$: let $x(\pi) \in \mathcal{B}^B$ and let us suppose $x(\pi) \in \mathcal{B}_{A \cup \{i\}}$. Then $x(\pi) \in \mathcal{B}_B \cap \mathcal{B}_{A \cup \{i\}} = \mathcal{B}_{A \cup \{i\} \cup B}$. Since $A \cup \{i\} \cup B \supset B$, this is in opposition to $x(\pi) \in \mathcal{B}^B$, and thus $x(\pi) \notin \mathcal{B}_{A \cup \{i\}}$.

Now (5.4) becomes

$$\sum_{\substack{B \in D(K) \\ B \supseteq A \cup \{i\}}} a(w)_A^B v_{A \cup \{i\}} = \sum_{\substack{0 \le s \le t+1 \\ s \ne p_A(i)}} (-1)^{s-1+p_A(i)} \sum_{\substack{B \in D(K) \\ B \supseteq A \cup \{i\}}} a(w)^B_{(A \cup \{i\}) \setminus \{j_s\}} v_{A \cup \{i\}},$$

and for every $B \in D(K)$, $B \supseteq A$

$$a(w)^B_A = \sum_{\substack{0\le s\le t+1\\ s\ne p_A(i)}} (-1)^{s-1+p_A(i)} a(w)^B_{(A\cup\{i\})\setminus\{j_s\}} \tag{5.5}$$

follows.

We want to show the existence of

$$z = \sum_{I_t\in D_t(K)} z_{I_t} \in \sum_{I_t\in D_t(K)} W_{I_t}$$

such that $\varphi_t(z) = w$, which implies that for every $I_{t+1} \in D_{t+1}(K)$

$$\varphi_t(z)_{I_{t+1}} = a(w)_{I_{t+1}} v_{I_{t+1}} \tag{5.6}$$

holds.

For arbitrary $I_t \in D_t(K)$ let us define

$$a(z)_{I_t} = \sum_{\substack{\{i\}\in D_1(K)\\ i\notin I_t}} \sum_{\substack{B\in D(K)\\ B\supseteq I_t\cup\{i\}}} \frac{(-1)^{p_{I_t}(i)}}{|B|} a(w)^B_{I_t\cup\{i\}},$$

for $p_{I_t}(i)$ such that $i = j'_{p_{I_t}(i)}$ in $I_t \cup \{i\} = \{j'_0, \dots, j'_t\}$ for some $\{j'_0, \dots, j'_t\} \in D_{t+1}(K)$.

Let us prove (5.6) for $A \in D_{t+1}(K)$ we fixed before. We calculate:

$$\begin{aligned}\varphi_t(z)_A &= \sum_{0\le r\le t} (-1)^r a(z)_{A\setminus\{i_r\}} v_A\\ &= \sum_{0\le r\le t} (-1)^r \sum_{\substack{\{i\}\in D_1(K)\\ i\notin A\setminus\{i_r\}}} \sum_{\substack{B\in D(K)\\ B\supseteq (A\setminus\{i_r\})\cup\{i\}}} \frac{(-1)^{p_{A\setminus\{i_r\}}(i)}}{|B|} a(w)^B_{(A\setminus\{i_r\})\cup\{i\}} v_A \end{aligned}\tag{5.7}$$

and conclude (similarly as before) that nontrivial summands are only those indexed by $B \in D(K)$, $B \supseteq (A \setminus \{i_r\}) \cup \{i\}$, for which also $B \supseteq A$ holds. We discuss two possibilities: if $i = i_r$, then $B \supseteq A$ and no additional conditions apply. But, in the case of $i \ne i_r$ we must request $i_r \in B$ in order for $B \supseteq A$ to be true (and then we have $B \supseteq A \cup \{i\}$).

