FILTER FAMILIES WITH MINIMUM IMPULSE RESPONSE MOMENTS

Mladen Vucic and Hrvoje Babic

Faculty of Electrical Engineering and Computing
Unska 3, Zagreb, HR10000, Croatia

ABSTRACT

The lowpass systems with minimum higher order moments of the impulse response are presented. The optimization of the transfer function parameters is carried out for functions ranging from the second to the tenth order, with zeros at infinity. Besides the system order, no other requirements are set to restrict the frequency domain behavior of the system. Time and frequency domain properties of the obtained systems for various moments are given and compared.

1 INTRODUCTION

In many applications the systems with small time spread of the impulse response as well as small ringing are required. There are methods for design of such filters in the frequency and time domain, and they can be found in [1]. Most of them approximate the shape of the prescribed impulse response, while others optimize a particular property of the system time response that may be described by an integral criterion. If the criterion can be expressed from the system parameters through simple relations, it can be used not only for characterization but for optimization procedure as well. Here we propose the use of the integral criterion to find a class of filters with minimum time spread of the impulse response h(t).

The first and the second order moments can be used as the integral criteria having the mentioned property. In particular, when impulse response is nonnegative, $h(t)\geq 0$, its two moments, centroid and standard deviation are

$$t_d = \frac{m_1}{m_0} = \frac{\int_0^\infty t\,h(t)\,dt}{\int_0^\infty h(t)\,dt} \;;\quad \tau^2 = 2\pi\frac{\int_0^\infty (t-t_d)^2\,h(t)\,dt}{\int_0^\infty h(t)\,dt} \;. \tag{1}$$

They can be attractive definitions of the delay, t_d, and the rise time, τ, for analytical purposes [2], [3].

However, if h(t) is not nonnegative, the resulting central moment can become small, not only because of the small time spread, but because the positive, $h(t)>0$, and the negative contribution, $h(t)<0$, of the impulse response may cancel each other. It seams that the choice of the absolute value $|h(t)|$ in (1) is better, but unfortunately, it is not easy to work with. The central moment of $|h(t)|^2$, which gives the power spread along the time axis, is more tractable. Therefore, we will use even central moments of this function to minimize the response spread.

2 MOMENTS AND TRANSFER FUNCTIONS

The n-th order moments of the squared impulse response around centroid t_m are given by

$$m_n = \int_0^\infty (t-t_m)^n\, h^2(t)\,dt \;. \tag{2}$$

We define a measure of impulse response spread by the central moment (2), normalized to impulse response energy, which is, in fact, the zeroth moment

$$E_n = \frac{m_n}{m_0} \;. \tag{3}$$

For the optimization procedure in the complex domain, the criterion E_n should be expressed by the transfer function poles, p_i, and zeros, z_i. The impulse response of the N-th order filter with simple poles is given by

$$h(t) = \sum_{r=1}^{N} K_r e^{p_r t} \;,\quad K_r = H_0 \frac{\prod_{i=1}^{M}(p_r - z_i)}{\prod_{\substack{j=1\\ j\neq r}}^{N}(p_r - p_j)} \;, \tag{4}$$

where the pole residues are K_r, r=1,2,...,N. Now, the n-th moment can be expressed as function of poles, zeros and residues as

$$m_n = (-1)^{n+1}\sum_{i=1}^{N}\sum_{j=1}^{N} K_i K_j \sum_{k=0}^{n}\frac{n!}{k!}\frac{t_m^k}{(p_i+p_j)^{n-k+1}} \;. \tag{5}$$

3 OPTIMIZATION PROCEDURE

The positions of poles and zeros of causal filters with the most compact impulse response can be found by solving the problem

$$\min_{z_i, p_j} E_n\left[z_i, p_j\right] \;. \tag{6}$$

Although expressions (3) to (6) can be applied to any stable linear system with simple poles and zeros, in this paper only all-pole transfer functions are considered. Furthermore, in our optimization procedure the frequency ω_p and quality factor Q_p of poles were used, instead of the poles, p_j. This variable set enables the pole position on the whole complex plane. Using ω_p and Q_p, an all-pole transfer function can be written in the form

Table I. Transfer function parameters for t_m=1.

