INTEGER TO INTEGER MAPPING WAVELET FILTER BANK WITH ADAPTIVE NUMBER OF ZERO MOMENTS

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ABSTRACT

An efficient realization of two-channel wavelet filter bank that maps integers to integers with adaptive number of zero moments is presented. Filters with more zero moments result in better representation of smooth parts of the analyzed signal, while less zero moments is better for transients and singularities. Proposed realization is based on the lifting scheme that enables mapping integer signals to integer wavelet coefficients, preserving perfect reconstruction property. The realization is derived from a method of fixed wavelet filter bank design, using Lagrange interpolation of samples in time domain. Adaptation criterion is computed from integer wavelet coefficients, which is under some restrictions reproducible on the reconstruction side. Quantization introduces non-predictable components of the wavelet coefficients, thus influencing behavior of the adaptation algorithm. Adaptation on interval is used to reduce variance of the filter parameters.

1. INTRODUCTION

The number of zero moments of a fixed filter bank is usually chosen as a trade-off between filter complexity and decomposition properties. More vanishing moments correspond to more regularity, which results in a compact representation of smooth and correlated parts of the analyzed signal [1]. But, longer filters cause ripple effects on discontinuities, where shorter filters are more suitable. We want to change the number of zero moments on both filters in the bank at each step of decomposition. Moreover, our goal is to map integer signals to integer wavelet coefficients, which have important applications in lossless coding [2]. The two-channel PR filter bank should form wavelet tree or wavelet packet tree, so the convergence and some degree of regularity of the limit wavelet functions and scales must remain. The adaptation criterion is computed from the wavelet coefficients, in purpose to achieve more compact representation of the analyzed signal. Such time-variant decompositions are not wavelets in a strict sense, rather a generalized wavelet construct.

Sweldens 96 [3] constructed biorthogonal wavelet filter banks based on the lifting scheme. Odd samples are estimated from evens using Lagrange interpolation polynomials of chosen order. In section 2 we give the construction of the proposed adaptive filter bank. We use variable odd order interpolating polynomials corresponding to the even length FIR filters.

In section 3 we discuss the adaptation criterion. To ensure the convergence and minimum regularity, filters are split in a fixed and a variable part. Number of zero moments is adjusted to minimize the prediction error. Quantization of the wavelet coefficients results in non-predictable components of the signal, causing intensive variance of filter parameters. Averaging on interval is used to reduce the variance.

2. ADAPTIVE FILTER BANK STRUCTURE

2.1 Lifting scheme

Daubechies and Sweldens 98 [4] show that any two-band FIR filter bank can be factored in a set of lifting steps, using Euclidean algorithm. Polyphase matrix is factored in a cascade of triangular sub-matrices, where each sub-matrix corresponds to a lifting or a dual lifting step. Its all-ones diagonal form guaranties existence of the inverse sub-matrix.

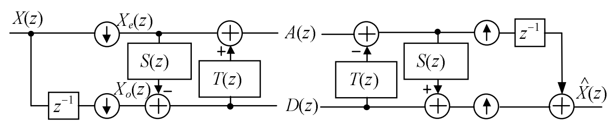

Figure 1. Two-channel PR filter bank factored in lifting and dual lifting steps.

Polyphase matrix Π of the filter bank from **Figure 1** is factored in 2 triangular sub-matrices, prediction (**P**) and update step (**U**):

$$\mathbf{\Pi}(z) = \mathbf{U}(z) \cdot \mathbf{P}(z) = \begin{bmatrix} 1 & 0 \\ T(z) & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -S(z) \\ 0 & 1 \end{bmatrix}.$$

Determinant of sub-matrices **P** and **U** is always 1, for arbitrary transfer functions S and T. Inverse sub-matrix is obtained by a simple transposition followed by the change in sign, which enables easy construction of time-variant and non-linear PR filter banks.

$$H(z) = z^{-1} - 1 \cdot S(z^2)\,, \tag{2.1}$$

$$L(z) = 1 + H(z) \cdot T(z^2)\,. \tag{2.2}$$

We limit to FIR lifting steps $S(z)$ and $T(z)$, with $2N$ taps:

$$S(z) = s_0 z^{N-1} + \ldots + s_{N-1} + s_N z^{-1} + \ldots + s_{2N-1} z^{-N}\,, \tag{2.3}$$

$$T(z) = t_0 z^{N} + \ldots + s_{N-1} z + s_N + \ldots + s_{2N-1} z^{-N+1}\,, \tag{2.4}$$

Prediction of odd samples from neighboring even using linear (II) and cubic (IV) interpolation polynomials of samples is illustrated below:

$$d_{\mathrm{II}}[k] = x_o[k] - \frac{x_e[k-1] + x_e[k]}{2}$$

$$d_{\mathrm{IV}}[k] = x_o[k] - \frac{1}{16}(-x_e[k-2] + 9x_e[k-1] + 9x_e[k] - x_e[k+1])$$

$$S_{\mathrm{II}} = \frac{1}{2}(1 + z^{-1}) \qquad S_{\mathrm{IV}} = \frac{1}{16}(-z + 9 + 9z^{-1} - z^{-2})$$

We use prediction filters of the Lagrange interpolation type, with even number of zero moments that change in time.

