RATIONAL TRANSFER FUNCTIONS WITH MINIMUM IMPULSE RESPONSE MOMENTS

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ABSTRACT

The lowpass systems with minimum higher order moments of the impulse response are presented. The systems have the largest possible energy concentration in time. The optimization of the transfer function parameters is carried out for the functions ranging from the second to the tenth order, with finite zeros. The optimum pole-zero positions suitable for filter design are given together with properties of the systems.

Keywords: filter design, impulse response moment, integral criterion, time domain optimization

1 INTRODUCTION

In many applications the systems with small time spread of the impulse response as well as small and short ringing are required. One approach is optimization of a particular property of the system time response that may be described by an integral criterion. If the criterion can be expressed by the system parameters through simple relations, it can be used not only for characterization but for optimization procedure as well. Here we propose the use of the impulse response moments as the integral criterion to find a class of filters with minimum time spread of the impulse response and small and short ringing. The first and the second order moments, namely centroid and standard deviation of the impulse response can be used, when impulse response is nonnegative, $h(t)\geq 0$, [1]:

$$t_d = \frac{m_1}{m_0} = \frac{\int_0^\infty t\, h(t)\, dt}{\int_0^\infty h(t)\, dt} \quad \text{and} \quad \tau^2 = 2\pi \frac{\int_0^\infty (t-t_d)^2\, h(t)\, dt}{\int_0^\infty h(t)\, dt} \; . \tag{1}$$

These expressions can be used as integral criteria for system optimization, [2].

However, if h(t) is not nonnegative, the resulting central moment can become small, not only because of the small time spread, but because the positive, $h(t)>0$, and the negative contribution, $h(t)<0$, in the integrals (1) may partly cancel each other. It seams that the choice of the absolute value $|h(t)|$ in (1) is better, but unfortunately, it is not easy to work with, i.e. express response parameters by the transfer function parameters. The central moment of $|h(t)|^2$, which gives the power spread along the time axis, is more tractable. Therefore, we will use even central moments of this function to minimize the impulse response spread. Using central moments of the higher order as the integral criterion the ringing with smaller amplitude and duration can be achieved. The n-th order moments of the squared impulse response around centroid t_m are given by

$$m_n = \int_0^\infty (t - t_m)^n\, h^2(t)\, dt \; . \tag{2}$$

The parabola $(t-t_m)^n$ is in fact the weighting function that will "punish" ringing more efficiently for higher n.

2 MOMENT AND TRANSFER FUNCTIONS

We define a measure of impulse response spread by the central moment (2), normalized to impulse response energy, which is, in fact, the zeroth moment

$$E_n = \frac{m_n}{m_0} \; . \tag{3}$$

For the optimization procedure in the complex domain, the criterion E_n should be expressed by the transfer function poles p_i and zeros z_i. The impulse response of the N-th order filter with simple poles is given by

$$h(t) = \sum_{r=1}^{N} K_r e^{p_r t} \;, \quad K_r = H_0 \frac{\prod_{i=1}^{M} (p_r - z_i)}{\prod_{\substack{j=1 \\ j\neq r}}^{N} (p_r - p_j)} \;, \tag{4}$$