Therefore (5.7) can be rewritten as follows:

$$\begin{aligned}\varphi_t(z)_A &= \sum_{0\le r\le t} (-1)^r \sum_{\substack{B\in D(K)\\ B\supseteq A}} \frac{(-1)^r}{|B|} a(w)^B_A v_A + \sum_{0\le r\le t} (-1)^r \sum_{\substack{\{i\}\in D_1(K)\\ i\notin A}} \sum_{\substack{B\in D(K)\\ B\supseteq A\cup\{i\}}} \frac{(-1)^{p_{A\setminus\{i_r\}}(i)}}{|B|} a(w)^B_{(A\setminus\{i_r\})\cup\{i\}} v_A\\ &= \sum_{\substack{B\in D(K)\\ B\supseteq A}} \sum_{0\le r\le t} \frac{1}{|B|} a(w)^B_A v_A + \sum_{\substack{\{i\}\in D_1(K)\\ i\notin A}} \sum_{\substack{B\in D(K)\\ B\supseteq A\cup\{i\}}} \frac{1}{|B|} \sum_{0\le r\le t} (-1)^{r+p_{A\setminus\{i_r\}}(i)} a(w)^B_{(A\setminus\{i_r\})\cup\{i\}} v_A.\end{aligned}$$

Let us now demonstrate that for every $\{i\} \in D_1(K)$ such that $i \notin A$ and every $B \in D(K)$ such that $B \supseteq A \cup \{i\}$

$$\sum_{0\le r\le t} (-1)^{r+p_{A\setminus\{i_r\}}(i)} a(w)^B_{(A\setminus\{i_r\})\cup\{i\}} = \sum_{\substack{0\le s\le t+1\\ s\ne p_A(i)}} (-1)^{s-1+p_A(i)} a(w)^B_{(A\cup\{i\})\setminus\{j_s\}} \tag{5.8}$$

holds, with $\{j_0, \dots, j_{t+1}\} = A \cup \{i\}$ for some $\{j_0, \dots, j_{t+1}\} \in D_{t+2}(K)$.

Since both sides of (5.8) contain $t+1$ summands, it is enough to show the equality between the corresponding summands. Depending on the choice of $0 \le s \le t + 1$, $s \ne p_A(i)$, we distinguish two cases:

1. For chosen $s < p_A(i)$ the corresponding right-hand side summand equals the left-hand side summand that corresponds to $r = s$. Namely, for $r = s < p_A(i)$ we have $p_{A\setminus\{i_r\}}(i) = p_A(i) - 1$ and $i_r = j_s$, and thus

$$\begin{aligned}&r + p_{A\setminus\{i_r\}}(i) = s + p_{A\setminus\{i_s\}}(i) = s + p_A(i) - 1 = s - 1 + p_A(i)\\ &(A \setminus \{i_r\}) \cup \{i\} = (A \setminus \{i_s\}) \cup \{i\} = (A \cup \{i\}) \setminus \{i_s\} = (A \cup \{i\}) \setminus \{j_s\}.\end{aligned}$$

2. For chosen $s > p_A(i)$ the corresponding right-hand side summand equals the left-hand side summand corresponding to $r = s - 1$, because $s \ge p_A(i) + 1 \Rightarrow r \ge p_A(i)$ implies that $p_{A\setminus\{i_r\}}(i) = p_A(i)$ and $i_r = i_{s-1} = j_s$. Therefore:

$$\begin{aligned}&r + p_{A\setminus\{i_r\}}(i) = s + p_{A\setminus\{i_{s-1}\}}(i) = s - 1 + p_A(i)\\ &(A \setminus \{i_r\}) \cup \{i\} = (A \setminus \{i_{s-1}\}) \cup \{i\} = (A \cup \{i\}) \setminus \{j_s\}.\end{aligned}$$

Using (5.5) and (5.8) we now transform $\varphi_t(z)_A$ as follows:

$$\begin{aligned}\varphi_t(z)_A &= \sum_{\substack{B\in D(K)\\ B\supseteq A}} \frac{t+1}{|B|} a(w)_A^B v_A + \sum_{\substack{\{i\}\in D_1(K)\\ i\notin A}} \sum_{\substack{B\in D(K)\\ B\supseteq A\cup\{i\}}} \frac{1}{|B|} a(w)_A^B v_A \\ &= a(w)_A^A v_A + \sum_{\substack{B\in D(K)\\ B\supset A}} \frac{t+1}{|B|} a(w)_A^B v_A + \sum_{\substack{\{i\}\in D_1(K)\\ i\notin A}} \sum_{\substack{B\in D(K)\\ B\supseteq A\cup\{i\}}} \frac{1}{|B|} a(w)_A^B v_A.\end{aligned}$$

