Adaptation Criteria for Wavelet Filter Banks
With Variable Zero Moments

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Abstract—In this paper, we compare different criteria of
adaptation in the proposed two-channel wavelet filter bank
with variable number of zero moments. Generally, filters with
more zero moments are more appropriate for representing
smooth parts of the analyzed signal, while shorter filters are
better for transients and singularities. Depending on the
criterion, filter banks that change number of zero moments at
each step of decomposition can outperform fixed banks in a
number of applications. When applied to signals, variable zero
moment wavelets result in lower ripple on edges and good
concentration of wavelet coefficients in smooth parts. Error
signal is formed from filtered wavelet coefficients and
therefore available on the reconstruction side. We discuss
different adaptation algorithms based on spectral properties,
adaptation interval and causality. Chosen criterion determines
decomposition properties, as well as its reproducibility on the
reconstruction side. Results on synthetic and real signals are
presented.

Index Terms—adaptation criteria, variable zero moments,
wavelets, time-varying filter banks

I. INTRODUCTION

The number of vanishing moments of a fixed wavelet filter
bank is usually chosen as a compromise between filter
complexity and concentration of the wavelet coefficients.
More zero moments correspond to more regularity, which
gives better description of smooth and correlated parts of
the analyzed signal [1][2]. But, it results in longer filters
that cause spread of wavelet coefficients on sharp edges of
the analyzed signal. On the other hand, shorter filters are
more suitable for compact representation of transients and
singularities, as well as parts of the signal with narrower
correlation of samples.

In this paper we describe a wavelet filter bank that adapts
the number of zero moments. Adaptation is performed on
both filters in the bank at each step of decomposition. The
adaptation criterion is computed from wavelet coefficients.
In some cases it is reproducible on the reconstruction side.
We expect benefits of using adaptive number of zero
moments in many applications, such as [6].

In section II we describe the construction of the proposed
adaptive filter bank. Sweldens 96 [3] proposed a
construction of biorthogonal wavelet filter banks based on
the lifting scheme, using interpolation of samples in the
time domain. A short review is given in paragraph II.A.

Even samples are estimated from odd samples using Lagrange
interpolation functions of chosen order. In the proposed
scheme, we consider odd order Lagrange polynomials
Corresponding to the even length FIR filters.

In paragraph II.B we give the proposed factorization of the
adjustable lifting step. In paragraph II.C adjustable dual
lifting step is introduced.

In section III we discuss the adaptation criteria and
results. Proposed filter bank is applied on a synthetic and a
real-world signal. Decomposition of synthetic signal is
almost optimal. Unpredictable components of the real-
world signals cause modification of memoryless adaptation
criterion. It should be computed on interval. Depending on
the choice of interval boundaries, algorithm is reproducible
or not reproducible on the reconstruction side. It is shown
that the entropy of the wavelet coefficients computed by
adaptive filter bank is lower when compared to fixed banks.

II. FILTER BANK STRUCTURE

A. Lifting scheme

Lifting scheme enables easy construction of perfect
reconstruction time-variant and non-linear filter banks.
Daubechies and Sweldens [4] show that any two-band FIR
filter bank can be factored in a set of lifting steps, using
Euclidean algorithm. Associated polyphase matrix is
factored in a cascade of triangular sub-matrices. An inverse
sub-matrix is obtained by a simple transposition followed
by the change in sign.

![Two-channel PR filter bank factored in lifting and dual lifting steps.](image)

Figure 1. Two-channel PR filter bank factored in lifting and dual lifting steps.

The polyphase matrix of the filter bank from Figure 1 is
factored in 2 triangular sub-matrices:

\[
P(z) = \begin{bmatrix}
1 & 0 \\
T(z) & 1
\end{bmatrix} \begin{bmatrix}
1 - S(z) \\
0 & 1
\end{bmatrix}.
\]

\[
H(z) = z^{-1} - 1 - S(z^2),
\]

\[
\hat{H}(z) = 1 + H(z) \cdot T(z^2).
\]

