FILTERS WITH MINIMUM IMPULSE RESPONSE MOMENT

Mladen Vucic and Hrvoje Babic
Faculty of Electrical Engineering and Computing
Unska 3, Zagreb, HR10000, Croatia
E-mail: mladen.vucic@fer.hr and hrvoje.babic@fer.hr

Abstract
The lowpass systems with minimum second order moment of the impulse response are presented. Therefore the obtained systems have the largest possible energy concentration in time. The optimization of the transfer function parameters is carried out for the functions ranging from the second to the tenth order, with finite zeros. The optimum pole-zero positions suitable for filter design are given together with properties of the systems.

Key words: filter, impulse response moment, integral criterion, time domain optimization

1 INTRODUCTION

In many applications the systems with small time spread of the impulse response as well as small and short ringing are required. There are methods for design of such filters in the frequency and time domain, and they can be found in [1]. Most of them approximate the shape of the prescribed impulse response, while others optimize a particular property of the system time response that may be described by an integral criterion. If the criterion can be expressed by the system parameters through simple relations, it can be used not only for characterization but for optimization procedure as well. Here we propose the use of the integral criterion to find a class of filters with minimum time spread of the impulse response h(t).

The first and the second order moments can be used as integral criteria having the mentioned property. In particular, when impulse response is nonnegative, $h(t)\geq 0$, its two moments, centroid and standard deviation are

$$t_d = \frac{m_1}{m_0} = \frac{\int_0^\infty t\, h(t)\, dt}{\int_0^\infty h(t)\, dt} \quad \text{and} \quad \tau^2 = 2\pi \frac{\int_0^\infty (t-t_d)^2\, h(t)\, dt}{\int_0^\infty h(t)\, dt} \, . \tag{1}$$

They can be attractive definitions of the delay t_d, and the rise time τ, for analytical purposes [2], [3].

However, if h(t) is not nonnegative, the resulting central moment can become small, not only because of the small time spread, but because the positive, $h(t)>0$, and the negative contribution, $h(t)<0$, of the impulse response may cancel each other. It seams that the choice of the absolute value $|h(t)|$ in (1) is better, but unfortunately, it is not easy to work with. The central moment of $|h(t)|^2$, which gives the power spread along the time axis, is more tractable. Therefore, we will use second central moment of this function to minimize the response spread.

2 MOMENT AND TRANSFER FUNCTIONS

The second order moment of the squared impulse response around centroid t_m is given by

$$m_2 = \int_0^\infty (t - t_m)^2 h^2(t) dt \ . \tag{2}$$

We define a measure of impulse response spread by the central moment (2), normalized to impulse response energy

$$E_2 = \frac{m_2}{E_0} \ . \tag{3}$$

For the optimization procedure in the complex domain, the criterion E_2 should be expressed by the transfer function poles, p_j, and zeros, z_i. The impulse response of the N-th order filter with simple poles and M<N finite zeros is given by

$$h(t) = \sum_{r=1}^{N} K_r e^{p_r t} \quad , \quad K_r = H_0 \frac{\prod_{i=1}^{M} (p_r - z_i)}{\prod_{\substack{j=1 \\ j \neq r}}^{N} (p_r - p_j)} \ . \tag{4}$$

The pole residues are K_r, r=1,2,...,N. Now, the second moment and the impulse response total energy can be expressed as function of poles, zeros and residues as

$$m_2 = -\sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \sum_{k=0}^{2} \frac{2}{k!} \frac{t_m^k}{(p_i + p_j)^{3-k}} \quad \text{and} \quad E_0 = -\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{K_i K_j}{p_i + p_j} \ . \tag{5}$$

Besides the system order N and number of zeros M, no other requirements are set to restrict the frequency domain behavior of the system.

3 OPTIMIZATION PROCEDURE

The positions of poles and zeros of causal filters with the most compact impulse response can be found by solving the problem

$$\min_{z_i, p_j} E_2 [z_i, p_j] \ . \tag{6}$$

In our optimization procedure the frequency ω_p and ω_z and quality factor Q_p and Q_z were used, instead of the poles and zeros, p_j and z_i. Using ω_p and Q_p and ω_z and Q_z, a rational transfer function with even number of zeros can be written in the form

$$H(s) = H_0 \frac{\prod_{i=1}^{M/2} (s^2 - \frac{\omega_{zi}}{Q_{zi}} s + \omega_{zi}^2)}{\prod_{j=1}^{N/2} (s^2 + \frac{\omega_{pj}}{Q_{pj}} s + \omega_{pj}^2)} \quad \text{and} \quad H(s) = H_0 \frac{\prod_{i=1}^{M/2} (s^2 + \frac{\omega_{zi}}{Q_{zi}} s + \omega_{zi}^2)}{(s + \omega_{p0}) \prod_{j=1}^{(N-1)/2} (s^2 + \frac{\omega_{pj}}{Q_{pj}} s + \omega_{pj}^2)} \tag{7}$$

for N even and N odd, respectively.