	n=2			n=4		
N	ω_p	Q_p	ω_{3dB}	ω_p	Q_p	ω_{3dB}
2	1.7678	0.7071	1.7678	1.9943	0.6159	1.7054
3	2.8795 1.5717	1.0847	2.2154	2.9794 1.9808	0.8416	2.0805
4	4.0839 2.1632	1.5467 0.6307	2.5971	4.0890 2.5084	1.1164 0.5751	2.4353
5	5.3175 3.1916 1.9159	2.0923 0.9050	2.9763	5.2551 3.3986 2.4255	1.4293 0.7436	2.7786
6	6.5588 4.3744 2.4272	2.7327 1.2451 0.6024	3.4322	6.4488 4.4642 2.8730	1.7798 0.9585 0.5589	3.1249
7	7.7991 5.6164 3.4078 2.1575	3.4823 1.6360 0.8311	3.6434	7.6561 5.6151 3.7057 2.7522	2.1703 1.2040 0.6993	3.4232
8	9.0342 6.8829 4.5760 2.6228	4.3577 2.0785 1.1190 0.5875	4.1107	8.8693 6.8104 4.7421 3.1515	2.6044 1.4759 0.8835 0.5500	3.7497
9	10.2624 8.1587 5.8217 3.5712 2.3422	5.3772 2.5777 1.4467 0.7898	4.3113	10.0841 8.0299 5.8823 3.9451 3.0084	3.0862 1.7738 1.0953 0.6735	4.0276
10	11.4829 9.4359 7.1023 4.7281 2.7773	6.5605 3.1405 1.8113 1.0475 0.5781	4.6906	11.2979 9.2628 7.0786 4.9601 3.3755	3.6203 2.0990 1.3288 0.8387 0.5444	4.3237

Table II. Transfer function parameters for t_m=1.

	n=6			n=8		
N	ω_p	Q_p	ω_{3dB}	ω_p	Q_p	ω_{3dB}
2	2.2565	0.5726	1.7543	2.5348	0.5493	1.8616
3	3.1486 2.3619	0.7232	2.0478	3.3598 2.7217	0.6569 0.5000	2.0730
4	4.1760 2.8601	0.9104 0.5488	2.3559	4.3189 3.2066	0.7944 0.5343	2.3322
5	5.2770 3.6565 2.8860	1.1226 0.6632	2.6622	5.3609 3.9388 3.3091	0.9512 0.6174 0.5000	2.6028
6	6.4203 4.6284 3.3064	1.3563 0.8136 0.5385	2.9662	6.4553 4.8372 3.7190	1.1233 0.7296 0.5274	2.8757
7	7.5888 5.6998 4.0423 3.2849	1.6112 0.9872 0.6341	3.2553	7.5838 5.8411 4.3908 3.7689	1.3095 0.8607 0.5971 0.5000	3.1438
8	8.7730 6.8299 4.9759 3.6552	1.8892 1.1790 0.7640 0.5328	3.5427	8.7357 6.9131 5.2468 4.1288	1.5093 1.0060 0.6943 0.5234	3.4081
9	9.9715 7.9992 6.0279 4.3458 3.5956	2.1933 1.3889 0.9172 0.6173	3.8162	9.9013 8.0291 6.2238 4.7588 4.1449	1.7227 1.1627 0.8096 0.5846 0.5000	3.6655
10	11.1682 9.1923 7.1529 5.2549 3.9347	2.5100 1.6103 1.0852 0.7342 0.5292	4.0842	11.0776 9.1751 7.2788 5.5835 4.4706	1.9508 1.3305 0.9381 0.6716 0.5207	3.9175

$$H(s) = \frac{H_0}{\prod_{i=1}^{N/2}\left(s^2 + \frac{\omega_{pi}}{Q_{pi}}s + \omega_{pi}^2\right)} , \tag{7}$$

for N even, or

$$H(s) = \frac{H_0}{(s+\omega_{p0})\prod_{i=1}^{(N-1)/2}\left(s^2 + \frac{\omega_{pi}}{Q_{pi}}s + \omega_{pi}^2\right)} , \tag{8}$$

if N is odd.

In a stable system, ω_p and Q_p are positive. Square values of goal function variables were employed rather than constrained optimization procedure. Finally, optimum system poles were found as

$$\min_{\omega_p, Q_p} E_n\left[\omega_p^2, Q_p^2\right] . \tag{9}$$

For searching minimum Quasi-Newton method with BFGS formula for Hessian matrix update [4] was used to obtain the minimum. In each iteration, a bisection type line search was performed followed by quadratic interpolation. In situations when Hessian matrix had irregular inverse, a steepest descent method was forced by setting Hessian matrix to identity.

