# Contemporary Philosophical Issues: Truth, Justice and Beauty

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**Mathematics, Logics, Structures**

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# A. Koslow: *A Structuralist Theory of Logic*

- The lynch-pin of the theory is the definition of logical operators with respect to implication relations, i.e. relative to implication structures.

- What is an implication structure?

# Implication structure

- $(S, \Rightarrow)$; $S \neq \varnothing$, "$\Rightarrow$" is an implication relation.
- An implication relation is any relation that satisfies the following conditions:

(1) *Reflexivity*: $A \Rightarrow A$, for every $A$ in $S$

(2) *Projection*: $A_1, A_2, \ldots, A_n \Rightarrow A_k$, for every $k$=1, …, n,

and for all $A_i$ in $S$ ($i$ = 1, …,n)

(3) *Simplification*: If $A_1, A_1, A_2, \ldots, A_n \Rightarrow B$,

then $A_1, A_2, \ldots, A_n \Rightarrow B$, for each $A_i$

($i$ = 1, …,n) and $B$ in $S$

(4) *Permutation*: If $A_1, A_2, \ldots, A_n \Rightarrow B$, then

$A_{f(1)}, A_{f(2)}, \ldots, A_{f(n)} \Rightarrow B$, for any

# Examples

- Semantic consequence
- Syntactic notion of deducibility
- Set inclusion
- Implication structures on names
- Erotetic (interrogative) logic
- ...

- Implication structures – comparable to equivalence relations

# Logical operators

- Defined with respect to an implication structure

  i.e.,

- *any* explanation of the operators is based *exclusively* on the notion of implication structure:

Because the elements of an implication structure need not be syntactical objects having a special sign design, and they need not have some special semantic value, an explanation of what can count as hypothetical, disjunctions, negations, and quantifies items (existential or universal) can proceed in a way that is free of such restrictions.

(Koslow, 1992, *A Structuralist Theory of Logic*)

# Logical operators - examples

- **Conjunction**: given an implication structure ($S$, $\Rightarrow$), the conjunction operator is a function $C$ that assigns to any elements $A$ and $B$ of $S$, a subset $C_{\Rightarrow}(A, B)$ of $S$ containing all those members (if any) that satisfy the following conditions:
- ($C$1) $C_{\Rightarrow}(A, B) \Rightarrow A$ and $C_{\Rightarrow}(A, B) \Rightarrow B$
- ($C$2) $C_{\Rightarrow}(A, B)$ is the weakest member of the implication structure that satisfies ($C$1);

  i.e. if $T$ is any member of the implication struc. such that $T \Rightarrow A$, $T \Rightarrow B$, then $T \Rightarrow C_{\Rightarrow}(A, B)$.

# Logical operators - example

- **Hypothetical**: for any elements $A$ and $B$ in the implication structure $(S, \Rightarrow)$, $H_{\Rightarrow}(A, B)$ is the hypothetical having $A$ as the antecedent and $B$ as a consequent, if and only if the following conditions are fulfilled:
- $(H1)$ $A, H_{\Rightarrow}(A, B) \Rightarrow B$
- $(H2)$ $H_{\Rightarrow}(A, B)$ is the weakest element satisfying the condition $(H1)$. It means that, for any element $T$ of the implication structure such that $A, T \Rightarrow B$, it follows that

$T \Rightarrow H_{\Rightarrow}(A, B)$

# May the operators fail to exist?

- Let us take the implication structure $(S, \Rightarrow)$, in which $S=\{A, B, C, D\}$ and the implication relation is given in the following way:

A

B

C D

- In such a structure, the hypothetical $H_{\Rightarrow}(A, B)$ does not exist;

namely, $H_{\Rightarrow}(A, B)$ is, by definition, the weakest member $T$ of $S$ such that: $A, T \Rightarrow B$. Since $A \Rightarrow B$, the condition is fulfilled by any element of $S$, but there is no weakest element.

$C$ cannot be the weakest element since

$A, D \Rightarrow B$, while $D \not\Rightarrow C$ (see the condition ($H2$).

$D$ cannot be the weakest one for the same reason.

