SYSTEMS WITH MINIMUM TIME-BANDWIDTH PRODUCT BASED ON THE HIGHER ORDER MOMENTS

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ABSTRACT

Time functions whose duration and bandwidth are as small as possible have practical as well as theoretical interest. Many criteria are employed for calculation of such time functions. Moments of the impulse response and moments of the frequency response can be used for characterization of response time and frequency band. The second moments were used for the definition of the uncertainty principle. In this paper response functions of finite order systems are characterized not only by the second but also with fourth, sixth and eight moments. To find out properties of obtained functions and values of time frequency band products, a numerical optimization is carried out for all-pole transfer functions, up to the eight orders. Properties of obtained systems are given and compared.

Keywords: filter design, time-bandwidth product, impulse response moments, higher order moments

1 INTRODUCTION

The transmission rate of digital signal trough the given channel bandwidth will be determined by the signal pulse duration [1]. As short signals require wide bandwidth, the optimum is a compromise given by minimum of the time and frequency product

$$P_n = \alpha_n \beta_n \ . \tag{1}$$

Various measures might be used for time spread α_n and frequency spread β_n [2]. Here we use high order moments n=2, 4, 6 and 8 for spread characterization and also as the integral criterion for optimization

$$\alpha_n^n = \frac{\int_{-\infty}^{\infty} (t - t_m)^n \, h^2(t)\,dt}{\int_{-\infty}^{\infty} h^2(t)\,dt} \ , \tag{2}$$

$$\beta_n^n = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^n \left|H(\omega)\right|^2 d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|H(\omega)\right|^2 d\omega} \tag{3}$$

Uncertainty principle is defined by the second moments. For noncausal signals the product satisfy expression [2]

$$P_2 \geq \frac{1}{2} \ . \tag{4}$$

The equality value is obtained for Gauss function. For causal signals a similar relation and optimum functions has been found [3]. The real, finite order systems will only approximate the mentioned optimum functions obtained with the second moment [3], [4]. We have found that the use of higher moments could give additional insight in the systems with minimum time-frequency products. It could also give real systems with small and short response ringing.

Higher moments, similar as the second moment, enable a simple mathematical treatment and optimization.

2 MOMENTS AND TRANSFER FUNCTIONS

Time spread and bandwidth definitions (2) and (3) are suitable for causal functions with corresponding change of integral limits. Thus, we define a measure of impulse response spread by the n-th order central moment, and bandwidth by the n-th order moment, both normalized to the impulse response energy.

For optimization procedure in the complex domain, the criterion (1) should be expressed by the transfer function poles p_i, and zeros, z_i. The N-th order filter impulse response for simple poles, and M<N zeros, is given by

$$h(t)=\sum_{r=1}^{N} K_r e^{p_r t} \quad , \quad K_r = H_0 \frac{\prod_{i=1}^{M}(p_r - z_i)}{\prod_{\substack{j=1\\ j\neq r}}^{N}(p_r - p_j)} \quad , \tag{5}$$

where the pole residues are K_r, r=1,2,...,N. Now, the n-th moment of the impulse response can be expressed as function of poles, zeros and residues as

$$m_n = (-1)^{n+1} \sum_{i=1}^{N} \sum_{j=1}^{N} K_i K_j \sum_{k=0}^{n} \frac{n!}{k!} \frac{t_m^k}{(p_i + p_j)^{n-k+1}} \quad . \tag{6}$$

Impulse response energy can be obtained using (6) assuming n=0.

The n-th moment of the frequency response can be expressed by the impulse response derivative, using Parseval's relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^n |H(\omega)|^2 d\omega = \int_{-\infty}^{\infty} \left[h^{(n/2)}(t)\right]^2 dt \quad . \tag{7}$$

Expression (7) can be computed using (6) assuming transformed derivative

$$L\left\{h^{(n/2)}(t)\right\} = s^{n/2} H(s) \ , \quad n=2,\ 4,\ 6,\ 8 \tag{8}$$

where residues Q_r of the transform (8) are given by

$$h^{(n/2)}(t)=\sum_{r=1}^{N} Q_r e^{p_r t} \quad , \quad Q_r = H_0 \frac{p_r^{n/2}\prod_{i=1}^{M}(p_r - z_i)}{\prod_{\substack{j=1\\ j\neq r}}^{N}(p_r - p_j)} \quad , \tag{9}$$