The last sum above obviously goes on all $B \in D(K)$, $B \supset A$, and every such fixed B occurs exactly $|B| - |A|$ times. Thus we have

$$\begin{aligned}\varphi_t(z)_A &= a(w)_A^A v_A + \sum_{\substack{B\in D(K)\\ B\supset A}} \frac{t+1}{|B|} a(w)_A^B v_A + \sum_{\substack{B\in D(K)\\ B\supset A}} \frac{|B|-|A|}{|B|} a(w)_A^B v_A \\ &= a(w)_A^A v_A + \sum_{\substack{B\in D(K)\\ B\supset A}} a(w)_A^B v_A = \sum_{\substack{B\in D(K)\\ B\supseteq A}} a(w)_A^B v_A = a(w)_A v_A,\end{aligned}$$

which finally proves that $Ker(\varphi_{t+1}) \subseteq Im(\varphi_t)$, $t = 0, \dots, m-2$.

6. Recurrence relations for formal characters

6.1. Formal characters and systems of recurrences

Fix k and $\Lambda = k_0\Lambda_0 + \cdots + k_\ell \Lambda_\ell$ such that $k_0 + \cdots + k_\ell = k$. Define the formal character $\chi(W)$ of a subspace $W = W(\Lambda)$ as

$$\chi(W)(z_1, \dots, z_\ell; q) = \sum \dim W^{m,n_1,\dots,n_\ell} q^m z_1^{n_1} \cdots z_\ell^{n_\ell},$$

where $W^{m,n_1,\dots,n_\ell}$ denotes a subspace of W spanned by monomial vectors $x(\pi)v_\Lambda = \dots x_{\gamma_1}(-2)^{a_\ell} x_{\gamma_\ell}(-1)^{a_{\ell-1}} \dots x_{\gamma_1}(-1)^{a_0} v_\Lambda \in \mathcal{B}$ such that $x(\pi)$ is of degree $d(x(\pi)) = m$ and weight $w(x(\pi)) = n_1\gamma_1 + \cdots + n_\ell\gamma_\ell$, with $d(x(\pi))$ and $w(x(\pi))$ given by

$$d(x(\pi)) = \sum_{j=0}^{\infty} \sum_{i=1}^{\ell} (j+1) \cdot a_{i+j\cdot\ell-1}$$

$$w(x(\pi)) = \sum_{j=0}^{\infty} \sum_{i=1}^{\ell} \gamma_i \cdot a_{i+j\cdot\ell-1}.$$

Note that for every fixed choice of integers n_i, $i = 1, \dots, \ell$, and m, such that

$$n_i \geq k_{i-1}, \quad i = 1, \dots, \ell$$
$$m \geq n_1 + \cdots + n_\ell + k_0 + \cdots + k_{\ell-1}$$

and every

$$x(\pi)v_{k_\ell,k_0,\dots,k_{\ell-1}} \in W_{k_\ell,k_0,\dots,k_{\ell-1}}^{m-(n_1+\cdots+n_\ell)-(k_0+\cdots+k_{\ell-1}),n_1-k_0,\dots,n_\ell-k_{\ell-1}}$$

we have

$$[\omega]^{\otimes k}(x(\pi)v_{k_\ell,k_0,\dots,k_{\ell-1}}) = x(\pi^-)x_{\gamma_\ell}(-1)^{k_{\ell-1}} \cdots x_{\gamma_2}(-1)^{k_1} x_{\gamma_1}(-1)^{k_0} v \in W^{m,n_1,\dots,n_\ell}.$$

Also, if we now apply mappings $\varphi_0, \varphi_1, \dots, \varphi_{m-1}$ consecutively to the above right-hand side vector, neither the degree nor the weight of monomial part of that vector will change. Taking this in consideration and by using Theorem 5.1 we conclude that the following equality among the dimensions of weight subspaces holds:

$$\begin{aligned}&\dim W_{k_\ell,k_0,\dots,k_{\ell-1}}^{m-(n_1+\cdots+n_\ell)-(k_0+\cdots+k_{\ell-1}),n_1-k_0,\dots,n_\ell-k_{\ell-1}} - \dim W^{m,n_1,\dots,n_\ell} \\ &\quad + \sum_{I_1\in D_1(K)} \dim W_{I_1}^{m,n_1,\dots,n_\ell} + \cdots + (-1)^{m-1} \dim W_{I_m}^{m,n_1,\dots,n_\ell} = 0.\end{aligned}$$