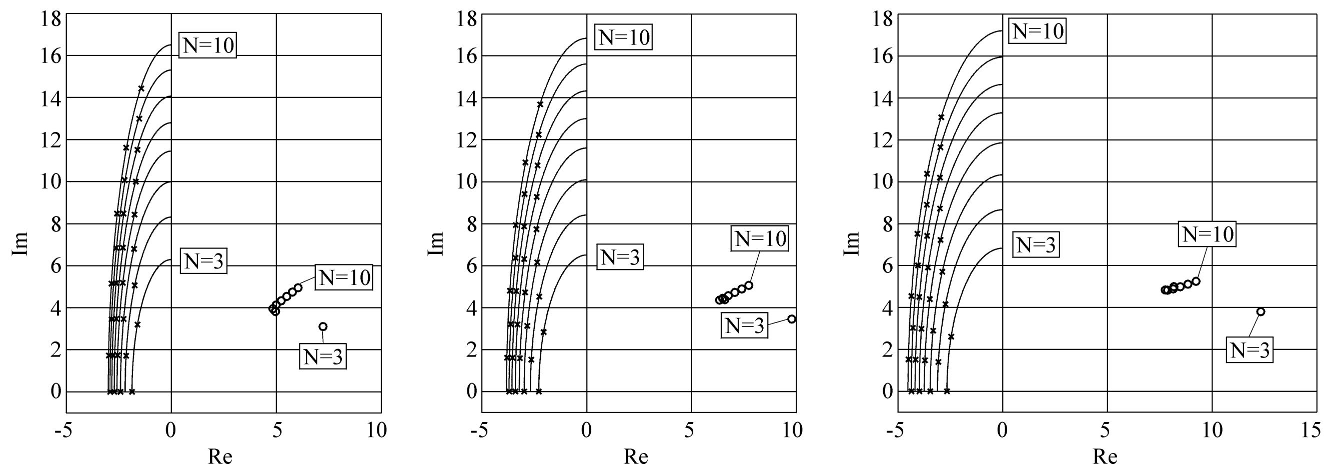

Figure 1 Zero pole positions of the systems with minimum second, fourth and sixth order moments, with one pair of complex zeros, t_m=1.

where the pole residues are K_r, r=1,2,...,N. Now, the n-th central moment can be expressed as function of poles, zeros and residues as

$$m_n = (-1)^{n+1} \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \sum_{k=0}^{n} \frac{n!}{k!} \frac{t_m^k}{(p_i + p_j)^{n-k+1}} \ . \quad (5)$$

3 OPTIMIZATION PROCEDURE

The positions of poles and zeros of causal filters with the most compact impulse response can be found by solving the problem

$$\min_{z_i, p_j} E_2[z_i, p_j] \ . \quad (6)$$

In our optimization procedure the frequency ω_p and ω_z and quality factor Q_p and Q_z were used, instead of the poles and zeros, p_j and z_i. Using ω_p, Q_p, ω_z and Q_z a rational transfer function with even number of zeros can be written in the form

$$H(s) = H_0 \frac{\prod_{i=1}^{M/2} (s^2 - \frac{\omega_{zi}}{Q_{zi}} s + \omega_{zi}^2)}{\prod_{j=1}^{N/2} (s^2 + \frac{\omega_{pj}}{Q_{pj}} s + \omega_{pj}^2)} \quad \text{and}$$

$$H(s) = H_0 \frac{\prod_{i=1}^{M/2} (s^2 - \frac{\omega_{zi}}{Q_{zi}} s + \omega_{zi}^2)}{(s + \omega_{p0}) \prod_{j=1}^{(N-1)/2} (s^2 + \frac{\omega_{pj}}{Q_{pj}} s + \omega_{pj}^2)} \quad (7)$$

for N even and N odd, respectively.

In a stable system ω_p and Q_p are positive. Positive values of ω_z and Q_z in (7) will give zeros in the right half plane. Square values of goal function variables were employed rather than constrained optimization procedure. Finally, optimum system poles and zeros were found as

$$\min_{\omega_p, Q_p, \omega_z, Q_z} E_2[\omega_p^2, Q_p^2, \omega_z^2, Q_p^2] \ . \quad (8)$$

Optimization will force impulse response to concentrate around t_m and practically extend to $2t_m$. The parameter t_m is chosen to be 1 that will not change the generality of the solution.

For searching minimum Quasi-Newton method with BFGS formula for Hessian matrix update [3] was used. We have obtained previously parameters of the all pole transfer functions [4]. Here we optimize transfer functions with one pair of complex zeros for systems up to the tenth order. We found in [5] that more zeros than one pair do not improve the system properties significantly.

4 OPTIMIZATION RESULTS

Numerical values of the rational transfer function parameters ω_p, Q_p, ω_z and Q_z for one pair of complex zeros are given in Table 1 to Table 4. For the all pole transfer functions, the numerical values of pole positions can be found in [4].

