Finite 2-groups all of whose proper subgroups have commutator groups of order ≤ 2.

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1. Introduction and known results

Denote by $\mathcal{G}_2$ the set of finite 2-groups, and for any group G denote by $\mathcal{M}(G)$ the set of its maximal subgroups. We consider here the groups which satisfy the condition stated in the title, which is equivalent with the following one:

$$G \in \mathcal{G}_2 \text{ and if } M \in \mathcal{M}(G) \text{ then } |M'| \leq 2. \tag{1}$$

We prove the folowing result, which describes the structure of such groups.

Theorem. Let G be a 2-group whose all proper subgroups have commutator groups of order ≤ 2. Then we have one of the following cases:

1) if $|G'| = 2$ then $G = H_1 * \cdots * H_n \cdot Z(G)$, H_i $(i = 1, 2, \ldots, n)$ being minimal nonabelian subgroups of G, and $*$ denoting the central product.

2) $d(G) = 3$, $G' \cong E_8$, $\Phi(G) \leq Z(G)$, $G = \langle a_1, a_2, a_3\rangle$, $[a_2, a_3] = z_1$, $[a_3, a_1] = z_2$, $[a_1, a_2] = z_3$, $G' = \langle z_1, z_2, z_3\rangle$. The maximal subgroups of G are all nonabelian and for maximal $M, N \leq G$, $M \neq N$ it is $M' \neq N'$.

3) $d(G) = 3$, $G' \cong E_4$, $\Phi(G) \leq Z(G)$, $G = \langle a_1, a_2, a_3\rangle$, $[a_2, a_3] = z_1$, $[a_3, a_1] = z_2$, $[a_1, a_2] = 1$, $G' = \langle z_1, z_2\rangle$. There is one abelian maximal subgroup $M_1 = \langle a_1, a_2, \Phi(G)\rangle$ and the remaining 6 are nonabelian and divided in 3 pairs, each pair having the same commutator group.

4) $d(G) = 2$, $G' \cong E_4$ or $G' \cong Z_4$, $G = \langle a_1, a_2\rangle$. There are 3 maximal subgroups M_1, M_2, M_3 such that $K = M_1'M_2'M_3' = \langle z\rangle \cong Z_2$, $K < G'$ and G/K is minimal nonabelian 2-group.

In the proof of this theorem we shall use following known results:

Proposition 1.(Janko [2,Proposition 1.7]) Let G be a nonabelian finite 2-group possesing an abelian maximal subgroup. Then $|G| = 2|G'| \cdot |Z(G)|$.

Proposition 2.(A. Mann, see Berkovich [1]) Let G be a finite 2-group, M, N any two different maximal subgroups of G. Then $|G' : M'N'| \leq 2$.

Proof. Since $M, N \lhd G$, therefore $M'N' \trianglelefteq G$. Also $M'N' \leq G' \leq \Phi(G)$. Thus $M/M'N'$ and $N/M'N'$ are abelian and maximal in $\overline{G} = G/M'N'$. Obviously, $|\overline{G} : Z(\overline{G})| \leq 4$, and so by Proposition 1, $\overline{G}' = |G'/M'N'| \leq 2$.

Proposition 3.(Janko [3,Proposition 1.6]) Let G be a 2-group with $|G'| = 2$. If H is a minimal nonabelian subgroup of G, then $G = HC_G(H)$ and $|G : C_G(H)| = 4$.

Proof. Each minimal nonabelian group H is 2-generated:
For $x_1, x_2 \in H$, $[x_1, x_2] \neq 1$ is $\langle x_1, x_2\rangle$ nonabelian and thus $H = \langle x_1, x_2\rangle$ because of its minimality. Denote $C_i = C_G(x_i)$, $i = 1, 2$. Because of $|G'| = 2$ and $|x^G| = |G : C_G(x)| = |\{x^g | g \in G\}| = |\{x[x, g] | g \in G\}| \leq |G'| = 2$, we have $|C : C_i| = 2$ for $i = 1, 2$. Considering $C = C_1 \bigcap C_2$, we have $C = C_G(H)$, $|G : C| \leq 4$, $H \bigcap C = Z(H)$. Since H is nonabelian $|H : Z(H)| \geqslant 4$ and so $|HC| = (|C| \cdot |H|) : |H \bigcap C| \geqslant 4|C|$. Therefore $|G : C| = |H : Z(H)| = 4$ and $G = HC$.

Proposition 4. Let G be a finite 2-group, $G' \leq Z(G)$ and $exp\ G' = 2$. Then $\Phi(G) \leq Z(G)$.

Proof. Let $x, g \in G$. Then $[x, g^2] = [x, g][x, g]^g = [x, g]^2 = 1$, as $[x, g] \in G'$. Since $\Phi(G) = \mho_1(G) = \langle g^2 | g \in G\rangle$ for any 2-group G and $g^2 \in Z(G)$ for all $g \in G$, we have $\Phi(G) \leq Z(G)$.