In a stable system poles ω_p and Q_p are positive. Positive values of ω_z and Q_z in (7) will give zeros in the right half plane. Square values of goal function variables were employed rather than constrained optimization procedure. Finally, optimum system poles and zeros were found as

$$\min_{\omega_p, Q_p, \omega_z, Q_z} E_2\left[\omega_p^2, Q_p^2, \omega_z^2, Q_p^2\right] . \tag{8}$$

For searching minimum Quasi-Newton method with BFGS formula for Hessian matrix update [4] was used. In each iteration, a bisection type line search was performed followed by quadratic interpolation. In situations when Hessian matrix had irregular inverse, a steepest descent method was forced by setting Hessian matrix to identity.

To obtained causal filters with the most compact impulse response, the optimization is carried out for the second moment, and system orders from N=2 to N=10, and two and four zeros. The parameter t_m is chosen to be 1 that will not change the generality of the solution.

4 OPTIMIZATION RESULTS

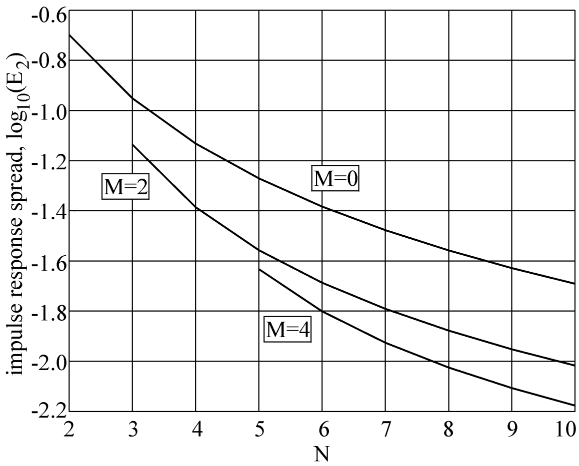

Figure 1 Impulse response spread of obtained systems.

The impulse response spread of obtained systems is shown in Figure 1 for various numbers of zeros. The presence of zeros apparently reduces the energy spread in time.

Numerical values of the rational transfer function parameters ω_p, Q_p, ω_z and Q_z are given in Table 1 and Table 2. For the all pole transfer functions, the numerical values can be found in [6].

For the rational transfer functions with t_m=1, the examples of pole-zero positions are shown in Figure 2. It is interesting to note that the poles are located very closely to ellipses that have the joint center at the complex plane origin, similar as in the case of the systems with symmetric impulse response [5]. Zeros are grouped around the point 5+4j for the systems with one pair of zeros, and around points 6+4j and 5+11j for the systems with two pairs of zeros.

The impulse response of the all-pole system is shown in Figure 3. It is a bell-shaped response, with small and short ringing. The impulse response of the filters with one pair of complex zeros is given in Figure 4. The presence of zeros improves the response symmetry and causes the ringing before the main pulse. The corresponding step responses have small overshoots and undershoots, as shown in Figure 5 and Figure 6. In both cases the overshoots are below 3% for N>2. Generally, the step response overshoot is smaller than 1.2% for odd order systems and N≤9.

The optimization results with presence of two pairs of zeros are similar in character to the systems described above. The impulse response with t_m=1 has somewhat smaller time spread and larger leading edge ringing. The step response consequently has shorter rise-time for larger number of zeros. Such transfer functions can be suitable in the design of linear delay networks.

The step responses normalized to ω_{3dB}=1 will have rise-time quasi independent of number of zeros for the given system order. It takes values 2.17 s to 2.43 s for 2<N<10.

Table 1 Transfer function parameters of systems with one pair of complex zeros.