To ensure convergence of moment integral (3), the number of zeros and poles should satisfy inequality

$$2M + n \le 2N - 2 \text{, or } N \ge M + 1 + n/2 \quad . \tag{10}$$

3 OPTIMIZATION PROCEDURE

Poles and zeros positions of causal filters with minimum time-bandwidth product can be found solving the problem

$$\min_{z_i, p_j} P_n^n\left[z_i, p_j\right] \quad . \tag{11}$$

Although expressions (1) to (11) can be applied to any linear stable system with simple poles and zeros, in this paper only all-pole transfer functions are considered. Furthermore, in our optimization procedure the frequency ω_p and quality factor Q_p of poles were used, instead of the poles, p_j. This variable set enables the pole position on the whole complex plane. Using ω_p and Q_p, an all-pole transfer function can be written in the form

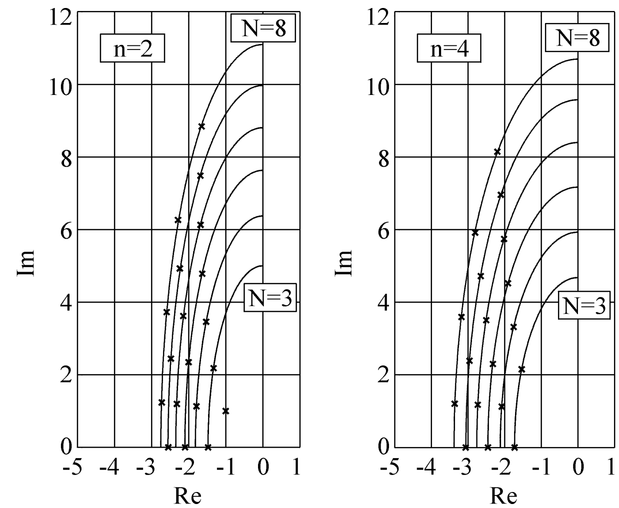

Figure 1. Pole positions of the optimum systems with n=2 and n=4, normalized to t_m=1.

$$H(s)=\frac{H_0}{(s+\omega_{p0})\prod_{i=1}^{(N-1)/2}(s^2+\frac{\omega_{pi}}{Q_{pi}}s+\omega_{pi}^2)} \quad , \tag{12}$$

when N is odd, and similar when N is even.

In a stable system, ω_p and Q_p are positive. A square values of goal function variables, rather than constrained optimization procedure were used. Finally, optimum system poles were found as

$$\min_{\omega_p, Q_p} P_n^n\left[\omega_p^2, Q_p^2\right] \quad . \tag{13}$$

For searching minimum Quasi-Newton method with BFGS formula for Hessian matrix update was used [5]. In each iteration a bisection type line search was performed followed by quadratic interpolation. In situations when Hessian matrix had irregular inverse, a steepest descent method was forced by setting Hessian matrix to identity.

To get causal filters with the minimum time-bandwidth product, the optimization is carried out for moments n=2, 4, 6 and 8, and system order, N=2 to 8. The parameter is taken t_m=1 what will not change the generality of the problem.

4 OPTIMIZATION RESULTS

The numerical values of the pole parameters ω_p and Q_p, are given in Table I and Table II.

For all-pole transfer functions with t_m=1, the examples of pole position are shown in Figure 1. It is interesting to note that poles are very nearly placed on ellipses with ellipses center located at the complex plane origin, similar as in the case of the systems with symmetric impulse response [6].

Table I. Transfer function parameters for t_m=1.

	n=2			n=4		
N	ω_p	Q_p	ω_{3dB}	ω_p	Q_p	ω_{3dB}
2	1.4142	0.7071	1.4142			
3	2.5543 1.4767	0.9611	1.8233	2.6445 1.7286	0.8616	1.8671
4	3.7816 2.1158	1.2387 0.5914	2.1697	3.7542 2.3641	1.0694 0.5685	2.2212
5	5.0551 3.0847 2.0967	1.5460 0.7702	2.5029	4.9082 3.2689 2.4569	1.2876 0.7040	2.5534
6	6.3547 4.2073 2.6087	1.8903 0.9802 0.5626	2.8122	6.0861 4.3071 2.9785	1.5101 0.8608 0.5446	2.8701
7	7.6694 5.4122 3.4795 2.5499	2.2798 1.2100 0.7019	3.0736	7.2713 5.4141 3.8019 3.0616	1.7277 1.0212 0.6426	3.1538
8	8.9926 6.6676 4.5385 2.9909	2.7233 1.4586 0.8756 0.5491	3.3523	8.4373 6.5484 4.7976 3.5755	1.9187 1.1675 0.7547 0.5312	3.4241

4.1 The fourth moment system

To illustrate behavior of the considered class of systems, the complete data are given for the system with minimum product of fourth moments in the time and the frequency domain. Impulse response is shown in Figure 2. It is a bell-shaped response, with small time spread and undershoots. For some applications it is interesting to see the step response which is shown in Figure 3 for systems normalized to t_m=1. The overshoots are bellow 0.70 %.