For the rational transfer functions with t_m=1, the examples of pole-zero positions are shown in Figure 1. It is interesting to note that the poles are located very closely to ellipses that have the common center at the complex plane origin. To illustrate the behavior of the filter class with one pair of complex zeros, the response curves in the time and frequency domain are given for the systems with minimum fourth order moment.

Table 1 Transfer function parameters of systems with one pair of complex zeros, n=2.

N	ω_p	Q_p	ω_z	Q_z	ω_{3dB}
3	3.5801 1.8684	1.1122	7.8547	0.5442	3.0765
4	5.3578 2.7472	1.5391 0.6394	6.2596	0.6314	3.9624
5	7.0263 4.1418 2.4171	1.9830 0.9131	6.2471	0.6459	4.5278
6	8.6126 5.6760 3.0909	2.4651 1.2291 0.6056	6.4843	0.6491	5.5917
7	10.1409 7.2342 4.3469 2.7114	3.0016 1.5695 0.8318	6.7937	0.6493	6.2190
8	11.6260 8.7817 5.8056 3.2921	3.6055 1.9335 1.1035 0.5885	7.1269	0.6486	6.8803
9	13.0770 10.3086 7.3278 4.4696 2.9080	4.2889 2.3254 1.3982 0.7878	7.4664	0.6477	7.4870
10	14.5002 11.8130 8.8636 5.8829 3.4345	5.0642 2.7506 1.7111 1.0333 0.5781	7.8050	0.6467	8.2661

Table 2 Transfer function parameters of systems with one pair of complex zeros, n=4.

N	ω_p	Q_p	ω_z	Q_z	ω_{3dB}
3	3.5028 2.2828	0.8525	10.3705	0.5303	2.5989
4	5.0583 3.0500	1.1168 0.5775	7.9079	0.6004	3.2554
5	6.5958 4.2213 2.9815	1.3955 0.7442	7.6887	0.6069	3.8460
6	8.1025 5.5686 3.5583	1.6913 0.9477 0.5589	7.8578	0.6059	4.4629
7	9.5794 6.9845 4.5923 3.4026	2.0085 1.1711 0.6955	8.1532	0.6037	4.9717
8	11.0307 8.4243 5.8522 3.8873	2.3513 1.4102 0.8699 0.5490	8.4980	0.6014	5.5012
9	12.4589 9.8687 7.2157 4.8410 3.6982	2.7229 1.6646 1.0650 0.6682	8.8639	0.5994	5.9888
10	13.8675 11.3081 8.6259 6.0470 4.1286	3.1283 1.9355 1.2744 0.8246 0.5430	9.2370	0.5976	6.4530

The impulse response is given in Figure 2. Compared to the all-pole systems [4] the presence of zeros improves the response symmetry, decreases the response spread and causes the ringing before the main pulse. The impulse response here is also bell-shaped with shorter duration and somewhat larger ringing than in the all-pole case [4]. The step response is shown in Figure 3. The overshoot is smaller than 1% already at n=4. Thus we may consider the step response very nearly monotonic for n≥4.

The amplitude and the group delay responses, normalized to $\omega_{3dB}=1$, are shown in Figure 4 and Figure 5 in the form suitable for comparison with the classic filter approximations, given for example in [6]. The amplitude response is quasi gaussian. The group delay curves illustrate approximation of a constant despite no requirements were given in the frequency domain. The bandwidth of quasi-constant group delay is extending well beyond cutoff frequency ω_{3dB}. Compared to [4] the zeros improve the average slope of the group delay within the passband.

The optimization results for all moment orders are similar in character to the systems of fourth order described above. The impulse response shows smaller undershoots for higher moment n, as it can be seen in Figure 6. The step response overshoots are also smaller for higher order moments, as shown in Figure 7. For example at n=8 the overshoot is smaller than 0.17%. Therefore, the step response is almost monotonic for n>4 what is a consequence of used integral criterion.

Shorter rise-time causes the wider passband so the rise-time bandwidth product $t_r\omega_{3dB}$ is spread from 2.45 to 2.17 for n=2 and 2.19 to 2.17 for n=8, for various system orders N.

