WAVELET FILTER BANKS WITH ADAPTIVE NUMBER OF ZERO MOMENTS

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ABSTRACT

In this paper, an efficient realization of two-channel wavelet filter bank with adaptive number of zero moments is proposed. Described time variant wavelet filter bank is more suitable for analysis of non-stationary signals then fixed banks. Filters with more zero moments result in better representation of smooth parts of the analyzed signal, while less zero moments is better for transients and singularities. Proposed realization is based on the lifting scheme, derived from a method of fixed wavelet filter bank design, using Lagrange interpolation of samples in time domain. Adaptation criterion is calculated from wavelet coefficients, which is under some restrictions reproducible on the reconstruction side. Wavelet filter banks with adaptive number of zero moments outperforms fixed banks in a number of applications.

KEYWORDS

Adaptive zero moments, time varying filter banks, wavelets.

1. INTRODUCTION

Analytical properties of wavelet filter banks are closely related to convergence and regularity of the limit wavelet functions and scales. More zero moments correspond to more regularity, which gives better description of smooth and correlated parts of the analyzed signal [1][2]. But, it results in longer filters that cause ripple effects on sharp edges. On the other hand, shorter filters are more suitable for compact representation of transients and singularities, as well as parts of the analyzed signal with narrower correlation of samples.

The number of vanishing moments of a fixed filter bank is usually chosen as a compromise between filter complexity and concentration of the wavelet coefficients. Our goal is to change the number of zero moments on both filters in the bank at each step of decomposition. The two-channel PR filter bank should form wavelet tree or wavelet packet tree, so the convergence and some degree of regularity must remain. The adaptation criterion should be computed from wavelet coefficients, wishfully resulting in more compact representation of the analyzed signal. We expect benefits of using adaptive number of zero moments in many applications [6].

In section 2 we describe the construction of the proposed adaptive filter bank. Sweldens 96 [3] proposed a construction of biorthogonal wavelet filter banks based on the lifting scheme, using interpolation of samples in the time domain. A short review is given in paragraph 2.1. Even samples are estimated from odds using Lagrange interpolation functions of chosen order. In the proposed scheme, we consider odd order Lagrange polynomials corresponding to the even length FIR filters.

In paragraph 2.2 we give the proposed factorization of the adjustable lifting step. In paragraph 2 adjustable dual step is introduced. In section 3 we discuss the adaptation criterion. To ensure the convergence and minimum regularity, filters are split in a fixed and a variable part.

2. FILTER BANK STRUCTURE

2.1 Lifting scheme

The lifting scheme is related to the polyphase representation of filter banks, with polyphase matrix factored in a cascade of triangular sub-matrices. Each sub-matrix corresponds to a lifting or a dual lifting step. Its all-ones diagonal form guaranties existence of the inverse sub-matrix, even if lifting or dual lifting operators are not constant or linear. An inverse sub-matrix is obtained by a simple transposition followed by the change in sign. It enables easy construction of perfect reconstruction time-variant and non-linear filter banks. Daubechies and Sweldens 98 [4] show that any two-band FIR filter bank can be factored in a set of lifting steps, using Euclidean algorithm.

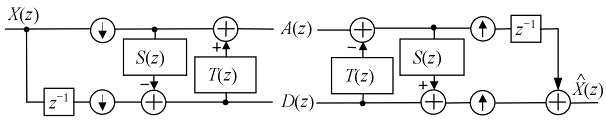

Figure 1. Two-channel PR filter bank factored in lifting and dual lifting steps.

