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The Reduction of Logic to Structures

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Abstract

The lynch-pin of the structuralist account of logic endorsed by Koslow has two components: one is the definition of an implication structure, while the other amounts to the definition of the logical operators as functions defined relative to an implication structure.
In this paper I first present the basic tenets of the structuralist account of logic. Then, in the discussion which follows I give reasons for rejecting certain definitions which form part of the theory and thirdly I point out some general difficulties arising from such an account of logic.

Introduction

The structuralist account of logic endorsed by Koslow (Koslow 1992, 2007) is one of the most appealing contemporary formulations of structuralism in logic.
But, what does structuralism in logic amount to? Is it analogous with structuralism in mathematics or other domains? And if yes, in which sense?
In this paper I try to answer these questions by presenting the basic tenets of Koslow's theory; I then analyse his views and offer reasons for holding that some aspects of his structuralist account are flawed. Finally, I try to show that Koslow's theory of logic fails to achieve a satisfactory answer to the question of a possible reduction of logic to structure(s).
One of the fields that is paradigmatically about structures is mathematics. This can be read in two ways: mathematics is about different structures such as the vector space structure, the natural number structure, the group structure etc., while the possibility of reducing mathematical theories to set theory, gives sense to viewing mathematics as about the (common) set-theoretic structure. Philosophically speaking, the (ontological) reduction of mathematical objects to structures leads to interesting results that aim to solve some ontological, as well as epistemological, problems in the philosophy of mathematics (see (Resnik 1997), (Shapiro 1997), (Hellman 2001)), even though it also brings to the surface some difficulties such as the problem of

structures that admit non-trivial automorphisms (for more details see for example (Hellman 2001, p.193)).

What about logic? Is there any analogy with mathematics, in the sense of being about (different) structures? Or is it maybe the case that logics share a universal structure? As is well known, different logics are based on different principles. Examples are legion. Let us just mention the case of relevance logic and its constraint of a necessary relevant connection between the premises and the conclusion in any argument, absolutely absent in classical logic.

Does it mean that the proposal of a common logical structure and, consequently, of a universal logic, is destined to fail? In this paper I will try to answer this question through a discussion of the tenets of the structuralist account of logic.

Koslow's structuralist account of logic

The lynch-pin of Koslow's structuralist account of logic is the notion of *implication structure* and the definition of logical (and modal) operators relative to an implication structure. Let us see what these definitions amount to and what results they imply.

An implication structure is any order pair: ((S, $\Rightarrow$); where S is a non-empty set, while "$\Rightarrow$" is an implication relation.

An implication relation is (implicitly) defined as any relation that satisfies the following conditions:

(1) *Reflexivity*: $A \Rightarrow A$, for each A in S

(2) *Projection*: $A_1, A_2, \ldots, A_n \Rightarrow A_k$, for every k=1, …, n, and for each A_i in S (i = 1, …,n)

(3) *Simplification*: If $A_1, A_1, A_2, \ldots, A_n \Rightarrow B$, then $A_1, A_2, \ldots, A_n \Rightarrow B$, for all A_i (i = 1, …,n) and B in S

(4) *Permutation*: If $A_1, A_2, \ldots, A_n \Rightarrow B$, then $A_{f(1)}, A_{f(2)}, \ldots, A_{f(n)} \Rightarrow B$, for any permutation f of $\{1, \ldots,n\}$

(5) *Dilution* (or *Thinning*): If $A_1, A_2, \ldots, A_n \Rightarrow B$, then $A_1, A_2, \ldots, A_n, C \Rightarrow B$, for any A_i (i = 1, …,n), B and C in S

(6) *Cut*: If $A_1, A_2, \ldots, A_n \Rightarrow B$, and $B, B_1, B_2, \ldots, B_m \Rightarrow C$, then $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m \Rightarrow C$, for any A_i, B_j, B and C (i, j = 1, …,n)

Someone might object that a different choice of constraints would be more fruitful and economical since clearly *Reflexivity* follows from *Projection*, and *Dilution* follows from *Projection* and *Cut*. Nevertheless, Koslow keeps the list of constraints for the sake of greater articulateness, based on Gentzen's theory.

Given the constraints, there are examples of implication relations that immediately come to mind, such as the notion of (semantic) consequence or the (syntactic) notion of deducibility for a set of sentences of some first-order logical theory. But, interestingly enough, these examples do not even remotely exhaust all the possibilities. Either in the sense of getting unusually defined logical operators**,** given a set of propositions of first-order logic or certain examples of logical operators defined on elements of S that are either not syntactic objects or truth bearers.