2.2 Lifting step with variable number of zero moments

We start from the filter bank structure given in **Figure 1** and equations (2.1) and (2.3). Two zero moments of the high-pass filter require $H(z)|_{z=1}=0$ and $H^{'}(z)|_{z=1}=0$, which is equivalent to conditions $\Sigma_k s_k=1$ and $\Sigma_k(N-1-k)s_k=-1/2$. We express central filter parameters from the outer ones: $s_N=1/2+\Sigma_{k\neq N}(N-1-k)s_k$, and $s_{N-1}=1/2-\Sigma_k(N-1-k)s_k-\Sigma_{k\neq N-1}s_k$. Now, we split the prediction filter in two additive components: fixed and "free" part: $S_{II}=S_0+S_{free}$, where $S_0(z)=(1+z^{-1})/2$. The same procedure is repeated for the next pair of zero moments: $H^{''}(z)|_{z=1}=0$ and $H^{'''}(z)|_{z=1}=0$, etc. For the simplicity, we limit ourselves to the linear phase prediction filters with 8-taps (N=4). The results are presented in **Table 2.1**.

Zeros	Central parameters of the prediction filter	Additive component of the fixed part
2	$s_3=\frac{1}{2}-(s_0+s_1+s_2)$	$S_0=\frac{1}{2}(1+z^{-1})$
4	$s_2=-\frac{1}{16}-(6s_0+3s_1)$	$S_1=-\frac{1}{16}(z-1-z^{-1}+z^{-2})$
6	$s_1=\frac{3}{256}-5s_0$	$S_2=\frac{3}{256}(z^2-3z+2+2z^{-1}-3z^{-2}+z^{-3})$
8	$s_0=-\frac{5}{2048}$	$S_3=-\frac{5}{2048}(z^3-5z^2+9z-5-5z^{-1}+9z^{-2}-5z^{-3}+z^{-4})$

Table 2.1. Additive components of the 8 tap linear phase prediction filter ($s_7=s_0$, $s_6=s_1$, $s_5=s_2$, $s_4=s_3$) providing 2, 4, 6 or 8 zero moments to the high-pass filter $H(z)$.

Prediction filter is always a sum of additive components: e.g. $S_{II}=S_0$ for 2 zero moments, $S_{IV}=S_0+S_1$ for 4, or $S_{VIII}=S_0+S_1+S_2+S_3$ for 8 zero moments.

Additive components can be factored and realized in a cascade:

$$S_0=\frac{1}{2}(1+z^{-1})$$

$$S_1=-\frac{1}{16}(1+z^{-1})z^{-1}(1-z)^2=-\frac{1}{8}(z^{-1}-2+z)\cdot S_0$$

$$S_2=\frac{3}{256}(1+z^{-1})[z^{-1}(1-z)^2]^2=\frac{3}{16}(z^{-1}-2+z)\cdot(-S_1)$$

$$S_3=-\frac{5}{2048}(1+z^{-1})[z^{-1}(1-z)^2]^3=-\frac{5}{24}(z^{-1}-2+z)\cdot S_2$$

Finally, the proposed realization of the lifting step is shown in **Figure 2**. Successive closing of switches S_{w2}, S_{w4}, S_{w6} and S_{w8} gives 2, 4, 6 or 8 zero moments of the high-pass filter, respectively. It corresponds to the prediction of odd samples from neighboring even samples, using linear, cubic, 5th or 7th order polynomial interpolation.

2.3 Dual lifting step with variable zero moments

Let the high-pass filter has at least 2 vanishing moments. From equations (2.2) and (2.4) two zero moments of the low-pass filter lead to conditions $L(z)|_{z=-1}=0$ and $L^{'}(z)|_{z=-1}=0$, or, in terms of filter parameters, $\Sigma_k t_k=1/2$ and $\Sigma_k(N-k)t_k=1/4$. Again, we express central filter parameters from the outer ones: $t_{N-1}=1/4-\Sigma_{k\neq N-1}(N-k)t_k$, and $t_N=1/4+\Sigma_k(N-k)t_k-\Sigma_{k\neq N}t_k$. Update filter is split in two additive components: fixed and "free" part T_0+T_{free}, where $T_0(z)=(z+1)/4$. We repeat the procedure for 4, 6 and 8 zero moments of the low-pass filter, times all switch positions $m\in\{1,2,3,4\}$ of the prediction filter. The factored results are presented in **Table 2.2**.