The polyphase matrix of the filter bank from **Figure 1** is factored in 2 triangular sub-matrices:

$$\mathbf{P}(z) = \begin{bmatrix} 1 & 0 \\ T(z) & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -S(z) \\ 0 & 1 \end{bmatrix} .$$

$$H(z) = z^{-1} - 1 \cdot S(z^2) \,, \tag{2.1}$$

$$L(z) = 1 + H(z) \cdot T(z^2) \,. \tag{2.2}$$

We limit to FIR lifting steps $S(z)$ and $T(z)$, with $2N$ taps:

$$S(z) = s_0 z^{N-1} + ... + s_{N-1} + s_N z^{-1} + ... + s_{2N-1} z^{-N} \,, \tag{2.3}$$

$$T(z) = t_0 z^N + ... + t_{N-1} z + t_N + ... + t_{2N-1} z^{-N+1} \,, \tag{2.4}$$

In this paper, a class of two-channel biorthogonal filter banks constructed by Lagrange interpolation method is used. Sweldens 96 [3] described a lifting scheme construction of Deslauriers - Dubuc filter banks [5] by interpolation of samples in the time domain. The illustration of linear (II) and cubic (IV) case is given below:

$$d_{\mathrm{II}}[k] = x_o[k] - \frac{x_e[k-1] + x_e[k]}{2}$$

$$d_{\mathrm{IV}}[k] = x_o[k] - \frac{1}{16}(-x_e[k-2] + 9x_e[k-1] + 9x_e[k] - x_e[k+1])$$

$$S_{\mathrm{II}} = \frac{1}{2}(1 + z^{-1}) \qquad S_{\mathrm{IV}} = \frac{1}{16}(-z + 9 + 9z^{-1} - z^{-2})$$

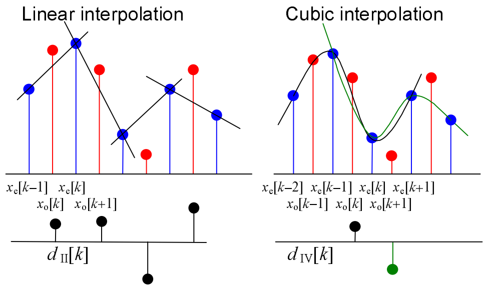

Figure 2. Lagrange interpolation of samples

We use S and T filters of the Lagrange interpolation type with even number of zero moments.

2.2 Adjustable lifting step

We start from the filter bank structure given in **Figure 1** and equations (2.1) and (2.3). Two zero moments of the high-pass filter are equivalent to the requirements:

$$H(z)\big|_{z=1} = 0, \qquad H'(z)\big|_{z=1} = 0.$$

These conditions decrease the freedom of choice of the prediction filter coefficients $\{s_k\}$, and lead to equations:

$$\sum_{k=0}^{2N-1} s_k = 1, \qquad \sum_{k=0}^{2N-1} (N-1-k)\, s_k = -\frac{1}{2}.$$

We express the central coefficients s_N and s_{N-1} from the outer ones:

$$s_N = \frac{1}{2} + \sum_{k \neq N} (N-1-k)\, s_k\,,$$

$$s_{N-1} = \frac{1}{2} - \sum_{k} (N-1-k)\, s_k \; - \sum_{k \neq N-1} s_k\,.$$

Now, we split the prediction filter into two additive components: fixed and "free" part: $S_{II}=S_0+S_{free}$, where

$$S_0(z) = \frac{1}{2} + \frac{1}{2} z^{-1},$$

$$S_{free}(z) = s_0 z^{-1}[z^N - Nz + (N-1)] +$$
$$+ s_1 z^{-1}[z^{N-1} - (N-1)z + (N-2)] + \ldots +$$
$$+ s_{N-2} z^{-1}[z^2 - 2z + 1] + s_{N+1} z^{-2}[1 + 1z^2 - 2z] +$$
$$+ \ldots + s_{2N-1} z^{-N}[1 + (N-1)z^N - Nz^{N-1}].$$

The next pair of zero moments additionally reduces freedom in coefficients of the prediction filter:

$$H''(z)\big|_{z=1} = 0, \qquad H'''(z)\big|_{z=1} = 0.$$

In general, expressions for central coefficients are getting more and more complex with each new pair of zeros. For the simplicity, we limit ourselves to the linear phase prediction filters with 8-taps (N=4). The results are presented in **Table 2.1**. Prediction filter is always a sum of additive components: e.g. $S_{II}=S_0$ for 2 zero moments, $S_{IV}=S_0+S_1$ for 4, or $S_{VIII}=S_0+S_1+S_2+S_3$ for 8 zero moments.

