Adaptation Criterions for Wavelet Filter Banks With Variable Zero Moments

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***Abstract*-- In this paper, we compare different criterions of adaptation in the proposed two-channel wavelet filter bank with variable number of zero moments. Generally, filters with more zero moments are more appropriate for representing smooth parts of the analyzed signal, while shorter filters are better for transients and singularities. Depending on the criterion, filter banks that change number of zero moments at each step of decomposition can outperform fixed banks in a number of applications. When applied to signals, variable zero moment wavelets result in low ripple on edges and good concentration of wavelet coefficients in smooth parts. Error signal is formed from filtered wavelet coefficients and therefore available on the reconstruction side. We discuss different adaptation algorithms based on spectral properties, adaptation interval and causality. Chosen criterion determines decomposition properties, as well as its reproducibility on the reconstruction side. Results on synthetic and real signals are presented.**

***Index Terms*— adaptation criterions, variable zero moments, wavelets, time-varying filter banks**

I. Introduction

The number of vanishing moments of a fixed wavelet filter bank is usually chosen as a compromise between filter complexity and concentration of the wavelet coefficients. More zero moments correspond to more regularity, which gives better description of smooth and correlated parts of the analyzed signal [1][2]. But, it results in longer filters that cause spread of wavelet coefficients on sharp edges of the analyzed signal. On the other hand, shorter filters are more suitable for compact representation of transients and singularities, as well as parts of the signal with narrower correlation of samples.

In this paper we describe a wavelet filter bank that adapts the number of zero moments. Adaptation is performed on both filters in the bank at each step of decomposition. The adaptation criterion is computed from wavelet coefficients. In some cases it is reproducible on the reconstruction side. We expect benefits of using adaptive number of zero moments in many applications, such us [6].

In section II we describe the construction of the proposed adaptive filter bank. Sweldens 96 [3] proposed a construction of biorthogonal wavelet filter banks based on the lifting scheme, using interpolation of samples in the time domain. A short review is given in paragraph II.A. Even samples are estimated from odds using Lagrange interpolation functions of chosen order. In the proposed scheme, we consider odd order Lagrange polynomials corresponding to the even length FIR filters.

In paragraph II.B we give the proposed factorization of the adjustable lifting step. In paragraph II.C adjustable dual lifting step is introduced.

In section III we discuss the adaptation criterions and results. Proposed filter bank is applied on a synthetic and a real-world signal. Decomposition of synthetic signal is almost optimal. Unpredictable components of the real-world signals cause modification of memoryless adaptation criterion. It should be computed on interval. Depending on the choice of interval boundaries, algorithm is reproducible or not reproducible on the reconstruction side. It is shown that the entropy of the wavelet coefficients computed by adaptive filter bank is lower when compared to fixed banks.

II. Filter Bank Structure

A. *Lifting scheme*

Lifting scheme enables easy construction of perfect reconstruction time-variant and non-linear filter banks. Daubechies and Sweldens [4] show that any two-band FIR filter bank can be factored in a set of lifting steps, using Euclidean algorithm. Associated polyphase matrix is factored in a cascade of triangular sub-matrices. An inverse sub-matrix is obtained by a simple transposition followed by the change in sign.

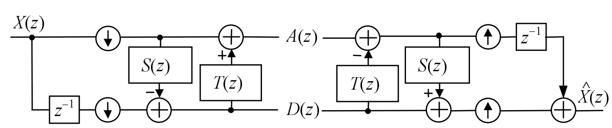

Figure 1. Two-channel PR filter bank factored in lifting and dual lifting steps.

The polyphase matrix of the filter bank from **Figure 1** is factored in 2 triangular sub-matrices:

$$\mathbf{P}(z) = \begin{bmatrix} 1 & 0 \\ T(z) & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -S(z) \\ 0 & 1 \end{bmatrix}.$$

$$H(z) = z^{-1} - 1 \cdot S(z^2), \tag{II.1}$$

$$L(z) = 1 + H(z) \cdot T(z^2). \tag{II.2}$$

We limit to FIR lifting steps $S(z)$ and $T(z)$, with $2N$ taps:

$$S(z) = s_0 z^{N-1} + ... + s_{N-1} + s_N z^{-1} + ... + s_{2N-1} z^{-N}, \tag{II.3}$$