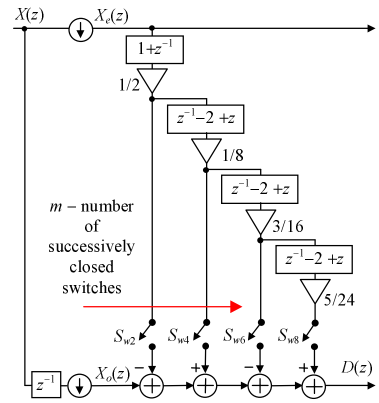

Figure 2. Lifting step of the high-pass filter with 0, 2, 4, 6 or 8 zero moments: *bior*(2×*m*).0.

$$T_0=A_0\frac{1}{4}(1+z),$$

$$T_1=-\frac{A_1}{32}(1+z)z^{-1}(1-z)^2=-\frac{A_1}{8}(z^{-1}-2+z)\cdot(\frac{S_0}{A_0}),$$

$$T_2=\frac{A_2\cdot 3}{512}(1+z)[z^{-1}(1-z)^2]^2=\frac{A_2\cdot 3}{16}(z^{-1}-2+z)\cdot(-\frac{S_1}{A_1}),$$

$$T_3=-\frac{A_3\cdot 5}{4096}(1+z)[z^{-1}(1-z)^2]^3=-\frac{A_3\cdot 5}{24}(z^{-1}-2+z)\cdot(\frac{S_2}{A_2}).$$

Table 2.2. Additive components of the 8 tap update filter with linear phase ($t_7=t_0$, $t_6=t_1$, $t_5=t_2$, $t_4=t_3$) providing 2, 4, 6 or 8 zero moments to the low-pass filter $L(z)$.

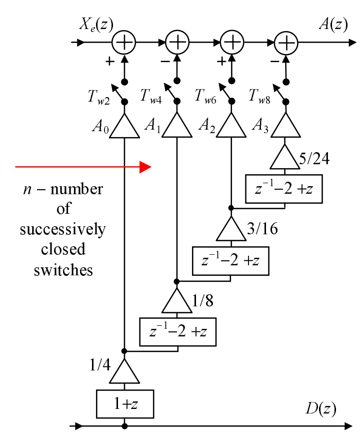

Figure 3. Dual lifting step of the low-pass filter with 0, 2, 4, 6 or 8 zero moments: *bior*(2×*m*).(2×*n*). If $n\leq m$, $A_0=A_1=A_2=A_3=1$.

Gain constants A_i depend on the number of zero moments of the HP filter, which is shown in the following table:

HP zeros →	2	4	6	8
A_0	**1**	**1**	**1**	**1**
A_1	3/2	**1**	**1**	**1**
A_2	5/3	3/2	**1**	**1**
A_3	7/4	3/2	3/2	**1**

Table 2.3. Gain A_i depends on the actual number of zero moments of the high-pass filter, unless $n < m$.

From **Table 2.3** we conclude that if the number of zero moments of the LP filter is less or equal to the number of zero moments of the HP filter, A_i equals 1 for all i=0–3. Hence, if $n \leq m$ – we have independent lifting and dual lifting switching networks. If we want more vanishing moments for the LP filter, we use the same FB structure with alternate signs of additive components. In that case LP and HP filter exchange their places.

3. Adaptive FB that maps integers to integers

To preserve convergence and minimum regularity of the generalized limit functions, as well as to split the FB in basic HP and LP channels, we fixed 2 vanishing moments on both filters. Thus, switches S_{w1} and T_{w1} are always closed. Of course, the bound between fixed and variable part of the filter bank can be arbitrary positioned.