Finally, the proposed realization of the lifting step is shown in **Figure 3**.

Successive closing of switches S_{w2}, S_{w4}, S_{w6} and S_{w8} gives 2, 4, 6 or 8 zero moments of the high-pass filter, respectively. It corresponds to the prediction of odd samples from neighboring even samples, using linear, cubic, 5th or 7th order polynomial interpolation.

Zeros	Central coefficients of the prediction filter	Additive component of the fixed part
2	$s_3 = \frac{1}{2} - (s_0 + s_1 + s_2)$	$S_0 = \frac{1}{2}(1 + z^{-1})$
4	$s_2 = -\frac{1}{16} - (6s_0 + 3s_1)$	$S_1 = -\frac{1}{16}(z - 1 - z^{-1} + z^{-2})$
6	$s_1 = \frac{3}{256} - 5s_0$	$S_2 = \frac{3}{256}(z^2 - 3z + 2 + 2z^{-1} - 3z^{-2} + z^{-3})$
8	$s_0 = -\frac{5}{2048}$	$S_3 = -\frac{5}{2048}(z^3 - 5z^2 + 9z - 5 - 5z^{-1} + 9z^{-2} - 5z^{-3} + z^{-4})$

Table 2.1. Additive components of the 8 tap linear phase prediction filter ($s_7=s_0$, $s_6=s_1$, $s_5=s_2$, $s_4=s_3$) providing 2, 4, 6 or 8 zero moments to the high-pass filter $H(z)$.

Additive components can be factored and realized in a cascade:

$$S_0 = \frac{1}{2}(1 + z^{-1})$$

$$S_1 = -\frac{1}{16}(1 + z^{-1}) z^{-1}(1-z)^2 = -\frac{1}{8} z^{-1}(1-z)^2 \cdot S_0$$

$$S_2 = \frac{3}{256}(1 + z^{-1})[z^{-1}(1-z)^2]^2 = \frac{3}{16} z^{-1}(1-z)^2 \cdot (-S_1)$$

$$S_3 = -\frac{5}{2048}(1 + z^{-1})[z^{-1}(1-z)^2]^3 = -\frac{5}{24} z^{-1}(1-z)^2 \cdot S_2$$

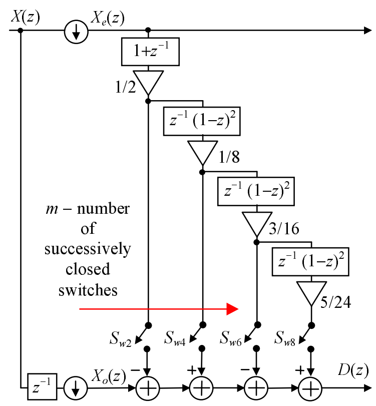

Figure 3. Lifting step of the high-pass filter with 0, 2, 4, 6 or 8 zero moments: *bior*(2×*m*).0. More zero moments corresponds to higher order Lagrange polynomial interpolation.

2.3 Adjustable dual lifting step

Let the high-pass filter have at least 2 vanishing moments. From **Figure 1**, equations (2.2) and (2.4) we have following expressions for the first 2 zero moments of the low-pass filter:

$$L(z)\big|_{z=-1} = 0, \qquad L'(z)\big|_{z=-1} = 0;$$

$$\sum_{k=0}^{2N-1} t_k = \frac{1}{2}, \qquad \sum_{k=0}^{2N-1} (N-k)\, t_k = \frac{1}{4}.$$

Now, we express the central coefficients of the update filter from its outer coefficients. Then we repeat the procedure for 4, 6 and 8 zero moments of the low-pass filter, times all switch positions $m \in \{1,2,3,4\}$ of the prediction filter. The factored results are presented in **Table 2.2**. Gain constants A_i depend on switch positions of the prediction filter.