N	ω_p	Q_p	ω_z	Q_z	ω_{3dB}
3	3.5801 1.8684	1.1122	7.8547	0.5442	3.0765
4	5.3578 2.7472	1.5391 0.6394	6.2596	0.6314	3.9624
5	7.0263 4.1418 2.4171	1.9830 0.9131	6.2471	0.6459	4.5278
6	8.6126 5.6760 3.0909	2.4651 1.2291 0.6056	6.4843	0.6491	5.5917
7	10.1409 7.2342 4.3469 2.7114	3.0016 1.5695 0.8318	6.7937	0.6493	6.2190
8	11.6260 8.7817 5.8056 3.2921	3.6055 1.9335 1.1035 0.5885	7.1269	0.6486	6.8803
9	13.0770 10.3086 7.3278 4.4696 2.9080	4.2889 2.3254 1.3982 0.7878	7.4664	0.6477	7.4870
10	14.5002 11.8130 8.8636 5.8829 3.4345	5.0642 2.7506 1.7111 1.0333 0.5781	7.8050	0.6467	8.2661

Table 2 Transfer function parameters of systems with two pairs of complex zeros.

N	ω_p	Q_p	ω_z	Q_z	ω_{3dB}
5	7.7238 4.5227 2.5976	2.0223 0.9259	11.2956 9.3981	0.7328 0.5223	5.0867
6	9.8576 6.4354 3.4550	2.4930 1.2483 0.6101	11.7152 6.6435	1.1242 0.5839	6.4912
7	11.7845 8.3135 4.9527 3.0461	2.9638 1.5823 0.8403	11.7924 6.4986	1.2158 0.5951	7.3700
8	13.5688 10.1356 6.6522 3.7363	3.4580 1.9288 1.1101 0.5907	12.1498 6.6145	1.2387 0.5994	8.1610
9	15.2574 11.9051 8.4011 5.0969 3.2904	3.9940 2.2903 1.3981 0.7912	12.6001 6.8130	1.2417 0.6012	8.8154
10	16.8789 13.6265 10.1504 6.7070 3.8942	4.5856 2.6690 1.6997 1.0342 0.5791	13.0835 7.0436	1.2380 0.6020	9.7009

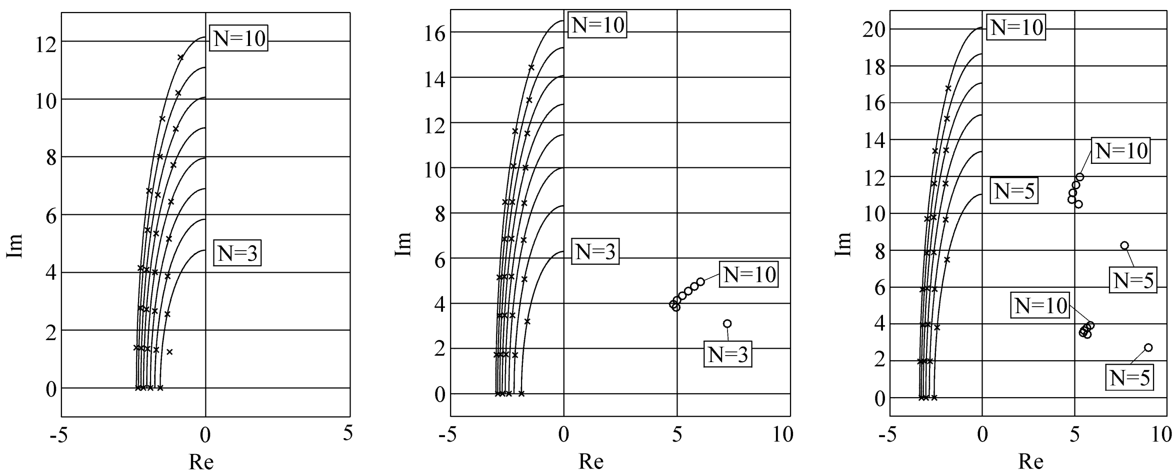

Figure 2 Zero pole positions of optimum systems with all zeros at the infinity, systems with one pair of complex zeros and systems with two pairs of complex zeros, t_m=1.

The amplitude and the group delay responses, normalized to ω_{3dB}=1, are shown in Figure 7, to Figure 10, in the form suitable for comparison with the classic filter approximations, given for example in [7]. The amplitude response is quasi gaussian. The group delay curves illustrate near constant approximations. Zeros improve the average slope of the group delay within the passband of the higher order systems.

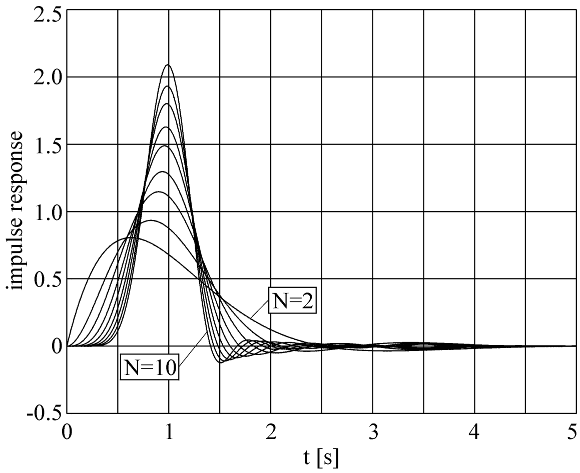

Figure 3 Impulse response of optimum systems with all zeros at the infinity, t_m=1.