We limit to FIR lifting steps \(S(z)\) and \(T(z)\), with 2\(N\) taps:

\[
S(z) = s_0 z^{-N} + s_1 z^{-N+1} + \ldots + s_{N-1} z^{-1} + s_N = 1.
\]

\[
T(z) = t_0 z^N + t_1 z^{N-1} + \ldots + t_{N-2} z^2 + t_{N-1} z + t_N = 1.
\]
In this paper, a class of two-channel biorthogonal filter banks constructed by Lagrange interpolation method is used. Sweldens [3] described a lifting scheme construction of Deslauriers - Dubuc filter banks [5] by interpolation of samples in the time domain. The illustration of linear (II) and cubic (IV) case is given below:

\[ d_{II}[k] = x_v[k] - \frac{x_c[k - 1] + x_c[k]}{2} \]
\[ d_{IV}[k] = x_v[k] - \frac{1}{16}(x_v[k - 2] + 9x_v[k - 1] + 8x_v[k] - x_v[k + 1]) \]
\[ S_{II} = \frac{1}{2}(1 + z^{-1}) \quad S_{IV} = \frac{1}{16}(-z + 9 + 9z^{-1} - z^{-2}) \]

We use \( S \) and \( T \) filters of the Lagrange interpolation type with variable even number of zero moments.

B. Adaptive lifting step

We start from the filter bank structure given in Figure 1 and equations (II.1) and (II.3). For the simplicity, we limit ourselves to the linear phase prediction filters with 6-taps (\( N=3 \)). Two zero moments of the high-pass filter are equivalent to the requirements:

\[ H(z)|_{z=1} = 0, \quad H'(z)|_{z=1} = 0. \]

These conditions decrease the freedom of choice of the prediction filter coefficients \( \{s_k\} \), and lead to equations:

\[ \sum_{k=0}^{5} s_k = 1, \quad \sum_{k=0}^{5} (2 - k) s_k = -\frac{1}{2}. \] (II.5)

The linear phase condition requires \( s_5 = s_0, \ s_4 = s_1, \ s_3 = s_2 \), so we have:

\[ \sum_{k=0}^{2} s_k = \frac{1}{2}, \quad \sum_{k=0}^{2} k s_k = \frac{1}{2}. \] (II.6)

Both conditions (II.5) reduce to the same expression (II.6). Hence:

\[ s_2 = \frac{1}{2} - \frac{1}{2} \sum_{k=0}^{2} s_k. \]

Now, we split the prediction filter into two additive components: fixed and “free” part: \( S_{II} = S_{f} + S_{free} \), where

\[ S_{f}(z) = \frac{1}{2} + \frac{1}{2} z^{-1}, \]
\[ S_{free}(z) = s_0 z^2 + s_1 z - (s_0 + s_1) - (s_0 + s_1) z^{-1} + s_2 z^{-2} + s_0 z^{-3}. \]

The next pair of zero moments additionally reduces freedom in choice of the prediction filter coefficients:

\[ H''(z)|_{z=1} = 0, \quad H''(z)|_{z=1} = 0, \]

as well as the last pair. The results are presented in Table II.1. Prediction filter is given as a sum of additive components: \( S_{II} = S_0 \) for 2 zero moments, \( S_{IV} = S_0 + S_1 \) for 4, and \( S_0 = S_0 + S_1 + S_2 \) for 6 zero moments.

<table>
<thead>
<tr>
<th>Zero</th>
<th>Fixed coefficients of the prediction filter</th>
<th>Additive component of the predictors fixed part</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( s_2 = \frac{1}{2} (s_0 + s_1) )</td>
<td>( S_0 = \frac{1}{2} (1 + z^{-1}) )</td>
</tr>
<tr>
<td>4</td>
<td>( s_1 = \frac{1}{16} - \frac{3}{16} s_0 )</td>
<td>( S_1 = \frac{1}{16} (z^{-1} - 1 + z^{-1} + z^{-2}) )</td>
</tr>
<tr>
<td>6</td>
<td>( s_0 = \frac{3}{256} )</td>
<td>( S_2 = \frac{3}{256} (z^{-2} - 3z + 2 + 2z^{-1} - 3z^{-2} + z^{-3}) )</td>
</tr>
</tbody>
</table>

Table II.1 Additive components of the 6 tap linear phase prediction filter (\( s_0 = s_0, \ s_1 = s_1, \ s_2 = s_2 \)) providing 2, 4 or 6 zero moments to the high-pass filter \( H(z) \).