Amplitude and group delay responses are shown in Figure 4 and Figure 5, respectively, in a form suitable for comparison with classic filter approximations, given for example in [7]. The amplitude response is quasi gaussian. The group delay curves illustrate an approximation of a constant. The bandwidth of quasi-constant group delay is extending well beyond cutoff frequency ω_{3dB}.

4.2 Comparison of systems with minimum 2nd, 4th, 6th and 8th moment

The optimization results for all moment orders are similar in character to the systems of fourth moment described above. Impulse response shows shorter ringing for the higher moments, n, as it is expected from the used criteria (3). The higher moments with weighting function of higher power, in fact, "punish" more the "tails" of the responses. Bandwidth is also wider for higher moments. Undershoot of the impulse response is smaller than 2 % and overshoot of the step response is smaller than 0.7 % for all moments and N≥3. Thus the step response is very nearly monotonic.

Table II. Transfer function parameters for t_m=1.

	n=6			n=8		
N	ω_p	Q_p	ω_{3dB}	ω_p	Q_p	ω_{3dB}
4	3.8479 2.6030	0.9339 0.5517	2.1919			
5	4.9399 3.4584 2.7602	1.1041 0.6567	2.4995	5.0367 3.6729 3.0613	0.9743 0.6235	2.4651
6	6.0663 4.4384 3.2653	1.2839 0.7838 0.5342	2.7988	6.1182 4.6065 3.5642	1.1175 0.7257 0.5268	2.7390
7	7.2120 5.4924 4.0287 3.3835	1.4685 0.9207 0.6133	3.0805	7.2296 5.6144 4.2844 3.7217	1.2691 0.8395 0.5904	3.0060
8	8.3648 6.5899 4.9535 3.8385	1.6530 1.0604 0.7133 0.5257	3.3510	8.3597 6.6733 5.1541 4.1556	1.4261 0.9601 0.6743 0.5207	3.2655

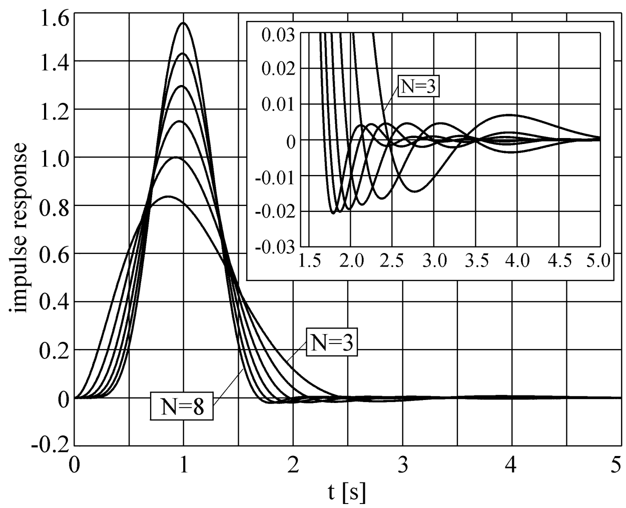

Figure 2. Impulse response of the optimum systems based on fourth moment, t_m=1.

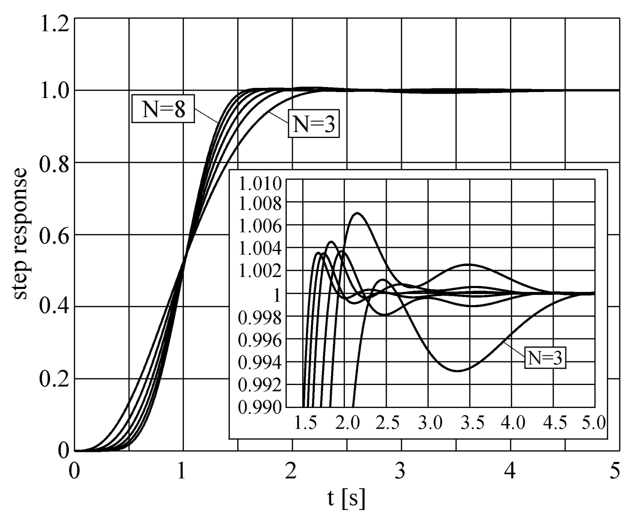

Figure 3. Step response of the optimum systems based on fourth moment, t_m=1.