The amplitude attenuation in the stop band is higher for lower moments, Figure 8. The group delay ripple is decreasing for higher moments and become smooth for n≥6 as can be seen from Figure 9.

Table 3 Transfer function parameters of systems with one pair of complex zeros, n=6.

N	ω_p	Q_p	ω_z	Q_z	ω_{3dB}
3	3.5794 2.6576	0.7284	12.8987	0.5232	2.4229
4	4.9706 3.3707	0.9115 0.5496	9.5801	0.5859	2.9558
5	6.3883 4.3985 3.4554	1.1084 0.6628	9.1467	0.5891	3.4561
6	7.8094 5.6053 3.9925	1.3171 0.8070 0.5382	9.2331	0.5863	3.9401
7	9.2242 6.9037 4.8905 3.9750	1.5380 0.9686 0.6310	9.4985	0.5830	4.3881
8	10.6294 8.2483 6.0064 4.4209	1.7728 1.1429 0.7547 0.5319	9.8391	0.5800	4.8232
9	12.0234 9.6155 7.2441 5.2408 4.3555	2.0223 1.3278 0.8964 0.6125	10.2158	0.5775	5.2350
10	13.4042 10.9915 8.5486 6.2989 4.7465	2.2878 1.5229 1.0503 0.7228 0.5279	10.6172	0.5752	5.6301

Table 4 Transfer function parameters of systems with one pair of complex zeros, n=8.

N	ω_p	Q_p	ω_z	Q_z	ω_{3dB}
3	3.7362 3.0089	0.6597	15.4225	0.5190	2.3743
4	5.0049 3.6931	0.7952 0.5347	11.2555	0.5776	2.8116
5	6.3214 4.6258 3.8768	0.9440 0.6169	10.6035	0.5791	3.2357
6	7.6625 5.7265 4.3969	1.1026 0.7253 0.5271	10.5986	0.5754	3.6460
7	9.0142 6.9284 5.2078 4.4734	1.2700 0.8492 0.5949	10.8269	0.5715	4.0378
8	10.3686 8.1896 6.2183 4.9042	1.4464 0.9840 0.6880 0.5227	11.1653	0.5680	4.4150
9	11.7252 9.4889 7.3574 5.6404 4.9247	1.6322 1.1274 0.7970 0.5817	11.5176	0.5655	4.7801
10	13.0849 10.8142 8.5788 6.5944 5.2992	1.8272 1.2791 0.9174 0.6649 0.5200	11.8499	0.5641	5.1373

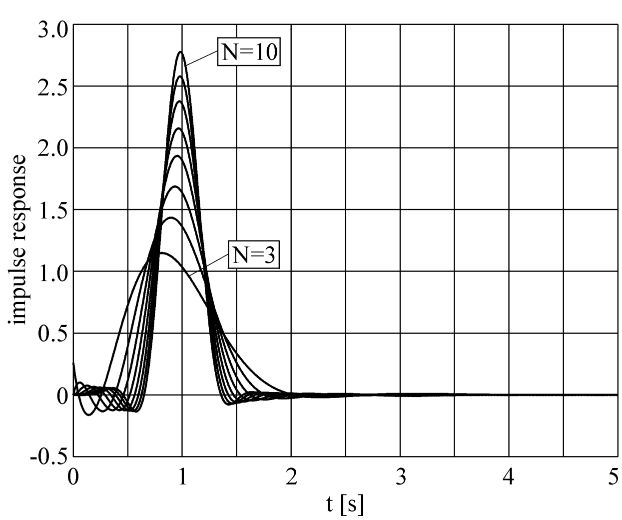

Figure 2. Impulse response of the optimum systems based on the fourth moment, M=2, t_m=1.