$$T_0 = A_0 \frac{1}{4}(1+z) ,$$

$$T_1 = -\frac{A_1}{32}(1+z)z^{-1}(1-z)^2 = -\frac{A_1}{8} z^{-1}(1-z)^2 \cdot (\frac{S_0}{A_0}) ,$$

$$T_2 = \frac{A_2 \cdot 3}{512}(1+z)[z^{-1}(1-z)^2]^2 = \frac{A_2 \cdot 3}{16} z^{-1}(1-z)^2 \cdot (-\frac{S_1}{A_1}) ,$$

$$T_3 = -\frac{A_3 \cdot 5}{4096}(1+z)[z^{-1}(1-z)^2]^3 = -\frac{A_3 \cdot 5}{24} z^{-1}(1-z)^2 \cdot (\frac{S_2}{A_2}) .$$

Table 2.2. Additive components of the 8 tap update filter with linear phase ($t_7=t_0$, $t_6=t_1$, $t_5=t_2$, $t_4=t_3$) providing 2, 4, 6 or 8 zero moments to the low-pass filter $L(z)$.

Gain constants A_i are shown in the following table:

HP zeros →	2	4	6	8
A_0	**1**	**1**	**1**	**1**
A_1	3/2	**1**	**1**	**1**
A_2	5/3	3/2	**1**	**1**
A_3	7/4	3/2	3/2	**1**

Table 2.3. Gain A_i depends on the actual number of zero moments of the high-pass filter, unless $n < m$.

An interesting conclusion comes from **Table 2.3**. If the number of zero moments of the LP filter is less or equal to the number of zero moments of the HP filter, A_i equals 1 for all i=0–3. Hence, if the number of closed switches in the update filter does not exceed the number of closed switches in the prediction filter, we have "independent" lifting and dual lifting switching networks.

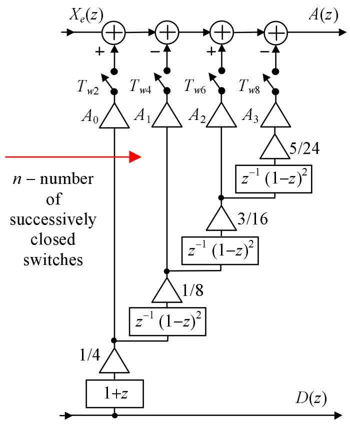

Figure 4. Dual lifting step of the low-pass filter with 0, 2, 4, 6 or 8 zero moments: $bior(2{\times}m).(2{\times}n)$. If $n \le m$, $A_0=A_1=A_2=A_3=1$.

3. Adaptation criterion and results

To preserve convergence and minimum regularity of the limit functions, we fixed 2 vanishing moments on both filters. Hence, switches S_{w1} and T_{w1} are always closed, and our filter bank is split in basic HP and LP channels. In order to change the number of zero moments of the HP filter, we used the criterion of the minimum absolute error. At each step of decomposition we choose such a number of closed switches m that gives the minimum absolute error $|e[k]|$. The error signal is computed from wavelet coefficients. The same procedure is applied for the LP filter, with additional condition $n \le m$.

Due to decimation, aliasing frequency of the analyzed signal $x[k]$ maps to the DC component of the wavelet coefficients $d[k]$. Signal DC is preserved in $a[k]$ coefficients. To avoid its influence on the criterion, we use high-pass filtered wavelet coefficients as the error input of the adaptation algorithm (Figure 5).

From **Figure 6** we see that the decomposition is almost optimal: the number of zero moments chosen by the adaptation criterion follows the properties of the analyzed signal $x[k]$. The filter order is low on polynomial edges thus decreasing the ringing effects. Discarding of some wavelet coefficients causes low and localized distortion of the restored signal.