To obtained causal filters with the most compact impulse response, the optimization is carried out for moments n=2, 4, 6 and 8, and system orders from N=2 to N=10. The parameter t_m is chosen to be 1 that will not change the generality of the solution.

4 OPTIMIZATION RESULTS

Numerical values of the pole parameters ω_p and Q_p, are given in Table I. and Table II.

For the all-pole transfer functions with t_m=1, the examples of pole positions are shown in Figure 1. It is interesting to note that the poles are located very closely to ellipses whose joint center is at the complex plane origin, similar as in the case of the systems with symmetric impulse response [5].

4.1 The system with minimum fourth moment

To illustrate the behavior of the filter class, the complete data are given for the system with minimum fourth order moment. Impulse response is shown in Figure 2. It is a bell-shaped

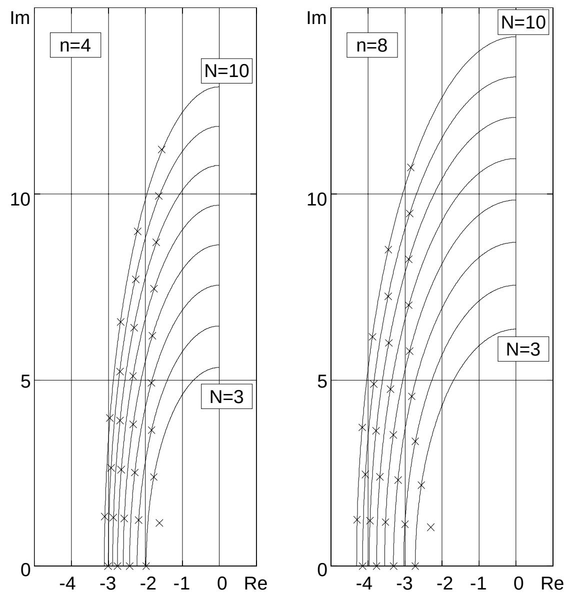

Figure 1. Pole positions of optimum systems with n=4 and n=8, normalized to t_m=1.

response, with small time spread and undershoot. In Figure 3. step responses are given for filters normalized to t_m=1. The overshoots are bellow 0.83 % for N≥3.

The amplitude and the group delay responses are shown in Figure 4. and Figure 5., respectively, in the form suitable for comparison with the classical filter approximations, given for example in [6]. The amplitude response is quasi gaussian. The group delay curves illustrate near constant approximations with small ripple. The bandwidth of quasi-constant group delay is extending well beyond cutoff frequency ω_{3dB}.

4.2 Comparison of systems with minimum 2nd, 4th, 6th and 8th moment

The optimization results for all moment orders are similar in character to the systems of fourth moment described above. The impulse response shows smaller time spread and undershoot for higher moments, n, as can be seen in Figure 6. The step response overshoot is also smaller for higher moments, as shown in Figure 7., therefore the step response is almost monotonic. In Figure 8. the rise time (10% - 90%) is given for filters normalized to ω_{3dB}=1. The rise time is spread between 2.15 s and 2.33 s. Generally, it is shorter for systems with higher n.

The amplitude attenuation in the stop band is higher for lower moments, and is generally higher than the response of Bessel, Gaussian or equiripple phase filters. The group delay is approximately constant. It has ripple for n=2 and 4 and no ripple for n=6 and 8.

5 CONCLUSION

By minimization of the higher order moments of the squared impulse response, a new class of finite order systems is obtained. The optimal pole positions are given in this paper together with properties of the systems, so, the filter design can be carried out. The obtained class of systems approximates quasi gaussian

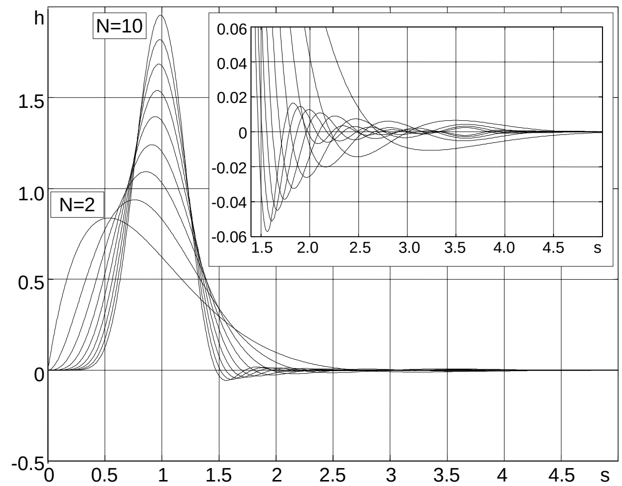

Figure 2. Impulse response of optimum systems with minimum fourth order moment, t_m=1.