> The logical operators can act in a broad variety of settings, sentential and otherwise. In particular, the actions of the operators on structures of sets, names, and interrogatives, to cite just some nonstandard examples, are mentioned because the items in these cases fail in an obvious way to be syntactical or fail to be truth-bearers. (Koslow 1992, p.9)

Set inclusion, for example, given any set of subsets of a non-empty set S, also fulfils all the mentioned constraints, so $(S, \subseteq)$ exemplifies the implication structure. Other examples may be found in the context of the theory of individuals and erotetic logic (Koslow 1992, p. 209-229).

Such a definition might seem to be needlessly general, especially given its non-economicity, but this point is not lost on Koslow since its generality is more a virtue than a limitation. Analogously, it is possible to get some rather weird group structure examples or mathematically uninteresting equivalence relations. Of course, such examples might be more or less philosophically, mathematically or logically interesting and fruitful.

Given the definition of implication structures, the logical operators are defined *relative* to such structures, i.e. as functions defined on structures. And here again, given the possibility of non-standard implication relations, the same applies to the operators as well.

Let us take the example of the hypothetical operator $H_{\Rightarrow}$. For any elements A and B in the implication structure $(S, \Rightarrow)$, $H_{\Rightarrow}(A, B)$ is the hypothetical having A as the antecedent and B as a consequent, if and only if the following conditions are fulfilled:

$(H1)$ $A, H_{\Rightarrow}(A, B) \Rightarrow B$

(H2) $H_{\Rightarrow}(A, B)$ is the weakest element satisfying the condition (1). It means that, for any element T of the implication structure such that A, $T \Rightarrow B$, it follows that $T \Rightarrow H_{\Rightarrow}(A, B)$

Such a definition leaves open *per se* the answer to the question as to whether the hypothetical, given an implication structure, may fail to exist or not. And the following example solves the dilemma positively.

Let us take the implication structure (S, $\Rightarrow$), in which $S=\{A, B, C, D\}$ and the implication relation is given in the following way:

$$\begin{tikzcd}
 & A \arrow[d, Rightarrow] & \\
 & B \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
C & & D
\end{tikzcd}$$

In such a structure, the hypothetical $H_{\Rightarrow}(A, B)$ does not exist (not to be confused with the fact that $A \Rightarrow B$); namely, $H_{\Rightarrow}(A, B)$ is, by definition, the weakest member T of S such that: A, $T \Rightarrow B$. Since $A \Rightarrow B$, the condition is fulfilled by any element of S, but there is no weakest element. C cannot be the weakest element since A, $D \Rightarrow B$, while $D \not\Rightarrow C$ (see the condition (H2) above). D cannot be the weakest for the same reason. (See Koslow (1992) for more examples).

As easily noted, such a definition does not put any constraints on truth conditions, syntactic features or others; as Koslow points out:

> There is no appeal to truth conditions, assertibility conditions, or any syntactical features or semantic values of the elements of the structure. (Koslow 1992, p. 78)

The fact that the elements of an implication structure are not necessarily syntactic objects having a special sign design or elements having a semantic value, is what make the explanation/definition of the logical operators free of such constraints.

The structuralist account of logic – critical remarks

I will concentrate on certain aspects of the structuralist account of logic and make a number of quite general remarks.

First, if we have a look at the six conditions (defined as (1)*Reflexivity*,…,(6)*Cut*) that any relation has to fulfil in order to be an implication relation, we may ask how is the left-hand side of the expression $A_1, \dots, A_n \Rightarrow B$ to be construed? In the case in which S

is a non-empty set of sets and the implication relation is set inclusion, the sequence $A_1, \dots, A_n$ is the intersection of sets (Koslow 1992, p. 53). Since the intersection of sets *is* their conjunction, it turns out that in order to interpret the sequence $A_1, \dots, A_n$, i.e. in order to determine that the implication relation is set inclusion, we ought to know what the intersection, i.e. the conjunction of sets is. It follows that in certain cases we ought to know how a certain logical operator is defined prior to having determined an implication relation on a non-empty set. According to the structuralist account of logic, it should be the other way round.

Secondly, one of the most interesting results of the structuralist account of logic is the definition of the operators for interrogatives:

Let Q be a set of interrogatives

S - a set of sentences inclusive of the sentential direct answers to the questions in Q

$S' = S \cup Q$.

A direct answer need not be a true one:

> We shall use the term "interrogative" to include any question that has a *direct answer*. The most important feature of the direct answers to a question is that they are statements that, whether they are true or false, tell the questioner exactly what he wants to know – neither more nor less. (Koslow 1992, p. 220)

The implication relation on S' ('$\Rightarrow_q$') is defined as follows (Koslow 1992, pp.218-229):

Let $M_1?, M_2?, \dots, M_n?$ and $R?$ be questions in Q,

$F_1, F_2, \dots, F_m$ and G be statements of S (the set of M's or the set of F's may be empty but not both), and

A_i be a direct answer to the question $M_i?$ (i=1,…,n).