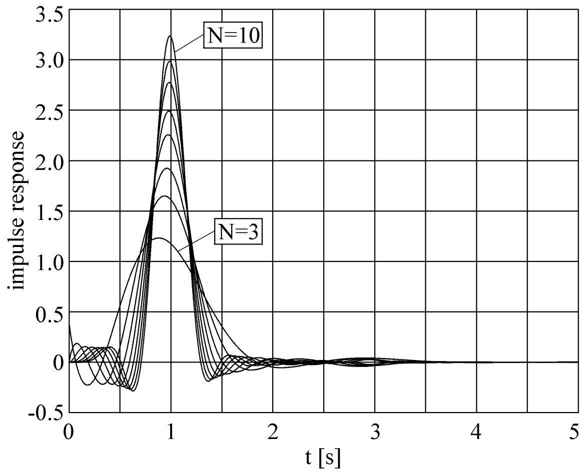

Figure 4 Impulse response of optimum systems with one pair of complex zeros, t_m=1.

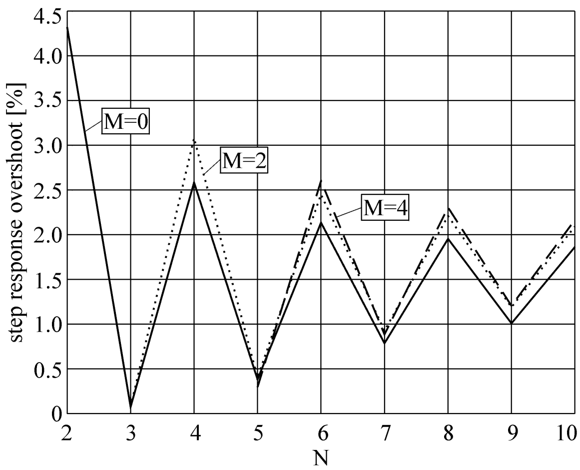

Figure 5 Step response overshoot of optimum systems.

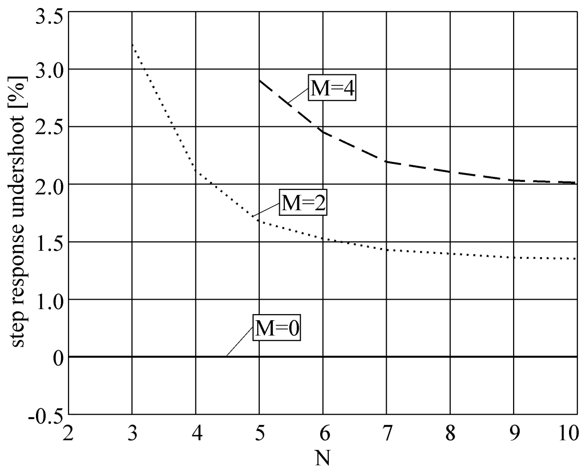

Figure 6 Step response undershoot of optimum systems.

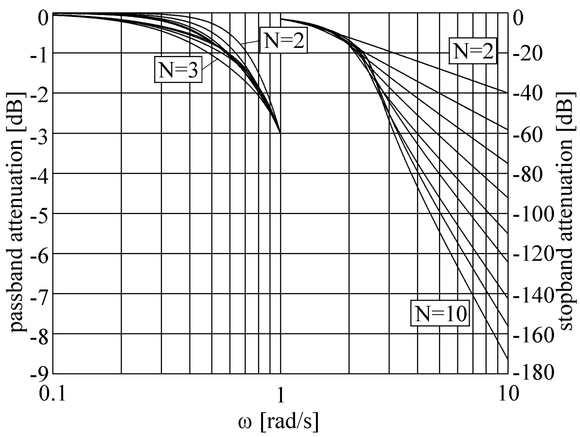

Figure 7 Amplitude response of optimum systems with all zeros at the infinity, ω_{3dB}=1.