$$T(z) = t_0 z^N + ... + t_{N-1} z + t_N + ... + t_{2N-1} z^{-N+1}. \tag{II.4}$$

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In this paper, a class of two-channel biorthogonal filter banks constructed by Lagrange interpolation method is used. Sweldens [3] described a lifting scheme construction of Deslauriers - Dubuc filter banks [5] by interpolation of samples in the time domain. The illustration of linear (II) and cubic (IV) case is given below:

$$d_{\mathrm{II}}[k]=x_o[k]-\frac{x_e[k-1]+x_e[k]}{2}$$

$$d_{\mathrm{IV}}[k]=x_o[k]-\frac{1}{16}(-x_e[k-2]+9x_e[k-1]+9x_e[k]-x_e[k+1])$$

$$S_{\mathrm{II}}=\frac{1}{2}(1+z^{-1}) \qquad S_{\mathrm{IV}}=\frac{1}{16}(-z+9+9z^{-1}-z^{-2})$$

We use S and T filters of the Lagrange interpolation type with variable even number of zero moments.

B. Adaptive lifting step

We start from the filter bank structure given in **Figure 1** and equations (II.1) and (II.3). For the simplicity, we limit ourselves to the linear phase prediction filters with 6-taps (N=3). Two zero moments of the high-pass filter are equivalent to the requirements:

$$H(z)\big|_{z=1}=0, \qquad H'(z)\big|_{z=1}=0.$$

These conditions decrease the freedom of choice of the prediction filter coefficients $\{s_k\}$, and lead to equations:

$$\sum_{k=0}^{5} s_k=1, \qquad \sum_{k=0}^{5}(2-k)\,s_k=-\frac{1}{2}. \qquad \text{(II.5)}$$

The linear phase condition requires $s_5=s_0$, $s_4=s_1$, $s_3=s_2$, so we have:

$$\sum_{k=0}^{2} s_k=\frac{1}{2}, \qquad \sum_{k=0}^{2} s_k=\frac{1}{2}. \qquad \text{(II.6)}$$

Both conditions (II.5) reduce to the same expression (II.6). Hence:

$$s_2=\frac{1}{2}-\sum_{k=0}^{1} s_k .$$

Now, we split the prediction filter into two additive components: fixed and “free” part: $S_{\mathrm{II}}=S_0+S_{\mathrm{free}}$, where

$$S_0(z)=\frac{1}{2}+\frac{1}{2}z^{-1},$$

$$S_{free}(z)=s_0z^2+s_1z-(s_0+s_1)-$$
$$-(s_0+s_1)z^{-1}+s_1z^{-2}+s_0z^{-3}.$$

The next pair of zero moments additionally reduces freedom in choice of the prediction filter coefficients:

$$H''(z)\big|_{z=1}=0, \qquad H'''(z)\big|_{z=1}=0,$$

as well as the last pair. The results are presented in **Table II.1**. Prediction filter is given as a sum of additive components: $S_{\mathrm{II}}=S_0$ for 2 zero moments, $S_{\mathrm{IV}}=S_0+S_1$ for 4, and $S_{\mathrm{VI}}=S_0+S_1+S_2$ for 6 zero moments.

Zeros	Fixed coefficients of the prediction filter	Additive component of the predictors fixed part
2	$s_2=\frac{1}{2}-(s_0+s_1)$	$S_0=\frac{1}{2}(1+z^{-1})$
4	$s_1=-\frac{1}{16}-3s_0$	$S_1=-\frac{1}{16}(z-1-z^{-1}+z^{-2})$
6	$s_0=\frac{3}{256}$	$S_2=\frac{3}{256}(z^2-3z+2+2z^{-1}-3z^{-2}+z^{-3})$

Table II.1. Additive components of the 6 tap linear phase prediction filter ($s_5=s_0$, $s_4=s_1$, $s_3=s_2$) providing 2, 4 or 6 zero moments to the high-pass filter $H(z)$.

Furthermore, additive components can be factored and realized in a cascade:

$$S_0=\frac{1}{2}(1+z^{-1})$$

$$S_1=-\frac{1}{16}(1+z^{-1})z^{-1}(1-z)^2=-\frac{1}{8}z^{-1}(1-z)^2\cdot S_0$$

$$S_2=\frac{3}{256}(1+z^{-1})[z^{-1}(1-z)^2]^2=\frac{3}{16}z^{-1}(1-z)^2\cdot(-S_1)$$

Finally, the proposed realization of the lifting step is shown in **Figure 2**. Successive closing of switches S_{w2}, S_{w4} and S_{w6} gives 2, 4 or 6 zero moments of the high-pass filter, respectively. It corresponds to the prediction of odd samples from neighboring even samples, using linear, cubic or 5th order polynomial interpolation.