Proposition 5. Let G be a 2-generated finite 2-group and $|G'| = 2$. Then G is minimal nonabelian.

Proof. Let $G = \langle a, b\rangle$. By Proposition 4, $\Phi(G) \leq Z(G)$. As $\Phi(G)$ is maximal in all 3 maximal subgroups $M_1 = \langle a, \Phi(G)\rangle$, $M_2 = \langle b, \Phi(G)\rangle$, $M_3 = \langle ab, \Phi(G)\rangle$ of G, they are all abelian and so G is minimal abelian.

2. Proof of the Theorem

We prove our Theorem in several steps.

(i) The case $|G'| = 2$*:*
Let $H_1 = \langle a_1, b_1\rangle$ be a minimal nonabelian subgroup of G.
Then, by Proposition 3, $G = H_1 C_G(H_1)$; if $C_G(H_1)$ is abelian, so $C_G(H_1) = Z(G)$ and we have $G = H_1 Z(G)$. Otherwise, let $H_2 = \langle a_2, b_2\rangle$ be a minimal nonabelian subgroup of $C_G(H_1)$. By the same Proposition 3 we have $C_G(H_1) = C_G(H_2) \cdot (C_G(H_1) \bigcap C_G(H_2)) = H_2 \cdot C_G(\langle H_1, H_2\rangle)$, and so $G = H_1 * H_2 \cdot C_G(\langle H_1, H_2\rangle)$. Continuing in the same way we get fnally $G = H_1 * H_2 * \cdots * H_n \cdot C_G(\langle H_1, \ldots, H_n\rangle)$, the last factor being abelian and so equal $Z(G)$. This proves the assertion 1) of the Theorem.

(ii) The order of G' *is at most 8.*
Proof. Let M, N be two different maximal subgroups of G. By assumption $|M'|, |N'| \leq 2$ and $M', N' \trianglelefteq G$ so $|M'N'| \leq 4$. By Proposition 2 it is $|G' : M'N'| \leq 2$ and so $|G'| \leq 8$.

In the following we denote $K = \langle M' | M \in \mathcal{M}(G)\rangle$, the group generated by commutator groups of all maximal subgroups of G. Obviously, $K \leq Z(G)$ and $\exp K = 2$.

(iii) If $G' = K$ *and* $|K| \geq 4$*, then* $d(G) = 3$*. Moreover* $\Phi(G) \leq Z(G)$*.*
Proof. Let M, N be maximal subgroups of G with $M'N' \cong E_4$, $M' = \langle z_1\rangle$, $N' = \langle z_2\rangle$. Then there exist elements $a, b \in M$, $c, d \in N$ such that $[a, b] = z_1$, $[c, d] = z_2$. Now, $H = \langle a, b, c, d\rangle \leq G$ and $H' \geq \langle z_1, z_2\rangle$, $|H'| \geq 4$. Consequently, $H = G = \langle a, b, c, d\rangle$. Consider $H_1 = \langle a, b, c\rangle$ and $H_2 = \langle b, c, d\rangle$. Now $[b, c] \leq \langle a, b, c\rangle' \bigcap \langle b, c, d\rangle'$. If H_1 and H_2 are different from G, then $[b, c] \leq H_1' \bigcap H_2' = \langle z_1\rangle \cap \langle z_1\rangle = 1$, so $[b, c] = 1$.
Similary, $[a, c] = [a, d] = [b, d] = 1$. Consider $H = \langle ac, b, d\rangle$. Here, $[ac, b] = [a, b] = z_1$, $[ac, d] = [c, d] = z_2$, so $|H_3'| > 2$ and therefore $H_3 = G = \langle ac, b, d\rangle$. We see that $d(G) \leq 3$. If $d(G) = 2$, then $G = \langle x_1, x_2\rangle$ and $[x_1, x_2] = z \in G' = K$, implying $G' = \langle [x_1, x_2]^G\rangle = \langle z\rangle \cong Z_2$, a contradiction. So, $d(G) = 3$. From Proposition 3 we see immediately that $\Phi(G) \leq Z(G)$.