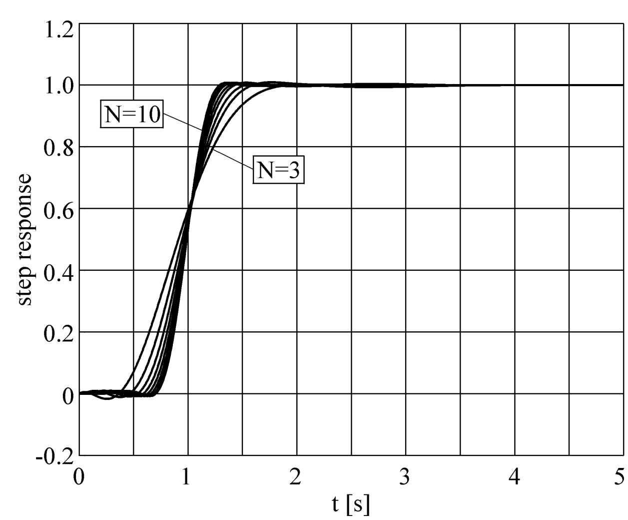

Figure 3. Step response of the optimum systems based on the fourth moment, M=2, t_m=1.

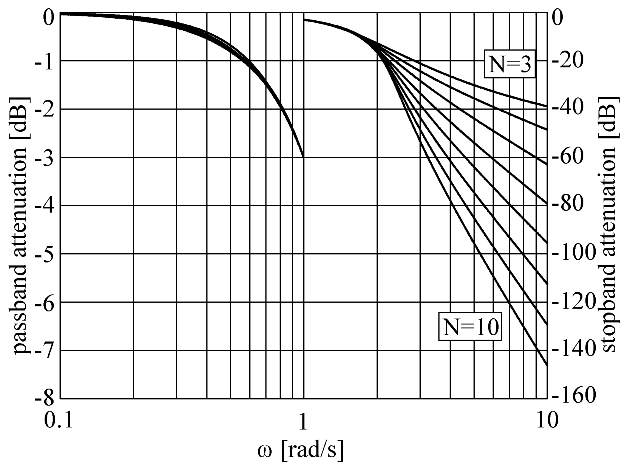

Figure 4. Amplitude response of the optimum systems based on the fourth moment, M=2, ω_{3dB}=1.

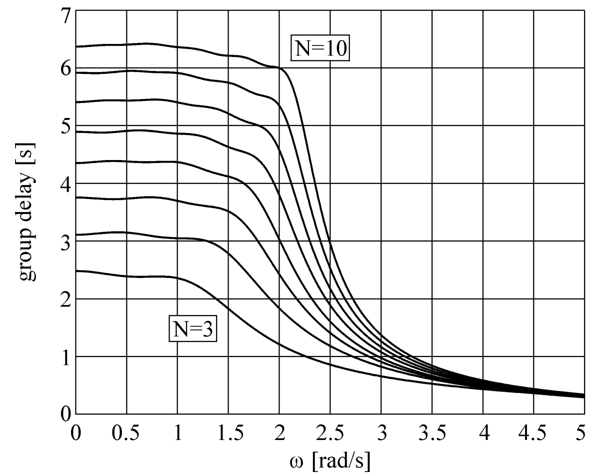

Figure 5. Group delay of the optimum systems based on the fourth moment, M=2, ω_{3dB}=1.

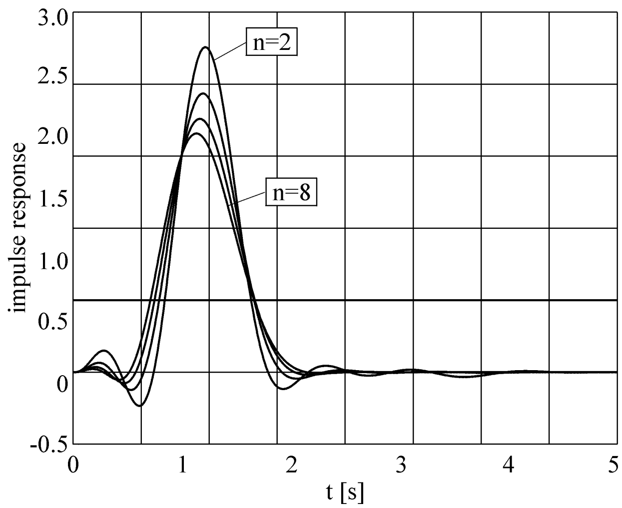

Figure 6. Impulse response of the optimum systems for various moment orders, N=6, M=2, t_m=1.