As an implication we have the following relation among characters of Feigin–Stoyanovsky's type subspaces at fixed level k:

$$\begin{aligned}&\chi(W)(z_1, \dots, z_\ell; q) - \sum_{I_1\in D_1(K)} \chi(W_{I_1})(z_1, \dots, z_\ell; q) \\ &\quad + \sum_{I_2\in D_2(K)} \chi(W_{I_2})(z_1, \dots, z_\ell; q) + \cdots + (-1)^m \chi(W_{I_m})(z_1, \dots, z_\ell; q) \\ &= (z_1q)^{k_0} \dots (z_\ell q)^{k_{\ell-1}} \chi(W_{k_\ell,k_0,\dots,k_{\ell-1}})(z_1q, \dots, z_\ell q; q). \qquad (6.1)\end{aligned}$$

Note that (6.1) can be abbreviated to

$$\sum_{I\in D(K)}(-1)^{|I|}\chi(W_I)(z_1,\ldots,z_\ell;q)=(z_1q)^{k_0}\ldots(z_\ell q)^{k_{\ell-1}}\chi(W_{k_\ell,k_0,\ldots,k_{\ell-1}})(z_1q,\ldots,z_\ell q;q). \tag{6.2}$$

Since (6.1) holds for $W = W(k_0\Lambda_0+\cdots+k_\ell\Lambda_\ell)$ for all possible choices of nonnegative integers $k_0,\ldots,k_\ell$ such that $k_0+\cdots+k_\ell=k$, we can say that (6.1) is a system of relations. Our attempt will be directed towards proving the uniqueness of solution for this system.

Example 6.1. *In the case of $\ell=2$, $k=2$, the above system reads:*

$$\begin{aligned}
\chi(W_{2,0,0})(z_1,z_2;q)&=\chi(W_{1,1,0})(z_1,z_2;q)+(z_1q)^2\chi(W_{0,2,0})(z_1q,z_2q;q)\\
\chi(W_{1,1,0})(z_1,z_2;q)&=\chi(W_{0,2,0})(z_1,z_2;q)+\chi(W_{1,0,1})(z_1,z_2;q)\\
&\quad-\chi(W_{0,1,1})(z_1,z_2;q)+(z_1q)(z_2q)\chi(W_{0,1,1})(z_1q,z_2q;q)\\
\chi(W_{1,0,1})(z_1,z_2;q)&=\chi(W_{0,1,1})(z_1,z_2;q)+z_1q\chi(W_{1,1,0})(z_1q,z_2q;q)\\
\chi(W_{0,2,0})(z_1,z_2;q)&=\chi(W_{0,1,1})(z_1,z_2;q)+(z_2q)^2\chi(W_{0,0,2})(z_1q,z_2q;q)\\
\chi(W_{0,1,1})(z_1,z_2;q)&=\chi(W_{0,0,2})(z_1,z_2;q)+z_2q\chi(W_{1,0,1})(z_1q,z_2q;q)\\
\chi(W_{0,0,2})(z_1,z_2;q)&=\chi(W_{2,0,0})(z_1q,z_2q;q).
\end{aligned}$$

Let us now write

$$\chi(W_{k_0,\ldots,k_\ell})(z_1,\ldots,z_\ell;q)=\sum_{n_1,\ldots,n_\ell\geq 0}A^{n_1,\ldots,n_\ell}_{k_0,\ldots,k_\ell}(q)z_1^{n_1}\ldots z_\ell^{n_\ell}, \tag{6.3}$$

where $A^{n_1,\ldots,n_\ell}_{k_0,\ldots,k_\ell}(q)$ denote formal series in one formal variable q. Putting (6.3) into (6.2) gives us (with labels analogous to those we used in the previous section) the following system:

$$\sum_{I\in D(K)}(-1)^{|I|}A_I^{n_1,\ldots,n_\ell}(q)=q^nA^{n_1-k_0,\ldots,n_\ell-k_{\ell-1}}_{k_\ell,k_0,\ldots,k_{\ell-1}}(q), \tag{6.4}$$

with $n=n_1+\cdots+n_\ell$.

In the next section we prove that system (6.4) has a unique solution, thus proving the solution for (6.2) is also unique.