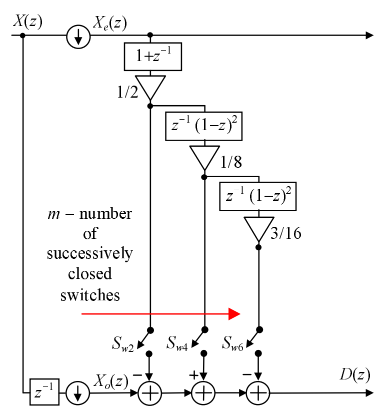

Figure 2. Lifting step of the high-pass filter with 0, 2, 4 or 6 zero moments: *bior*(2×*m*).0. More zero moments corresponds to higher order Lagrange polynomial interpolation.

If needed, the proposed adaptive realization can be easily extended to more zero moments.

C. Adaptive dual lifting step

Let the high-pass filter have at least 2 vanishing moments. From **Figure 1**, equations (II.2), (II.4) and N=3 we have following expressions for the first 2 zero moments of the low-pass filter:

$$L(z)\big|_{z=-1}=0, \qquad L'(z)\big|_{z=-1}=0;$$

$$\sum_{k=0}^{5} t_k=\frac{1}{2}, \qquad \sum_{k=0}^{5}(3-k)\,t_k=\frac{1}{4}. \qquad \text{(II.7)}$$

Again, the linear phase condition reduces expressions (II.7) to $\Sigma_{k=0,2}\, t_k = 1/4$. Now, we express t_3 from the rest of update filter coefficients. Then, we repeat the procedure for 4 and 6 zero moments of the low-pass filter, times all switch positions $m \in \{1,2,3\}$ of the prediction filter. The factored results are presented in **Table II.2**. Gain constants A_i depend on switch positions of the prediction filter.

$$T_0 = A_0 \frac{1}{4}(1+z)\ ,$$

$$T_1 = -\frac{A_1}{32}(1+z)z^{-1}(1-z)^2 = -\frac{A_1}{8} z^{-1}(1-z)^2 \cdot (\frac{S_0}{A_0})\ ,$$

$$T_2 = \frac{A_2 \cdot 3}{512}(1+z)[z^{-1}(1-z)^2]^2 = \frac{A_2 \cdot 3}{16} z^{-1}(1-z)^2 \cdot (-\frac{S_1}{A_1})\ ,$$

Table II.2. Additive components of the 6 tap update filter with linear phase (t_5=t_0, t_4=t_1, t_3=t_2) providing 2, 4 or 6 zero moments to the low-pass filter $L(z)$.

Gain constants A_i are shown in the following table:

HP zeros →	2	4	6
A_0	**1**	**1**	**1**
A_1	3/2	**1**	**1**
A_2	5/3	3/2	**1**

Table II.3. Gain A_i depends on the actual number of zero moments of the high-pass filter, unless $n < m$.

An interesting conclusion comes from **Table II.3**. If the number of zero moments of the LP filter is less or equal to the number of zero moments of the HP filter, A_i equals 1 for all $i \in \{1,2,3\}$. Hence, if the number of closed switches in the update filter does not exceed the number of closed switches in the prediction filter, we have "independent" lifting and dual lifting switching networks.

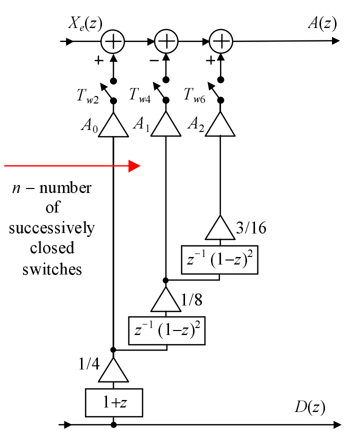

Figure 3. Dual lifting step of the low-pass filter with 0, 2, 4 or 6 zero moments: $bior(2{\times}m).(2{\times}n)$. If $n \le m$, A_0=A_1=A_2=1.