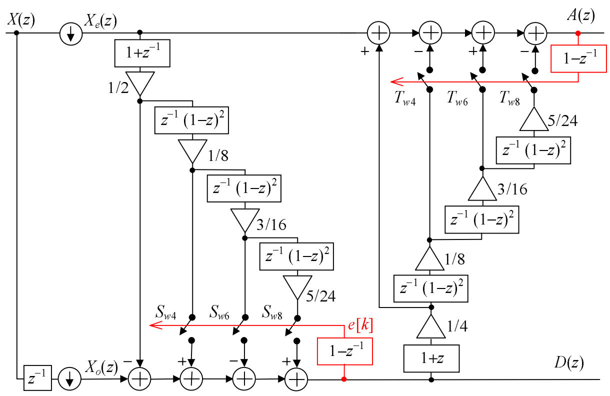

Figure 5. Wavelet filter bank with adaptive number of zero moments $bior(2{\times}m).(2{\times}n)$, $n \le m$.

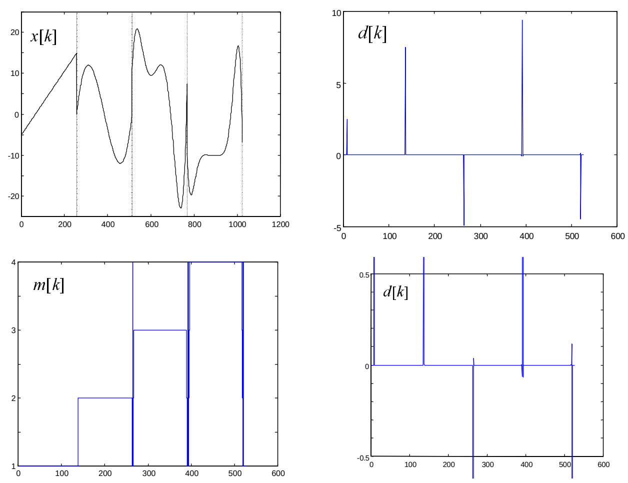

Figure 6. Top left: analyzed signal $x[k]$, composed from 4 polynomial sections of increasing order (1,3,5,7). Top right: wavelet coefficients $d[k]$ computed by time-variant filter bank. Bottom left: number of closed switches $m[k]$. Bottom right: magnified detail $d[k]$.

For the reference, **Figure 7** shows details of signal $x[k]$ computed by traditional fixed wavelets with 2 and 8 vanishing moments. Two zero moments are not enough for efficient representation of the high order polynomials. Large number of wavelet coefficients is different from zero. On the other hand, eight zero moments introduce ripple near polynomial edges. If we discard some non-zero sample, it results in distortion of the signal in a wide surrounding.

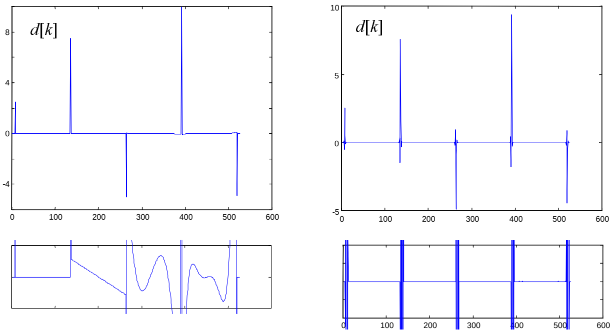

Figure 7. Left: wavelet coefficients $d[k]$ obtained from fixed filter bank *bior*2.2. Right: $d[k]$ from fixed filter bank *bior*8.8. Bottom: magnified details $d[k]$. Analyzed signal is $x[k]$ from **Figure 6**.

A three level wavelet decomposition is shown in **Figure 8**. Approximation coefficients contain almost undistorted polynomials, while details carry out the singularities.