Furthermore, additive components can be factored and realized in a cascade:

\[ S_0 = \frac{1}{2} (1 + z^{-1}) \]
\[ S_1 = -\frac{1}{16} (1 + z^{-1}) z^{-1} (1 - z)^2 = -\frac{1}{8} z^{-1} (1 - z)^2 \cdot S_0 \]
\[ S_2 = \frac{3}{256} (1 + z^{-1}) z^{-1} (1 - z)^2 = \frac{3}{16} z^{-1} (1 - z)^2 \cdot (S_1) \]

Finally, the proposed realization of the lifting step is shown in Figure 2. Successive closing of switches \( S_{k_2}, S_{k_4} \) and \( S_{k_6} \) gives 2, 4 or 6 zero moments of the high-pass filter, respectively. It corresponds to the prediction of odd samples from neighboring even samples, using linear, cubic or 5th order polynomial interpolation.

![Figure 2](image_url)

Figure 2. Lifting step of the high-pass filter with 0, 2, 4 or 6 zero moments: \( bior(2 \times m) \), \( 0 \). More zero moments corresponds to higher order Lagrange polynomial interpolation.

If needed, the proposed adaptive realization can be easily extended to more zero moments.

C. Adaptive dual lifting step

Let the high-pass filter have at least 2 vanishing moments. From Figure 1, equations (II.2), (II.4) and \( N=3 \) we have following expressions for the first 2 zero moments of the low-pass filter:

\[ L(z)|_{z=1} = 0, \quad L'(z)|_{z=1} = 0; \]
\[ \sum_{k=0}^{5} t_k = \frac{1}{2}, \quad \sum_{k=0}^{5} (3 - k) t_k = \frac{1}{4}. \] (II.7)
Again, the linear phase condition reduces expressions (II.7) to $\sum_{i=0}^{2} t_i = 1/4$. Now, we express $t_i$ from the rest of update filter coefficients. Then, we repeat the procedure for 4 and 6 zero moments of the low-pass filter, times all switch positions $m \in \{1,2,3\}$ of the prediction filter. The factorized results are presented in Table II.2. Gain constants $A_i$ depend on switch positions of the prediction filter.

$$T_0 = A_0 \frac{1}{4}(1 + z),$$

$$T_1 = \frac{A_1}{32}(1 + z)^2 - \frac{1}{8} z^{-1}(1 - z)^2 \left( \frac{S_0}{A_0} \right),$$

$$T_2 = \frac{A_2 - 3}{512} (1 + z)^3 \left( 1 - z^{-1} \right)^2 - \frac{A_2 - 3}{16} z^{-1}(1 - z)^2 \left( \frac{-S_1}{A_1} \right).$$

Table II.2. Additive components of the 6 tap update filter with linear phase ($t_5-t_6$, $t_4-t_5$, $t_3-t_4$) providing 2, 4 or 6 zero moments to the low-pass filter $H(z)$.

<table>
<thead>
<tr>
<th>Gain constants $A_i$ are shown in the following table:</th>
</tr>
</thead>
<tbody>
<tr>
<td>HP zeros $\rightarrow$</td>
</tr>
<tr>
<td>$A_0$</td>
</tr>
<tr>
<td>$A_1$</td>
</tr>
<tr>
<td>$A_2$</td>
</tr>
</tbody>
</table>

Table II.3. Gain $A_i$ depends on the actual number of zero moments of the high-pass filter, unless $n < m$.

An interesting conclusion comes from Table II.3. If the number of zero moments of the LP filter is less or equal to the number of zero moments of the HP filter, $A_i$ equals 1 for all $i \in \{1,2,3\}$. Hence, if the number of closed switches in the update filter does not exceed the number of closed switches in the prediction filter, we have “independent” lifting and dual lifting switching networks.

**Figure 3.** Dual lifting step of the low-pass filter with 0, 2, 4 or 6 zero moments: $bi or(2 \times m), (2 \times n)$. If $n \leq m$, $A_i = A_i - A_i - 1$.