III. ADAPTATION CRITERIONS AND RESULTS

At first, we will adapt the filters in order to minimize the absolute error of the prediction. We analyze a synthetic signal that is concatenated from 3 polynomials of different orders. Prediction error signal is calculated as the absolute value of the wavelet coefficients $d[k]$. The number of successively closed switches m is chosen to give minimum $|d[k]|$ at each step of decomposition.

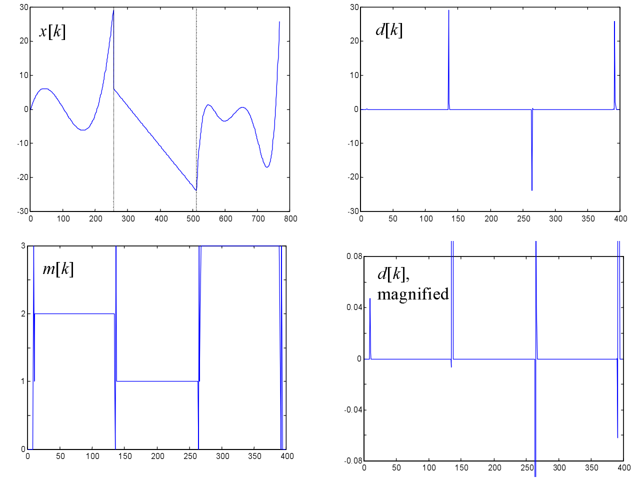

Figure 4. Top left: analyzed signal $x[k]$, composed from 3 polynomial sections of different order {3,1,5}. Top right: wavelet coefficients $d[k]$ computed by adaptive filter bank. Bottom left: number of closed switches $m[k]$. Bottom right: magnified detail $d[k]$.

We see that the adapted prediction order exactly follows the order of polynomials. Also, we notice that the prediction order on the interval boundaries is low, thus reducing the spread of the wavelet coefficients. Number of closed switches is exactly zero on discontinuities, where the filter bank degenerates to the polyphase decomposition. The decomposition is almost optimal, so the majority of the wavelet coefficients $d[k]$ is equal to zero.

The dual lifting step adapts in the narrowed range $n \in (0,m)$. Signal DC should be preserved in the approximation coefficients $a[k]$. Hence, the error signal is computed from the high-pass filtered coefficients $a[k]$:

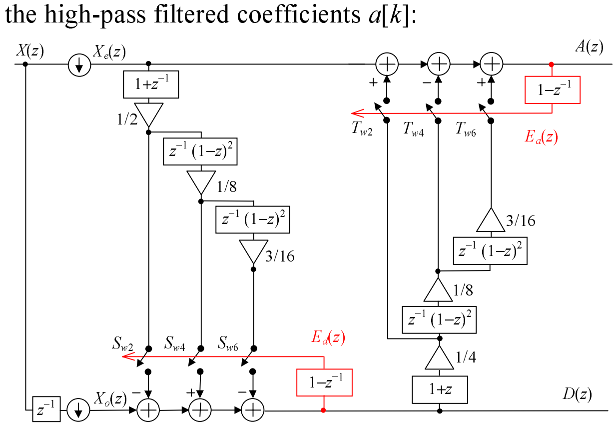

Figure 5. Wavelet filter bank with adaptive number of zero moments $bior(2{\times}m).(2{\times}n)$, $n \le m$.

Due to decimation, aliasing frequency of the analyzed signal $x[k]$ maps to the DC component of the wavelet coefficients $d[k]$. To avoid its influence on the adaptation criterion, we may use similar high-pass filtered scheme (**Figure 5**) for the lifting step, too.

The approximation coefficients $a[k]$ are very close to decimated version of the analyzed signal $x[k]$ (**Figure 6**).

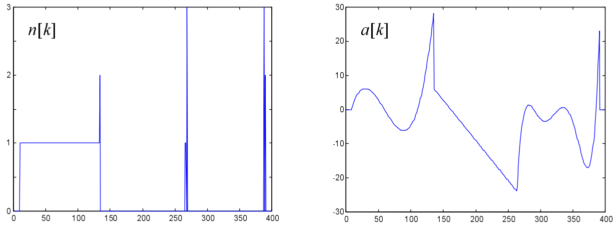

Figure 6. Left: number of closed switches $n[k]$. Right: approximation coefficients $a[k]$.

Now we introduce a real world signal – a short sequence of human speech. The spoken word consists from a consonant followed by a vowel (Croatian word “da” – means “yes”).