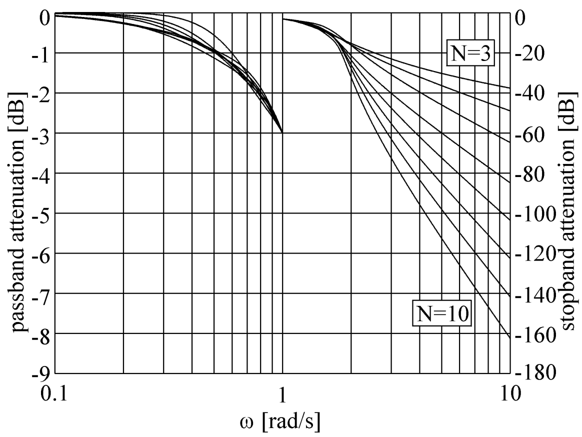

Figure 8 Amplitude response of optimum systems one pair of complex zeros, ω_{3dB}=1.

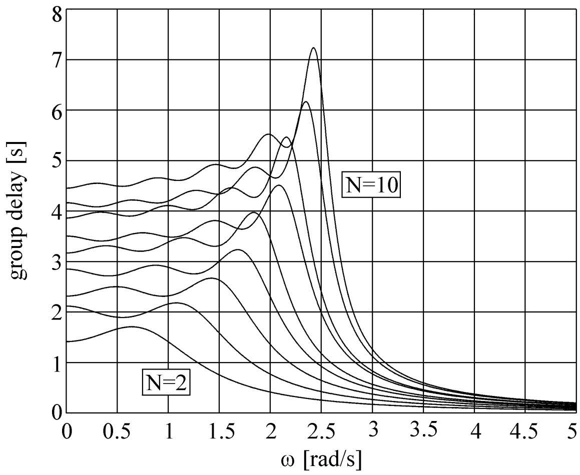

Figure 9 Group delay of optimum systems with all zeros at the infinity, ω_{3dB}=1.

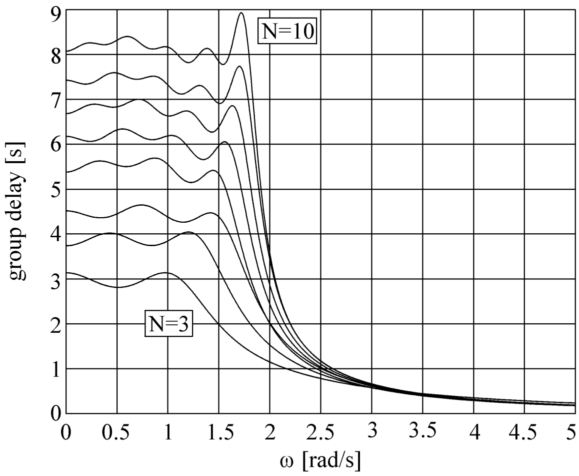

Figure 10 Group delay of optimum systems one pair of complex zeros, ω_{3dB}=1.

5 CONCLUSION

By minimization the second order moment of the squared impulse response, a new class of finite order systems is obtained. Based on the used criterion, the obtained filters have the largest possible energy concentration in time. The impulse response has small and short ringing. The amplitude response is quasi gaussian, with quasi constant group delay. The presence of zeros improves the impulse response energy concentration and the group delay.

6 ACKNOWLEDGEMENT

This study was made at the Department for Electronic Systems and Information Processing of the Faculty of Electrical Engineering and Computing, University of Zagreb, Croatia. It was supported by Ministry of Science and Technology of Croatia, under grants No. 036024 and No. 036124.

7 REFERENCES

[1] Su K. (1971), *Time-Domain Synthesis of Linear Network*, Prentice-Hall Inc.

[2] Elmore W. C. (1948), "Transient Response of Damped Linear Networks", *Journal of Appl. Physics*, Volume 19, January 1948, No. 1, pp. 55

[3] Babic H. (1968), "An Optimum Transfer Function for Amplifiers with Nondecreasing Step Response", *International Symposium on Network Theory - ISYNT*, Beograd, Yugoslavia, September 1968, pp. 66

[4] Fletcher R. (1980), *Practical Methods of Optimization*. Volume 1, John Wiley & Sons

[5] Vucic M., Babic H. (1998), "A Class of Systems with Symmetric Impulse Response", *Proceedings of 1998 IEEE Symposium on Circuit and Systems*, Volume 3, Monterey, CA, USA, May 31 - June 3 1998, pp. 485-488

[6] Vucic M., Babic H. (2000), "Filter Families with Minimum Impulse Response Moments", *Proceedings of 2000 IEEE Symposium on Circuit and Systems*, Geneva, Switzerland, May 28 - May 31 2000 (to be published)

[7] Zverev A. I. (1967), *Handbook of Filter Synthesis*, John Wiley & Sons