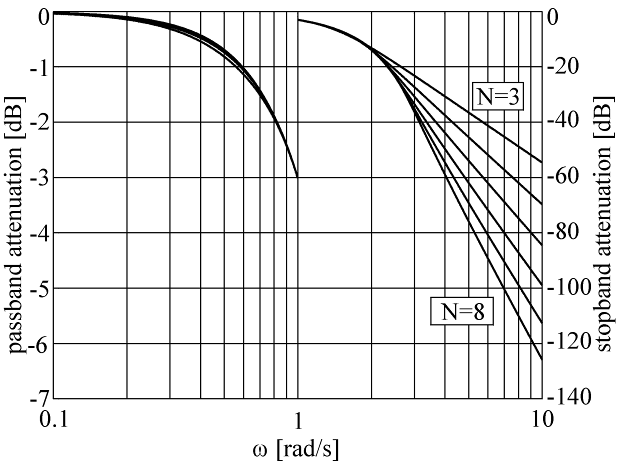

Figure 4. Amplitude response of the optimum systems based on fourth moment, ω_{3dB}=1.

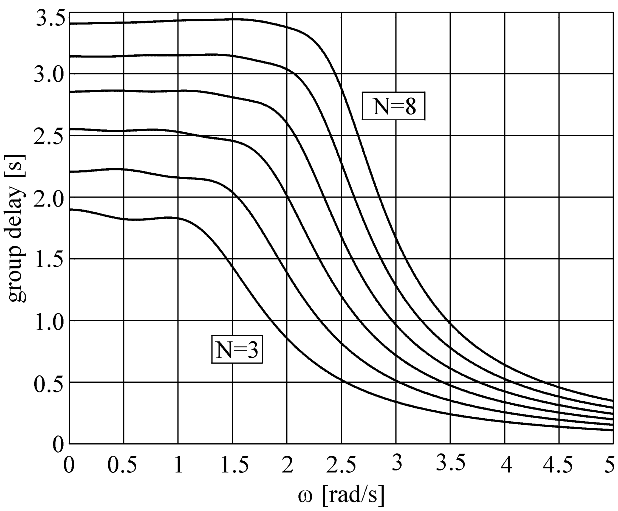

Figure 5. Group delay of the optimum systems based on fourth moment, ω_{3dB}=1.

Amplitude attenuation in stop band is higher for lower moments, and it is generally higher than response of Bessel and Gaussian filters, but smaller than at filters with equiripple phase filters [7] or filters with symmetric impulse response [6]. Group delay approximate constants with curves which are getting monotonic for higher moments.

The time-bandwidth product P_n, for various moments n, and system orders N, are shown in Figure 6. The products asymptotically converge to steady values for large N. For the second moment it is well-known limit p_2=1/2, (4), while other asymptotic values apparently depend on used moment order.

5 CONCLUSION

The numerical optimization of time-bandwidth product is minimized for systems of finite order. Product is expressed by higher moments of squared impulse response and squared frequency response. By this, the uncertainty principle for real systems is extended, and the new classes of systems with optimal properties are obtained. These classes for various moments approximate the quasi gaussian impulse response with negligible amplitude and duration of ringing. Amplitude responses is also gaussian, with nearly constant group delay in the passband.

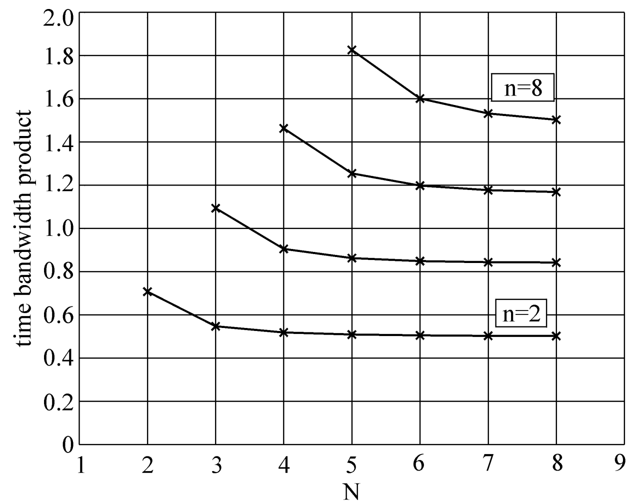

Figure 6. Time-bandwidth products for the optimum systems with n=2, 4, 6, and 8.

6 ACKNOWLEDGEMENT

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