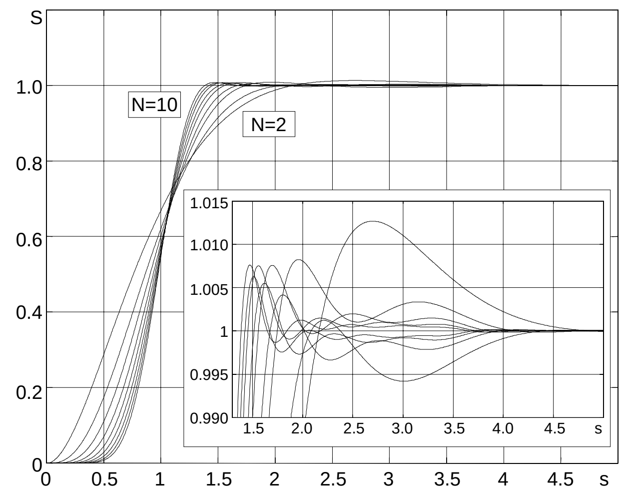

Figure 3. Step response of optimum systems with minimum fourth order moment, t_m=1.

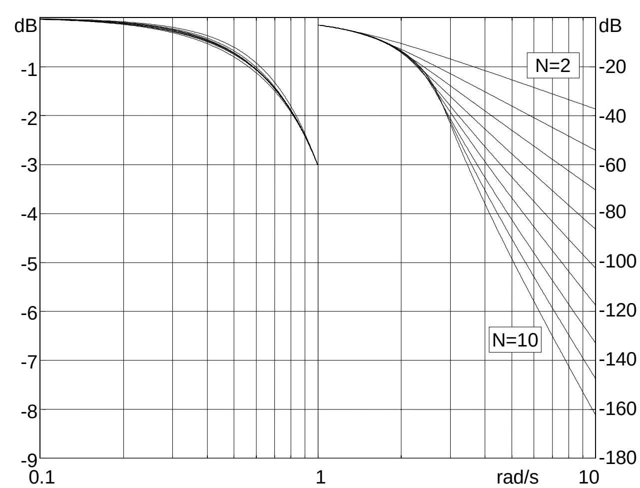

Figure 4. Amplitude response of the optimum systems with minimum fourth order moment, ω_{3dB}=1.

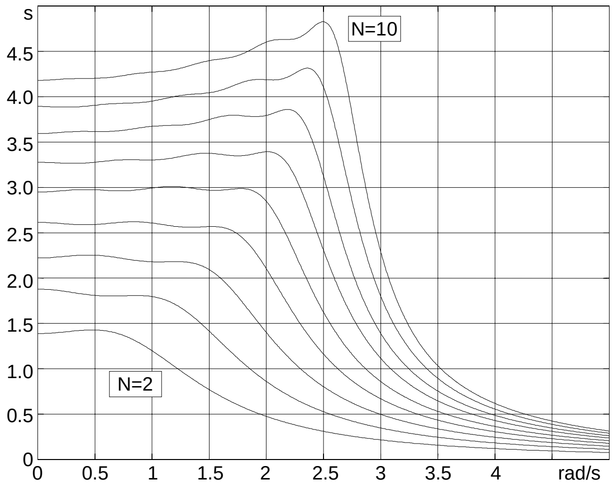

Figure 5. Group delay of the optimum systems with minimum fourth order moment, $\omega_{3dB}=1$.