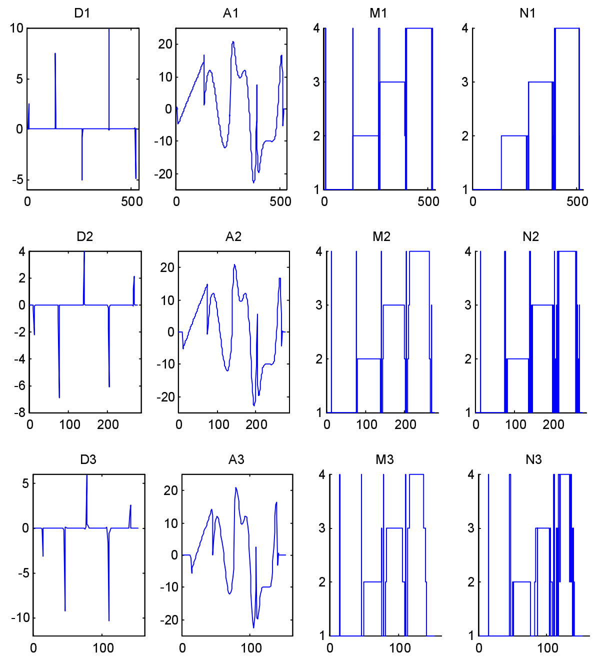

Figure 8. Three level decimated adaptive wavelet decomposition of signal $x[k]$ from **Figure 6**.

The contribution of each additional pair of zero moments can be quantified through the signal amplitude on switches. It is decreasing rapidly for most signals of interest. Hence, the structure in Figure 5 can be used to make decisions on the maximum reasonable number of vanishing moments for a given input process.

If we add some white noise to $x[k]$, the unpredictable noise components can excel the correlated parts of $x[k]$. The consequence is an intensive variance of the switches' positions. Additional filtering of the error signal can reduce the variance. We used averaging on 21 samples (minimum sum of absolute errors on interval: $\Sigma_{i=k-10}^{k+10}|e[i]|$.). The results are presented in Figure 9.

In general, we can reconstruct the analyzed signal from wavelet coefficients plus information on switch positions $m[k]$ and $n[k]$. They can be coded very efficiently. But, if the adaptation criterion is causal, e.g. if the current switch positions are determined exclusively from **previous** wavelet coefficients, the adaptation algorithm can be reproduced on the reconstruction side. In that case, perfect reconstruction does not require m and n to be separately transferred to the reconstruction side.

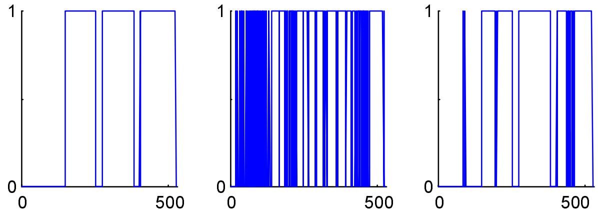

Figure 9. Positions of switch S_{w2}. Closed switch corresponds to value 1. Left: no additional noise in $x[k]$. Center: noise present. Right: noise present, error signal averaged on the interval of 20 samples.

4. SUMMARY

We give an efficient realization of the two-channel wavelet filter bank with adaptive number of zero moments. Prediction and update filters are implemented as a mixed cascade/parallel parallel set of filter sections, where each successive section brings the contribution of the higher order polynomial interpolation. A set of switches determines the desired number of zero moments at each step of decomposition or reconstruction. It is shown that, under some conditions, the same network can be used to enable simultaneous adaptation of zero moments of both filters in the bank. We used the least absolute error criterion, computed from filtered wavelet coefficients. Adaptive filter bank is applied on a synthetic signal. Wavelet coefficients get close to what we expect to be an optimal representation of the analyzed signal. Real world signals usually contain non-correlated components, inherent to the signal or caused by additive noise. They cause intensive variance of the switch positions. Averaging of the error signal decreases the variance. Described time variant wavelet filter bank is more suitable for analysis of non-stationary signals then fixed banks. It outperforms fixed wavelets in many applications.

5. ACKNOWLEDGEMENTS

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