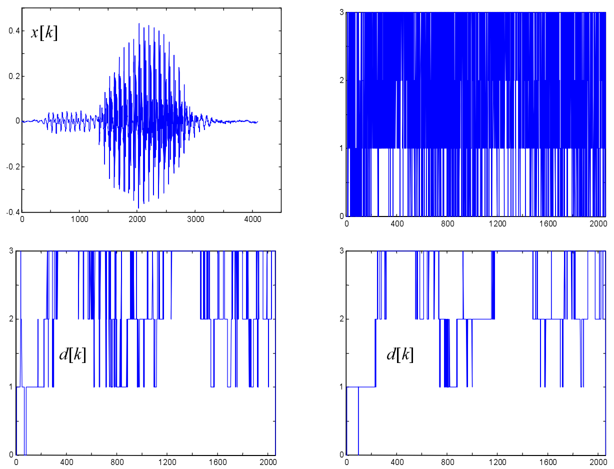

Figure 7. Top left: analyzed signal $x[k]$: word “da”. Top right: number of closed switches $m[k]$. Bottom: $m[k]$ adapted on interval. Left: $[k–15, k+15]$. Right: $[k–30, k+30]$.

Memoryless adaptation criterion causes intensive variations of switches’ positions $m[k]$ (**Figure 7**, top right). Unpredictable components of the real world signal x[k] make the adaptation algorithm switches too fast. Entropy of wavelet coefficients $d[k]$ increases when compared to the fixed wavelet analysis, which is exactly the opposite of our intention. Because of that, we use an adaptation criterion defined on interval $[k–K_1, k+K_2]$. The number of successively closed switches m is chosen to give the minimum value of $err[k]$, which is computed using expression

$$err[k] = \sum_{i=k-K_1}^{k+K_1} |e[i]|,$$

at each step of decomposition.

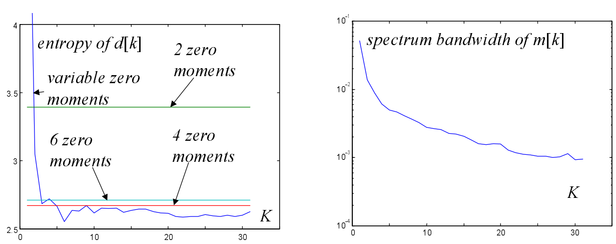

Figure 8. Left: Shannon entropy of the wavelet coefficients $d[k]$, compared to fixed wavelets. Right: Spectrum bandwidth of $m[k]$. Both variables depend on width of the adaptation interval: K_1=K_2=K, width = 2K.

Now, the adaptation is slower and more accurate (**Figure 7**, bottom left and right). The entropy of coefficients $d[k]$ computed by adaptive filter bank is better then of the fixed banks.

We introduce the spectrum bandwidth to measure the intensity of the switches’ variations $m[k]$: $\int \omega^2 |M(\omega)|^2 d\omega / \int |M(\omega)|^2 d\omega$. Clearly, switch variations are slower for wide adaptation intervals (**Figure 8**). The decision on the adaptation interval width is usually a trade-off between computational complexity and desirable properties of the criterion.

In general, we can reconstruct the analyzed signal from wavelet coefficients plus information on switch positions $m[k]$ and $n[k]$. They can be coded very efficiently. But, if the adaptation criterion is **causal**, e.g. if the current switch positions are determined exclusively from **previous** wavelet coefficients ($-K_1 \le K_2 < 0$), the adaptation algorithm can be reproduced on the reconstruction side. In that case, perfect reconstruction does not require m and n to be separately transferred to the reconstruction side.

IV. Conclusion

We describe an efficient realization of the two-channel wavelet filter bank with adaptive number of zero moments. A set of switches determines the desired number of zero moments at each step of decomposition or reconstruction. We used the least absolute error criterion, computed from filtered wavelet coefficients. Adaptive filter bank is applied on a synthetic and a real-world signal. Wavelet coefficients get close to what we expect to be an optimal representation of the analyzed signal. Real world signals usually contain non-correlated components, inherent to the observed phenomenon or caused by additive noise. They cause intensive variance of the filter bank switches. Adaptation on interval decreases the variance. Described time variant wavelet filter bank is more suitable for analysis of non-stationary signals then fixed banks. It gives lower entropy of the wavelet coefficients.

V. Acknowledgment

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VI. References

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