**Figure 4.** Top left: analyzed signal $x[k]$, composed from 3 polynomial sections of different order $(3,1,5)$. Top right: wavelet coefficients $d[k]$ computed by adaptive filter bank. Bottom left: number of closed switches $m[k]$. Bottom right: magnified detail $d[k]$.

We see that the adapted prediction order exactly follows the order of polynomials. Also, we notice that the prediction order on the interval boundaries is low, thus reducing the spread of the wavelet coefficients. Number of closed switches is exactly zero on discontinuities, where the filter bank degenerates to the polyphase decomposition. The decomposition is almost optimal, so the majority of the wavelet coefficients $d[k]$ is equal to zero.

The dual lifting step adapts in the narrowed range $n \in (0,m)$. Signal DC should be preserved in the approximation coefficients $a[k]$. Hence, the error signal is computed from the high-pass filtered coefficients $d[k]$.

**Figure 5.** Wavelet filter bank with adaptive number of zero moments $bi or(2 \times m), (2 \times n), n \leq m$.

Due to decimation, aliasing frequency of the analyzed signal $x[k]$ maps to the DC component of the wavelet coefficients $d[k]$. To avoid its influence on the adaptation criterion, we may use similar high-pass filtered scheme (Figure 5) for the lifting step, too.

The approximation coefficients $a[k]$ are very close to decimated version of the analyzed signal $x[k]$ (Figure 6).
Now we introduce a real world signal—a short sequence of human speech. The spoken word consists from a consonant followed by a vowel (Croatian word “da”—means “yes”).

Memoryless adaptation criterion causes intensive variations of switches’ positions \( m[k] \) (Figure 7, top right). Unpredictable components of the real world signal \( x[k] \) make the adaptation algorithm switches too fast. Entropy of wavelet coefficients \( d[k] \) increases when compared to the fixed wavelet analysis, which is exactly the opposite of our intention. Because of that, we use an adaptation criterion defined on interval \( [k-K_1, k+K_2] \). The number of successively closed switches \( m \) is chosen to give the minimum value of \( \text{err}[k] \), which is computed using expression

\[
\text{err}[k] = \sum_{j=-K_1}^{K_2} |d[j]|.
\]

at each step of decomposition.

Now, the adaptation is slower and more accurate (Figure 7, bottom left and right). The entropy of coefficients \( d[k] \) computed by adaptive filter bank is better then of the fixed banks.

We introduce the spectrum bandwidth to measure the intensity of the switches’ variations \( m[k] \):

\[
\omega_s^2 |\mathcal{M}(\omega)|^2 d\omega / |\mathcal{M}(\omega)|^2 d\omega
\]

Clearly, switch variations are slower for wide adaptation intervals (Figure 8). The decision on the adaptation interval width is usually a trade-off between computational complexity and desirable properties of the criterion.

In general, we can reconstruct the analyzed signal from wavelet coefficients plus information on switch positions \( m[k] \) and \( n[k] \). They can be coded very efficiently. But, if the adaptation criterion is causal, e.g. if the current switch positions are determined exclusively from previous wavelet coefficients \((-K_1 \leq K_2 < 0)\), the adaptation algorithm can be reproduced on the reconstruction side. In that case, perfect reconstruction does not require \( m \) and \( n \) to be separately transferred to the reconstruction side.

IV. CONCLUSION

We describe an efficient realization of the two-channel wavelet filter bank with adaptive number of zero moments. A set of switches determines the desired number of zero moments at each step of decomposition or reconstruction.

We used the least absolute error criterion, computed from filtered wavelet coefficients. Adaptive filter bank is applied on a synthetic and a real-world signal. Wavelet coefficients get close to what we expect to be an optimal representation of the analyzed signal. Real world signals usually contain non-correlated components, inherent to the observed phenomenon or caused by additive noise. They cause intensive variance of the filter bank switches. Adaptation on interval decreases the variance. Described time variant wavelet filter bank is more suitable for analysis of non-stationary signals then fixed banks. It gives lower entropy of the wavelet coefficients.

V. ACKNOWLEDGMENT

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VI. REFERENCES