Then

(1.) $F_1, F_2, \dots, F_m, M_1?, M_2?, \dots, M_n? \Rightarrow_q R?$ if and only if there is some direct answer B to the question $R?$ such that

$F_1, F_2, \dots, F_m, A_1, A_2, \dots, A_n \Rightarrow B$

(2.) $F_1, F_2, \dots, F_m, M_1?, M_2?, \dots, M_n? \Rightarrow_q G$ if and only if

$F_1, F_2, \dots, F_m, A_1, A_2, \dots, A_n \Rightarrow G$

Such a definition is problematic. Let us see why. Let the set of F's be empty (for the sake of simplicity), and let us examine the case in which a question implies a statement (the second condition in the definition). Let the statement G be any false statement, e.g. a false answer to the question *R?*. In this case, whether $M_1? \Rightarrow_q G$ or not depends on whether $A_1 \Rightarrow G$, and the latter depends on what answer A_1 (to the question $M_1?$) we choose. If the answer we choose is a false one, then $M_1? \Rightarrow_q G$, otherwise $M_1? \not\Rightarrow_q G$. More generally, the same problem appears whenever the statement G is false. In this case, given a collection of interrogatives $M_i?$ (i=1,…n), their respective direct answers A_i, and a set of true statements F_i (i=1,…n), there is nothing in Koslow's definition that allows us to uniquely determine whether $F_1, F_2, \ldots, F_m, M_1?, M_2?, \ldots, M_n? \Rightarrow_q G$ or not.

Thirdly, let us consider a more general difficulty with the theory. Even though we might expect that certain results that hold given the operators classically defined hold in non-standard cases too, Koslow shows it is not the case. Let us mention the case**S** of implication structures in which $(((A\rightarrow B)\rightarrow A)\rightarrow A)$ is not a thesis or examples of structures in which the hypothetical with false antecedents can sometimes be true, and sometimes false (Koslow 1992, pp. 83-90).

These are features of the system, not of the structure**,** so it is not odd that such results are not necessarily present in non-standard implication structures. Nevertheless, having a true hypothetical whenever the antecedent is false (or having certain expressions as thesis) is not a marginal result given the operators classically defined. Two questions arise at this point: how do we get from the semantic-and-syntactic-features-free definitions to the syntactic rules for formula formation or the (semantic) truth tables, giving results that do not follow from the structurally defined operators? And how can a system have so many basic features that are not, in some form or another, already present in the structure?

Such problems make us think that the characterization of the operators relative to the implication structure does not, in itself, obtain the semantic and syntactic results we expect to get in the standardly defined implication structures. For these reasons Koslow's structuralist account does not satisfactorily solve the task of characterizing the logical operators relative to implication structures and even though tempting in many philosophical and logical aspects, it makes us sceptical as to whether a general logical structure might, after all, be defined.

Literature

- Hamilton, A. G. 1978 *Logic for Mathematicians*, Cambridge: Cambridge University Press.
- Hellman, G. 2001 ‘Three Varieties of Mathematical Structuralism’, *Philosophia Mathematica*, (3) Vol. 9, 184-211.
- Hirschman, L. and Gaizauskas, R. 2001 ‘Natural language question answering: the view from here’, *Natural Language Engineering* **7** (4), 275-300.
- Koslow, A. 1992 *A Structuralist Theory of Logic*, New York: Cambridge University Press.
- Koslow, A. 1999 *The Implicational Nature of Logic*: *A Structuralist Account*, in Varzi, A. C. (ed.), European Review of Philosophy, The Nature of Logic, CSLI Publications, Vol. 4, 111-155.
- Koslow, A. 2007 ‘Structuralist Logic: Implications, Inferences and Consequences’, *Logica Universalis* 1, 167-181.
- Nolt, J. 1997 *Logics*, Belmont: Wadsworth Publishing Company.
- Resnik, M. D. 1997 *Mathematics as a Science of Patterns*, Oxford: Clarendon Press.
- Scott, D. 1971 ‘On Engendering an Illusion of Understanding’, *Journal of Philosophy*, 68: 787-807.
- Shapiro, S. 1997 *Philosophy of Mathematics: Structure and Ontology*, New York: Oxford University Press.
- Szabo, M. E. 1969 *The Collected Papers of Gerhard Gentzen*, Amsterdam: North Holland.