(iv) If $|K| \geq 4$ *then* $G' = K$*.*
Proof. For $|K| = 8$ it is trivial, as $|G'| \leq 8$. Suppose $|K| = 4$ and $|G'| = 8$. So $G' > K \cong E_4$. Let $a, b \in G$ such that $[a, b] = c \in G' \smallsetminus K$. If $\langle a, b\rangle \leq M$ for some maximal $M \leq G$ then $c = [a, b] \leq M' \leq K$. Thus $\langle a, b\rangle = G$ and $d(G) = 2$. There are 3 maximal subgroups in G:

$$M_1 = \langle a, \Phi(G)\rangle,\ M_2 = \langle b, \Phi(G)\rangle,\ \text{and}\ M_3 = \langle ab, \Phi(G)\rangle.$$

As $(G/K)' = G'/K \cong Z_2$, it follows $G'/K \leq Z(G/K)$ and so $[c,x] \in K \leq Z(G)$ for every $x \in G$. Therefore $[c,x^2] = [c,x][c,x]^x = [c,x]^2 = 1$. Now $[a,b^2] = [a,b][a,b]^b = cc^b \in M_1'$, $[a^2,b] = [a,b]^a[a,b] = cc^a \in M_2'$, $[a^2,ab] = [a^2,b] = cc^a \in M_3' \bigcap M_2'$ and $[ab,b^2] = [a,b^2]^b = c^bc^{b^2} = cc^b \in M_3' \bigcap M_1'$.
If $cc^a \neq 1$ or $cc^b \neq 1$, then $M_3' = M_2'$ or $M_3' = M_1'$, respectively, and so, by Proposition 2, $|G'| = 4$. Thus $cc^a = cc^b = 1$, and so $c^a = c^{-1} = c^b$, and $c^{ab} = c$. It follows that $G' = \langle c^G \rangle = \langle c \rangle$ is cyclic, a contradiction. Thus $G' = K$.

(v) Case $G' = K \cong E_8$
Since $d(G) = 3$ and $M' \neq N'$, $M'N' \cong E_4$ for different maximal subgroups M, N, each involution in G' generates commutator group for exactly one of 7 maximal subgroups in G. Thus we have, without loss of generality:

$$G = \langle a_1, a_2, a_3 |\ a_i^{m_i} = z_1^{\delta_i} z_2^{\varepsilon_i} z_3^{\zeta_i},\ [a_i, a_j] = z_k,\ \langle z_1, z_2, z_3 \rangle = G' \rangle,$$

where $i = 1,2,3$ and $\{i,j,k\} = \{1,2,3\}$, $\delta_i, \varepsilon_i, \zeta_i \in \{0,1\}$ and m_i being the order of $\bar{a}_i$ in $\bar{G} = G/K$. The established facts prove the part 2) of our Theorem.

(vi) Case $G' = K \cong E_4$.
Again, by (iii) $d(G) = 3$ and $\Phi(G) \leq Z(G)$.
By Proposition 2 there cannot exist more than one abelian maximal subgroup in G. Thus there exist two maximal subgroups M_1, M_2 with $M_1' = M_2' = \langle z \rangle$, $1 \neq z \in G'$. Denote by x_3 an element of $M_1 \bigcap M_2 \setminus \Phi(G)$ and $x_1 \in M_1 \setminus M_2$, $x_2 \in M_2 \setminus M_1$. So $M_1 = \langle x_1, x_3, \Phi(G) \rangle$, $M_2 = \langle x_2, x_3, \Phi(G) \rangle$. We have $[x_1, x_3] = [x_2, x_3] = z$ and $[x_1 x_2, x_3] = [x_1, x_3][x_2, x_3] = z \cdot z = 1$.
For $M_3 = \langle x_1 x_2, x_3, \Phi(G) \rangle$, which is also maximal in G, it is $M_3' = \langle [x_1 x_2, x_3] \rangle' = 1$. We see that there is a unique abelian maximal subgroup in G.

After some renaming of generators we get the following relations for G:

$$G = \langle a_1, a_2, a_3 |\ a_i^{m_i} = z_1^{\delta_i} z_2^{\varepsilon_i},\ [a_1, a_2] = 1,\ [a_1, a_3] = z_1,\ [a_2, a_3] = z_2 \rangle,$$

where $i = 1,2,3$, $\delta_i, \varepsilon_i \in \{0,1\}$ and m_i being the order of $\bar{a}_i$ in $\bar{G} = G/K$. One can easily check that besides $M = \langle a_1, a_2, \Phi(G) \rangle$ which is abelian, the other six maximal subgroups are all nonabelian and are divided in 3 pairs with commutator groups $\langle z_1 \rangle$, $\langle z_2 \rangle$ and $\langle z_1, z_2 \rangle$, respectively. This proves the part 3) of the Theorem.

(vii) Case $G' > K$.
By (iv) and Proposition 2 we have in this case $|G'| = 4$ and $|K| = 2$. If $[a,b] = c \in G' \setminus K$ then $G = \langle a, b \rangle$, $d(G) = 2$, since otherwise $[a,b]$ would be in $K = \langle M' | M \text{ maximal in } G \rangle$. Now $(G/K)' = G'/K \cong Z_2$ and G/K is 2-generated. Applying Proposition 5 we see that G/K is minimal nonabelian. This proves the part 4) of the Theorem.

The Theorem is proved.

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