6.2. Uniqueness of the solution

For fixed k we have the following result:

Proposition 6.2. *For all choices of nonnegative integers $k_0,\ldots,k_\ell$ such that $k_0+\cdots+k_\ell=k$ and all nonnegative integers $n_1,\ldots,n_\ell$ such that $n_i\geq k_{i-1}$, $i=1,\ldots,\ell$, the following assertion holds:*

$$A^{n_1,\ldots,n_\ell}_{k_0,\ldots,k_\ell}(q)=q^n\sum_{(a_0,\ldots,a_{\ell-1})\in\mathcal{B}}A^{n_1-a_0,\ldots,n_\ell-a_{\ell-1}}_{k-a,a_0,\ldots,a_{\ell-1}}(q), \tag{6.5}$$

having denoted that $a=a_0+\cdots+a_{\ell-1}$, $n=n_1+\cdots+n_\ell$, and that $\mathbf{a}:=(a_0,\ldots,a_{\ell-1})\in\mathcal{B}$ means that (3.2) holds.

Proof. Index the equality (6.5) by $(\ell+1)$-tuple of lower indices of the series appearing on the left-hand side, i.e. by $(k_0,\ldots,k_\ell)$. We prove the proposition by induction on these tuples, in the usual lexicographic ordering among them.

Let us first check that (6.5) is true if given for smallest such $(\ell+1)$-tuple, that is $(0,\ldots,0,k)$, which means that we want to check if

$$A^{n_1,\ldots,n_\ell}_{0,\ldots,0,k}(q)=q^n\sum_{\mathbf{a}\in\mathcal{B}_{0,\ldots,0,k}}A^{n_1-a_0,\ldots,n_\ell-a_{\ell-1}}_{k-a,a_0,\ldots,a_{\ell-1}}(q)$$

holds. But, we see immediately that $\mathcal{B}_{0,\ldots,0,k}=\{(0,\ldots,0)\}$, and therefore (6.5) in this case reads

$$A^{n_1,\ldots,n_\ell}_{0,\ldots,0,k}(q)=q^nA^{n_1,\ldots,n_\ell}_{k,0,\ldots,0}(q),$$

which is exactly the corresponding relation in (6.4), and the assertion follows.

Let us now fix some $(\ell+1)$-tuple $K=(k_0,\ldots,k_\ell)$. The equation in (6.4) corresponding to K is

$$\sum_{I\in D(K)}(-1)^{|I|}A_I^{n_1,\ldots,n_\ell}(q)=q^nA^{n_1-k_0,\ldots,n_\ell-k_{\ell-1}}_{k_\ell,k_0,\ldots,k_{\ell-1}}(q).$$

Series appearing on the left-hand side indexed by $I\neq\emptyset$, $I\in D(K)$, yield $(\ell+1)$-tuples strictly smaller than K. Therefore, by induction hypothesis, the proposition assertion holds for (6.5) corresponding to those tuples. Thus we obtain

$$A^{n_1,\dots,n_\ell}(q) = q^n \sum_{\substack{I\in D(K)\\ I\neq\emptyset}} \sum_{\mathbf{a}\in\mathcal{B}_I} (-1)^{|I|-1} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q) + q^n A^{n_1-k_0,\dots,n_\ell-k_{\ell-1}}_{k_\ell,k_0,\dots,k_{\ell-1}}(q). \tag{6.6}$$

Note that $\mathbf{a} \in \mathcal{B}_I$, $I \in D(K) \setminus \{\emptyset\}$, belong also to $\mathcal{B}_\emptyset = \mathcal{B}$ (an obvious consequence of Lemma 5.3). Moreover, it is not hard to see that those ℓ-tuples are *all* $\mathbf{a} \in \mathcal{B}$ except $\mathbf{a} = (k_0, \dots, k_{\ell-1})$. But, the last remaining summand on the right-hand side of (6.6) is exactly the one "produced" by this "missing" tuple. It only remains to show that each $\mathbf{a} \in \mathcal{B}$ appears only once on the right-hand side of (6.6). For arbitrary $\mathbf{a} \in \mathcal{B}$ there exists $I \in D(K)$ such that $\mathcal{B}^I$ (cf. Lemma 5.6). But then, because of Lemma 5.3, also $\mathbf{a} \in B_J$ holds for all $J \in D(K)$ such that $J \subseteq I$. This means that the factor multiplying $A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q)$ in (6.6) equals $\sum_{j=1}^{|I|}(-1)^{j-1}\binom{|I|}{j}$. But, this factor equals one:

$$1 - \sum_{j=1}^{|I|}(-1)^{j-1}\binom{|I|}{j} = \sum_{j=0}^{|I|}(-1)^{j}\binom{|I|}{j} = \sum_{j=0}^{|I|}(-1)^{j}1^{|I|-j}\binom{|I|}{j} = 0.$$