We introduce quantizers to achieve mapping of integer signals to integer wavelet coefficients:

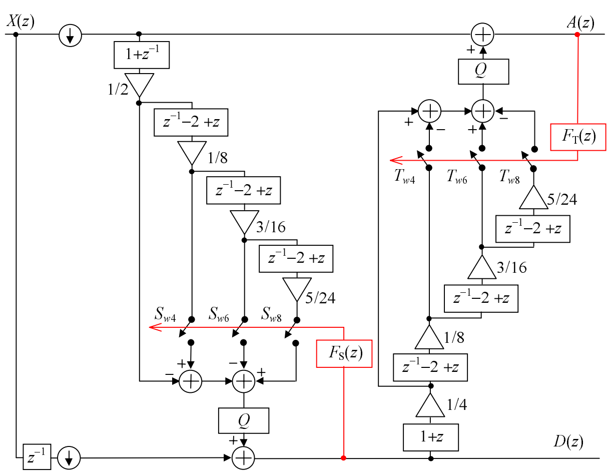

Figure 4. Wavelet filter bank that map integers to integers with adaptive number of zero moments $(2\times m).(2\times n)$, $n \leq m$.

Q can be any operator that maps real numbers to integers, either in uniform or non-equidistant way. Identical operator on the synthesis side ensures perfect reconstruction. We used $Q(u)=\text{int}(u+1/2)$.

In purpose to change the number of zero moments, we derived an error signal from wavelet coefficients $d[k]$ or $a[k]$. The adaptation criterion sets the switches to minimize the sum of absolute prediction errors, on interval $[k–K_1, k+K_2]$ around observed k.

$$e_d[k] = \sum_{n=k-K_1}^{k+K_1} |d[n]|, \qquad e_a[k] = \sum_{n=k-K_1}^{k+K_1} |a[n]|$$

Depending on interval bounds, we deal with memory-less (K_1=K_2=0), causal (K_1>0, $K_2 \leq -1$) or non-causal (e.g. K_1=K_2>0) adaptation criterion. In general, we can reconstruct the analyzed signal from wavelet coefficients plus information on switch positions $m[k]$ and $n[k]$. They can be coded very efficiently. But, if the adaptation criterion is causal, e.g. if the current switch positions are determined exclusively from **previous** wavelet coefficients, the adaptation algorithm can be reproduced on the reconstruction side. Then, m and n must not be separately transferred to the reconstruction side.

It is useful to exclude DC component from the wavelet coefficients: it originates either from aliasing or from DC component of the analyzed signal. To avoid its influence on the criterion, high-pass filtered wavelet coefficients may be used as the error input of the adaptation algorithm (**Figure 4**, where $F_S(z) = F_T(z) = 1–z^{-1}$).

We analyzed a test signal $x[k]$ composed from 4 polynomial sections of increasing order (1,3,5,7), uniformly quantized in 16 bits (**Figure 5**).

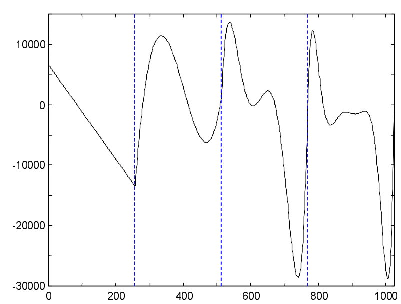

Figure 5. Analyzed 16 bit integer signal $x[k]$, composed from 4 polynomials of increasing order (1,3,5,7).

Figure 6 shows details of signal $x[k]$ computed by fixed integer wavelets with 2 and 8 vanishing moments (switches S_{w4}, S_{w6} and S_{w8} all open or closed). Two zero moments are not enough for efficient representation of the high order polynomials, while 8 zero moments introduce ripple near polynomial edges. Minimum variance of wavelet coefficients is ±1, due to quantization effects.

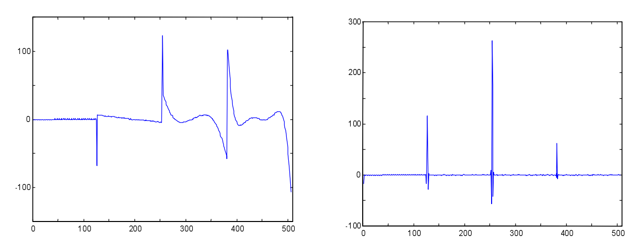

Figure 6. Integer details $d[k]$ computed by fixed wavelet filter bank. Left: 2 zero moments. Right: 8 zero moments.

In **Figure 7** details $d[k]$ are obtained by adaptive FB. Most of the wavelet coefficients are zeroed (±1), while there is no ripple near discontinuities.

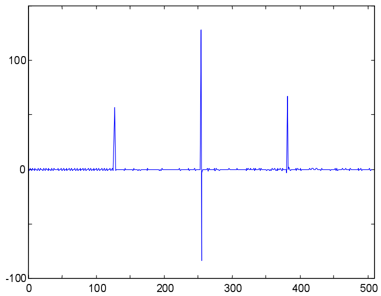

Figure 7. Integer details $d[k]$ computed by wavelet filter bank with variable number of zero moments: 2–8.