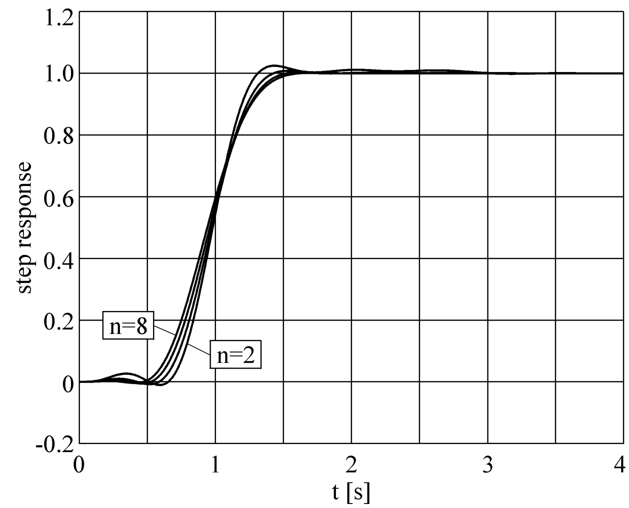

Figure 7. Step response of the optimum systems for various moment orders, N=6, M=2, t_m=1.

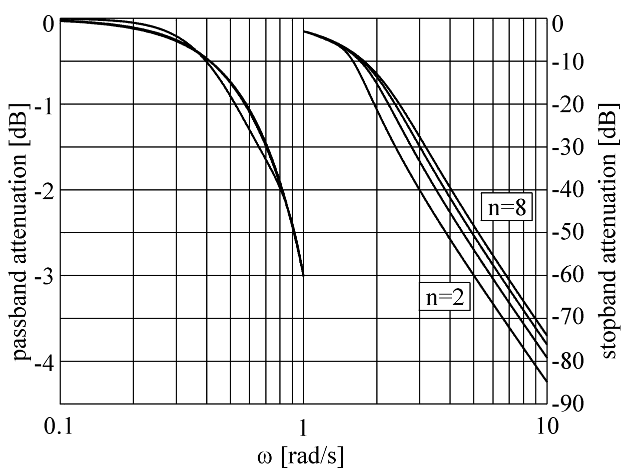

Figure 8. Amplitude response of the optimum systems for various moment orders, N=6, M=2, ω_{3dB}=1.

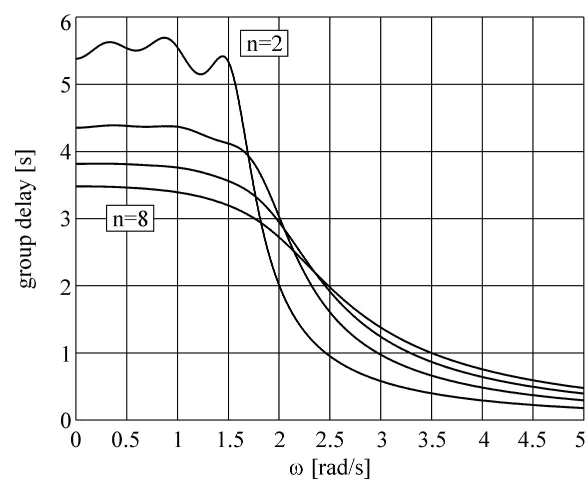

Figure 9. Group delay of the optimum systems for various moment orders, N=6, M=2, ω_{3dB}=1.

5 CONCLUSION

By minimization the higher order moments of the squared impulse response, a new class of finite order systems is obtained. Based on the used criterion, the obtained filters have the largest possible energy concentration in time. The impulse response is short and has small and short ringing giving very nearly monotonic step response for higher moments. The amplitude response is quasi gaussian, with nearly constant group delay within passband. The presence of zeros improves the impulse response energy concentration and the group delay as compared to the all-pole systems. The optimal pole-zero positions are given in this paper together with properties of the systems, so the filter design can be carried out. The obtained systems can be favorably compared to similar systems with linear phase optimized in the frequency domain.

6 ACKNOWLEDGEMENT

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