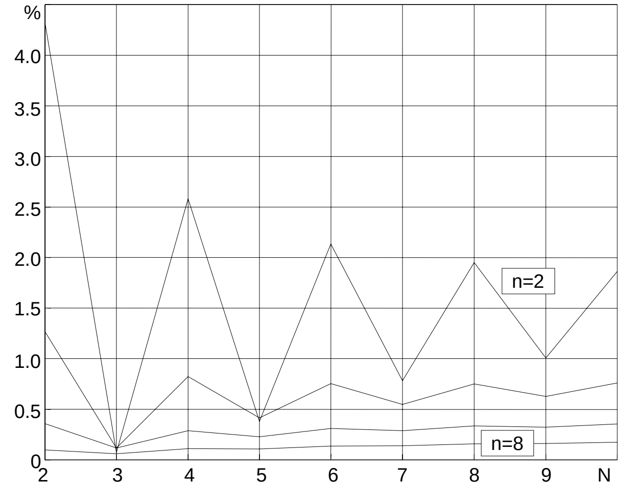

Figure 7. Step response overshoot of optimum systems with n=2, 4, 6, and 8.

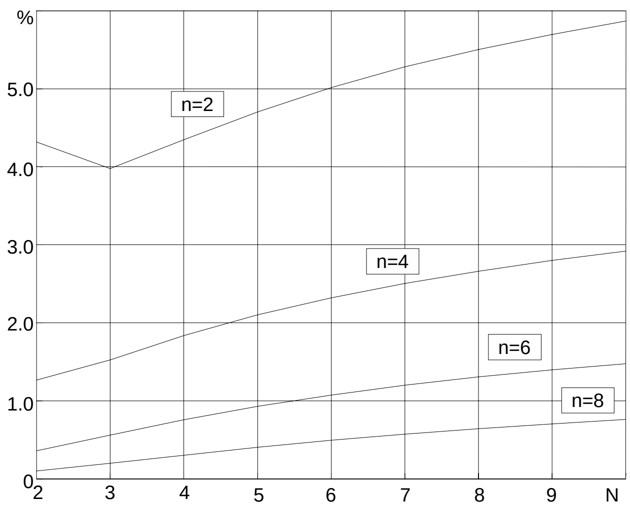

Figure 6. Impulse response undershoot of optimum systems with n=2, 4, 6, and 8.

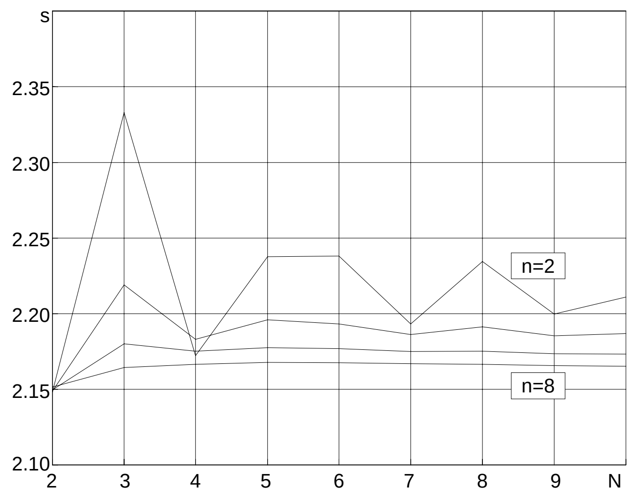

Figure 8. Step response rise time of optimum systems with n=2, 4, 6, and 8, normalized $\omega_{3dB}=1$.

amplitude with constant group delay within passband. The impulse response is very compact in time. It has small and short ringing, giving nearly monotonic step response. The properties of the obtained systems can be favorable compared to similar systems with linear phase optimized in the frequency domain.

6 ACKNOWLEDGEMENT

This study was made at the Department for Electronic Systems and Information Processing of the Faculty of Electrical Engineering and Computing, University of Zagreb, Croatia. It was supported by Ministry of Science and Technology of Croatia, under grants No. 036024 and No. 036124.

7 REFERENCES

[1] Su K. *Time-Domain Synthesis of Linear Network.* Prentice-Hall Inc. 1971

[2] Elmore W. C. "Transient Response of Damped Linear Networks". *Journal of Appl. Physics*. Volume 19, January 1948, No. 1, page 55

[3] Babic H. "An Optimum Transfer Function for Amplifiers with Nondecreasing Step Response". *International Symposium on Network Theory - ISYNT*. Beograd, Yugoslavia, September 1968 page 66

[4] Fletcher R. *Practical Methods of Optimization*. Volume 1, John Wiley & Sons, 1980

[5] Vucic M., Babic H. "A Class of Systems with Symmetric Impulse Response". *Proceedings of 1998 IEEE Symposium on Circuit and Systems*. Volume 3, Monterey, CA, USA, 1998, pages 485-488

[6] Zverev A. I. *Handbook of Filter Synthesis.* John Wiley & Sons, 1967