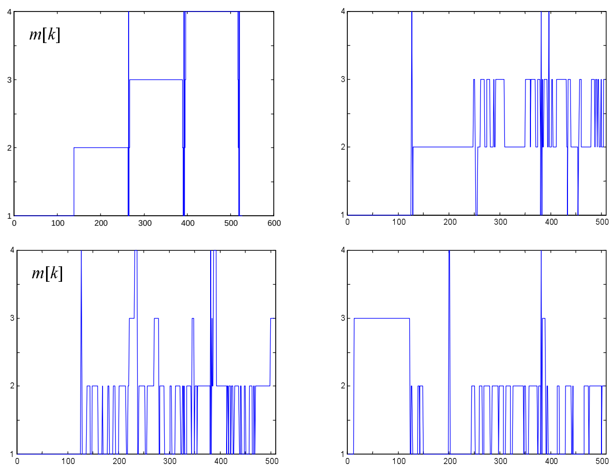

Figure 8. Number of closed switches $m[k]$ for different number of quantization levels. Top left: adaptation for continuous signal. Top right: 20 bits. Bottom left: 16 bits. Bottom right: 12 bits.

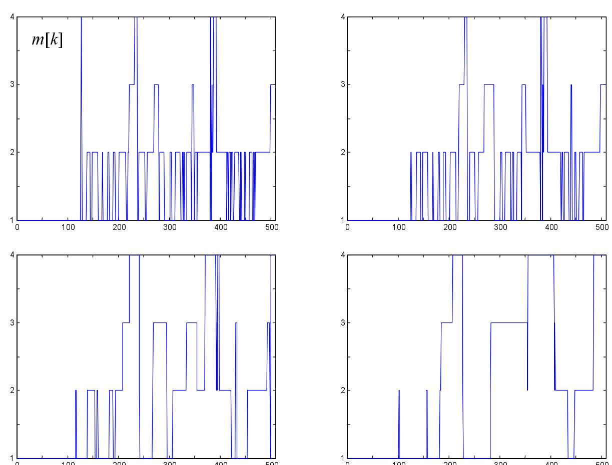

Figure 9. Number of closed switches $m[k]$ for different adaptation intervals. Top left: memory-less criterion. Top right: K_1=K_2=1. Bottom left: K_1=K_2=10. Bottom right: K_1=K_2=25.

Decomposition is almost optimal for continuous signal – the number of zero moments chosen by the adaptation criterion follows the properties of the analyzed signal $x[k]$. Filter order is low on polynomial edges thus decreasing the ringing effects. On the other hand, quantization introduces unpredictable components of the analyzed signal. It “confuses” the memory-less adaptation algorithm, especially for coarser quantization (**Figure 8**). Better results are achieved if the adaptation algorithm on interval $[k–K_1, k+K_2]$ is applied (**Figure 9**).

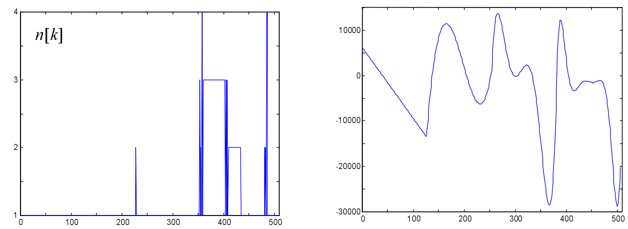

Figure 10. Left: number of closed switches $n[k]$ of the LP filter (K_1=K_2=25). Right: approximation coefficients $a[k]$.

Approximation coefficients are almost undistorted (but decimated) version of $x[k]$.

4. SUMMARY

We give an efficient realization of the two-channel wavelet filter bank that maps integers to integers, with adaptive number of zero moments. A set of switches determines the desired number of zero moments at each step of decomposition or reconstruction. With some limitations ($n≤m$), the same structure can be used to enable simultaneous adaptation of both filters in the bank. We used the least absolute sum criterion, computed from filtered wavelet coefficients. Adaptive filter bank is applied on a synthetic signal. Wavelet coefficients get close to what we expect to be an optimal representation of the analyzed signal, especially if the number of quantization levels is high. Quantized signals contain non-predictable components that cause intensive variance of the switch positions. Averaging of the error signal on an interval decreases the variance. Described time variant wavelet filter bank is more suitable for analysis of non-stationary signals then fixed banks.

It is shown that wavelet FB-s that map integers to integers have advantages in applications such as lossless compression [2]. Filter banks that change the number of zero moments in purpose to adapt to the signal properties may outperform fixed FB-s.

5. ACKNOWLEDGEMENTS

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