# Other non–standard examples

- (S, $\Rightarrow$), $S$ is a non-empty set of sets and the implication relation is set inclusion "$\subseteq$". In this case the conjunction of two sets $A$ and $B$, is some set

  $C_{\Rightarrow}(A, B)$ such that:
- ($C$1) $C_{\Rightarrow}(A, B) \subseteq A$ and $C_{\Rightarrow}(A, B) \subseteq B$
- ($C$2) $C_{\Rightarrow}(A, B)$ is the weakest member of the implication structure that satisfies ($C$1);
- i.e. if $T$ is any element of $S$ such that $T \subseteq A$,
- $T \subseteq B$, then $T \subseteq C_{\Rightarrow}(A, B)$.
- Given the definition, $C_{\Rightarrow}(A, B)$ corresponds to $A \cap B$. Hence, it turns out that the intersection of two sets is not just analogous to conjunction – it *is* the conjunction

# Stability and Distinctness

- Given that operators are defined as special functions *relative to* implication relations, such relativization to implication structures make the problems of stability and distinctness naturally arise.

# Stability

- The problem concerns the following issue: if $C_{\Rightarrow}(A, B)$ is a conjunction in one implication structure, will it be a conjunction in a different implication structure in which the same elements are included?

  And the answer is positive: given an implication structure, any conjunction of two elements will still be the conjunction of those elements in every conservative extension of the structure

# Distinctness

- The problem of distinctness is due to the general character of any implication structure and it amounts to distinguishing the operators from each other.

  Koslow defines two operators as distinct (given an implication structure) if and only if for some items in the structure the two yield at least two nonequivalent members of the structure.

# The structuralist account: Why not?

- **Set inclusion**
- satisfies the conditions (1)-(6) for any implication relation.
- the conjunction of sets = their intersection.
- what does the left-hand side of the conditions (1)-(6) amount to? The left-hand side = the intersection of sets:

  $A_1 \cap A_2 \cap \ldots \cap A_n$.

  We ought to know what the intersection of sets is prior to having defined the implication relation on sets even though formally the definition of conjunction, i.e. intersection should follow the one of implication structure.

- **Erotetic (interrogative) logic**

- We shall use the term "interrogative" to include any question that has a *direct answer*. The most important feature of the direct answers to a question is that they are statements that, whether they are true or false, tell the questioner exactly what he wants to know–neither more nor less.

(Koslow, 1992, *A Structuralist Theory of Logic*)

e.g. Belnap and Steel:
the direct answer…is what counts as completely, but just completely, answering the question.
(Belnap and Steel, 1976, *The Logic of Questions and Answers*)

- **Erotetic logic, the logic of questions and answers**
- an implication relation for interrogatives is defined in the following way:

Let $Q$ be a collection of interrogatives (every question is denoted by a capital letter followed by a question mark), and $S$ a set of sentences inclusive of the sentential direct answers to the questions in $Q$. We denote their union with $S^*$:
$S^* = S \cup Q$;

while "$\Rightarrow$" is an implication relation on sentences of $S$.

- What needs to be defined is an implication relation "$\Rightarrow_q$" on the set $S^*$, that involves just the questions of $Q$, or any combination of questions in $Q$ and statements in $S$.

- $M_1?, M_2?, \ldots, M_n?$ and $R?$ - questions in $Q$,
- $F_1, F_2, \ldots, F_m$ and $G$ - statements of $S$ (the set of $M$'s or the set of $F$'s may be empty but not both)
- $A_i$ - a direct answer to the question $M_i?$ (i=1,…,n)

Then:

(1.) $F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_q R?$ iff there is some direct answer $B$ to the question $R?$ such that

$$F_1, F_2, \ldots, F_m, A_1, A_2, \ldots, A_n \Rightarrow B$$

(2.) $F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_q G$ iff

$$F_1, F_2, \ldots, F_m, A_1, A_2, \ldots, A_n \Rightarrow G$$

# Problems?

- Example:
- *S* - collection of sentences in classical propositional logic (the implication relation - the standard semantic notion of logical consequence)
- *Q* - collection of interrogatives that, among others, includes e.g. $M_1$?: “*How many satellites does the planet Earth have?*” and *R?*: “*Does the number ‘3’ solve the equation ‘x-4=0’?*” and
- Let us say, for the sake of simplicity, that the set of *F’s* is empty

- The set $S$ is infinite, there are infinitely many possible direct answers to the question $M_1$?, given that the direct answer need not be the correct one.