From (6.6) we now get

$$A^{n_1,\dots,n_\ell}_{k_0,\dots,k_\ell}(q) = q^n \sum_{\mathbf{a}\in\mathcal{B}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q),$$

which completes the inductive proof. □

Proposition 6.3. *For all choices of nonnegative integers $k_0, \dots, k_\ell$ such that $k_0 + \dots + k_\ell = k$ and all nonnegative integers $n_1, \dots, n_\ell$ such that $n_i \geq k_{i-1}$, $i = 1, \dots, \ell$, the following assertion is true:*

$$A^{n_1,\dots,n_\ell}_{k_0,\dots,k_\ell}(q) = \frac{q^n}{1-q^n}\left(\sum_{\substack{\mathbf{a}\in\mathcal{B}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q) + q^n \sum_{\substack{\mathbf{a}\in\mathcal{B}_{k,0,\dots,0}\setminus\mathcal{B}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q)\right).$$

Proof. From (6.5) we have

$$\begin{aligned}
A^{n_1,\dots,n_\ell}_{k,0,\dots,0}(q) &= q^n \sum_{\mathbf{a}\in\mathcal{B}_{k,0,\dots,0}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q)\\
&= q^n \sum_{\substack{\mathbf{a}\in\mathcal{B}_{k,0,\dots,0}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q) + q^n A^{n_1,\dots,n_\ell}_{k,0,\dots,0}(q),
\end{aligned}$$

and it follows that

$$A^{n_1,\dots,n_\ell}_{k,0,\dots,0}(q) = \frac{q^n}{1-q^n} \sum_{\substack{\mathbf{a}\in\mathcal{B}_{k,0,\dots,0}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q). \tag{6.7}$$

Let us now fix nonnegative integers $k_0, \dots, k_\ell$ such that $k_0 + \dots + k_\ell = k$. From (6.5) by using (6.7) we get

$$\begin{aligned}
A^{n_1,\dots,n_\ell}_{k_0,\dots,k_\ell}(q) &= q^n \sum_{\mathbf{a}\in\mathcal{B}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q)\\
&= q^n \left(A^{n_1,\dots,n_\ell}_{k,0,\dots,0}(q) + \sum_{\substack{\mathbf{a}\in\mathcal{B}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q)\right)\\
&= q^n \left(\frac{q^n}{1-q^n}\sum_{\substack{\mathbf{a}\in\mathcal{B}_{k,0,\dots,0}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q) + \sum_{\substack{\mathbf{a}\in\mathcal{B}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q)\right)\\
&= \frac{q^n}{1-q^n}\left(q^n \sum_{\substack{\mathbf{a}\in\mathcal{B}_{k,0,\dots,0}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q) + (1-q^n)\sum_{\substack{\mathbf{a}\in\mathcal{B}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q)\right)\\
&= \frac{q^n}{1-q^n}\left(\sum_{\substack{\mathbf{a}\in\mathcal{B}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q) + q^n \sum_{\substack{\mathbf{a}\in\mathcal{B}_{k,0,\dots,0}\setminus\mathcal{B}\\ \mathbf{a}\neq(0,\dots,0)}} A^{n_1-a_0,\dots,n_\ell-a_{\ell-1}}_{k-a,a_0,\dots,a_{\ell-1}}(q)\right),
\end{aligned}$$

which proves the proposition. □

A direct consequence of this proposition is the fact that every $A^{n_1,\ldots,n_\ell}_{k_0,\ldots,k_\ell}(q)$ can be expressed as a linear combination (with coefficients being rational functions of formal variable q) of rational functions $A^{n_1-a_0,\ldots,n_\ell-a_{\ell-1}}_{k-a,a_0,\ldots,a_{\ell-1}}(q)$ indexed by ℓ-tuples $(n_1-a_0,\ldots,n_\ell-a_{\ell-1})$ of upper indices, which are all strictly smaller than $(n_1,\ldots,n_\ell)$, in the usual lexicographic ordering. This inductively guarantees the uniqueness of the solution for system (6.4).

Acknowledgement

This research was partially supported by the Ministry of Science and Technology of the Republic of Croatia, Project ID 037-0372794-2806.

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