- According to the definition of an implication relation “$\Rightarrow_q$”, it follows that:

$M_1? \Rightarrow_q R?$ iff there is some direct answer $B$ to the question $R?$

such that $A_1 \Rightarrow B$

Since we can choose as $A_1$ any answer to the question $M_1?$ that is not correct, it is (trivially) always the case that $M_1? \Rightarrow_q R?$

- given the possibility to do the same for any question $M_i?$

  it turns out that

  for any questions $M_i?$, $R?$ with a direct answer, we get: $M_1? \Rightarrow_{\mathrm{q}} R?$
- the application of such a definition?
- its fruitfulness?

- Same case – when a question implies a statement (the second condition in the definition)

- In this case, whether $M_1? \Rightarrow_{\mathrm{q}} G$ or not depends on whether $A_1 \Rightarrow G$,
and the latter depends on what answer $A_1$
(to the question $M_1$?) we *choose*.

- Here again, since the answer we choose may be a false one, it is always the case that
$M_1? \Rightarrow_{\mathrm{q}} G$,

# more generally...

- the same problem appears whenever the statement $G$ is a false one.

In this case, given a collection of interrogatives $M_i$? (i=1,…n), their respective direct answers $A_i$, and a set of true statements $F_i$ (i=1,…n), there is nothing in Koslow’s definition that allows us to uniquely determine whether

$$F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_{\mathrm{q}} G$$ or not,

or to rule out the possibility of discussing it in the first place.

- As soon as we choose at least one false answer, it follows that

$$A_i, F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_{\mathrm{q}} G$$

- while by choosing all the correct answers we get:

$$F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \not\Rightarrow_{\mathrm{q}} G$$

# Mathematics and Logic – (dis)analogy

- Koslow's programme might seem to be analogous to what the standard mathematical practice is, in the sense of defining a determinate structure that can be exemplified by totally different systems.
- Example: vector space.

  different objects – e.g., geometric vectors or real numbers - count as vectors, in the same way in which

  e.g. either the standard conjunction in classical propositional logic (that have the sign "$\wedge$") or the intersection of sets, both count as conjunctions $C_{\Rightarrow}(A, B)$. How far does the analogy go?

# in mathematics...

- the theory of vector spaces determines not just what a vector space (over a field) is, but it also allows the projection of many other properties from the structure to single templates (or systems), e.g. the existence a base for every finitely dimensional vector space.

# In the structuralist account of logic...

- It is not the case with Koslow's theory of implication structure. Let us observe one example. According to the definition, the conjunction operator $C$ on an implication structure $(S, \Rightarrow)$, is a *function* which assigns to any two elements $A$ and $B$ of $S$, a conjunction of them, i.e. a subset $C_{\Rightarrow}(A, B)$ of $S$:
  - $C$: $(S, \Rightarrow) \rightarrow (S, \Rightarrow)$
  - $C$: $A, B \mapsto C_{\Rightarrow}(A, B)$

- $C_{\Rightarrow}(A, B)$ is the subset of all those elements of $S$ (if they exist) that satisfy the conditions ($C$1) and ($C$2)
- Example: $C_{\Rightarrow}(A, B)$ is the standard logical conjunction operator in classical propositional logic in which the implication structure is the set of formulas of the language together with the "standard" implication.
- In this case the conjunction is *not* defined, as it is usually the case, through its truth tables nor through the Elimination and Introduction rules.

- Given Koslow's definition, we ought to be able to get such results for the conjunction operator out of Koslow's definition, because there is simply no other way in which we could do it.
- But, the conjunction is defined independently of any syntactic or semantic features, and it is unclear how this definition is to be combined with the syntactic rules for formula formation and (semantic) truth tables.

- Once the implication structure is defined and it turns out that the semantic concept of logical consequence in classical propositional logic exemplifies the structure as well as the conjunction "$\wedge$" fulfils the conjunction $C_{\Rightarrow}(A, B)$ requirements, none of the semantic properties of the conjunction "$\wedge$" follow from the structure.
- How can we define the truth table for it?
- How are such truth tables related to the characterisation of the operator within the theory?

- Koslow very clearly endorses the view that

  “the tasks of a logical theory of statements can be carried out without appeal to either syntax or semantics”

- nevertheless, in order to develop the logical theories we are interested in both the syntax and the semantics are necessary.

  Otherwise our classical logical theories are exemplified by Koslow’s structures just fragmentally. And different logics turn out to have just partial fragments reducible to or exemplified by the same structure.