

Comparison of Operator–Fitted methods for Singularly Perturbed Advection–Diffusion–Reaction Problems

Iva Kavčić¹, Mladen Rogina² and Tina Bosner²

¹Department of Geophysics, Faculty of Science, University of Zagreb
Horvatovac bb, 10000 Zagreb, Croatia
e-mail: ivakavc@gfz.hr

² Department of Mathematics, Faculty of Science, University of Zagreb
Bijenička cesta 30, 10000 Zagreb, Croatia
e-mail: {rogina,tinab}@math.hr

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Abstract. *We compare classical methods for singular perturbation problems, such as El–Mistikawy and Werle scheme and its modifications, to exponential spline collocation schemes. We discuss subtle differences that exist in applying this method to reaction–diffusion problems and advection–diffusion problems. If the advection–diffusion–reaction problem is specified in such a way that two boundary internal layers exist, collocation method is incapable of capturing only one boundary layer, which happens when no reaction term is present. Thus the existing collocation scheme in which the approximate solution is a projection to the space piecewisely spanned by $\{1, x, \exp(\pm px)\}$ is inferior to the generalization of El–Mistikawy and Werle method proposed by Ramos. We show how to remedy this situation by considering projections to spaces locally spanned by $\{1, x, x^2, \exp(px)\}$, where $p > 0$ is a tension parameter. Next, we exploit a unique feature of collocation methods, that is, the existence of special collocation points which yield better global convergence rates and double the convergence order at the knots.*

1 Theory

We compare some popular methods for solving a singularly perturbed boundary value problem

$$-\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x), \quad u(0) = \alpha, \quad u(1) = \beta, \quad (1)$$

where $\varepsilon, a(x) > 0$, $\varepsilon \ll a(x)$ and $b(x) \geq 0$. On one side is El–Mistikawy method, on the other a few collocation methods. Depending on the presence of advection or reaction term, there seems to be no unique answer as to the ‘best’ possible method in all the cases. As for El–Mistikawy method, a number of references exist, see [8] for more recent ones. This method

is in the class of operator fitted methods, in fact a projection of finite-element type to the space of splines associated with the dual of the operator $-\varepsilon D^2 + aD$, see [7]. Direct collocation methods are projections to spaces locally spanned by polynomials or combination of exponentials and polynomials, in practice those that belong to $\text{Ker}(D^4)$ or $\text{Ker}(D^2(D^2 - p^2))$, where $p > 0$ is the 'tension parameter', to be determined *a priori* from asymptotic arguments. We do not consider collocation by polynomial B-splines, since it would require Shiskin meshes for convergence, and is not in the class of operator-fitted methods. For collocation by combination of exponentials and powers, the one that projects to $L\{1, x, \exp(\pm px)\}$ [3, 2] is suitable whenever there is a reaction term present; if there is none, we propose to use $L\{1, x, x^2, \exp(px)\} = \text{Ker}(D^4 - pD^3)$ instead; the associated splines we call advection-diffusion splines, or AD-splines for short. In both cases on non-polynomial collocation, the underlying spline spaces are weak Chebyshev spaces [5], which allow fast evaluation of local basis by knot insertion [6]. For spline collocation, two variants exist, depending on the global smoothness: C^2 , in which case we collocate at the knots, and C^1 , where collocation points must be placed between the knots. In the C^1 case, following the ideas in [1], we can explicitly find collocation points which are optimal, *i.e.* yield maximal convergence rates at the knots. In the sequel, we shall consider C^1 collocation only.

2 Examples

Here we show only selected examples; the first one is the advection-diffusion problem

$$\varepsilon u''(x) - u(x)' = \exp(x), \quad u(0) = u(1) = 0, \quad (2)$$

$$u^{exact}(x) = \frac{e^{x/\varepsilon}(1-e) + e^x(e^{1/\varepsilon}-1) + e - e^{1/\varepsilon}}{(e^{1/\varepsilon}-1)(\varepsilon-1)}, \quad (3)$$

and the second one is the reaction-diffusion problem

$$-\varepsilon u''(x) + u(x) = 1, \quad u(0) = u(1) = 0, \quad (4)$$

$$u^{exact}(x) = \frac{e^{A(x-1)} - e^{-A(x-1)} - e^{Ax} + e^{-Ax} + e^A - e^{-A}}{e^A - e^{-A}}, \quad A = \frac{1}{\sqrt{\varepsilon}}. \quad (5)$$

Let h be the stepsize, $h = 1/N$, where N is the number of points for EMW (El-Mistikawy and Werle scheme), RAM (Ramos modification, see [4]) and FDM (Finite Difference Method) methods, and the number of knots for C1 and AD-splines.

The problems (2) and (4) were tested numerically by changing the parameters ε and N as

$$\varepsilon = 2^{-k}, \quad k = 2, 3, \dots, 14, \quad N = 2^m, \quad m = 4, 5, \dots, 12, \quad (6)$$

for every method except RAM. It seems that RAM method is not applicable to problem (2) where the reaction term, $b(x)$, is zero, despite the explicit condition $b(x) \geq 0$ in [4] (here it would be $-b(x) \geq 0$ because of the slightly different definitions of BVP (1)). Also, there are no numerical tests of RAM method in [4] applied to the pure advection-diffusion problem. However, RAM method is superior to classical EMW for the reaction-diffusion problems. Absolute errors for the above described test were calculated as

$$E_a = \max_i (u_i^{num} - u_i^{exact}), \quad i = 0, \dots, N. \quad (7)$$

Comparison between EMW and AD method reveals that overall error is in favor to AD, in the sense of the order of accuracy and behavior for small ε . Also, as it can be seen from Fig. 2,

AD is superior both to the classical C1 tension splines and EMW method in case of advection–diffusion problem (2). However, in case of pure reaction–diffusion problem (4), AD does not reproduce well the left boundary condition (not shown here). Contrary to this, RAM and C1 are far better for reaction–diffusion problem, with errors in favor to RAM method, especially for the combination of small ε and N (see Fig. 3), while EMW fails to reproduce both boundary conditions and is even worse than the simple FDM upwind method (also not shown).

3 Conclusions

As can be seen from the above examples, AD is superior for the pure advection–diffusion problem to classical C1 tension splines and EMW method (its max error is also much smaller than max error for C1 tension splines). However, in case of pure reaction–diffusion problem AD fails to reproduce properly the left boundary condition, what is expected. RAM and C1 are considerably better here. For the advection–diffusion–reaction problem AD method is better than EMW, C1 and RAM, especially for small ε , as long as the condition $a(x) > 0$ is met. Other methods are not so sensitive to the sign of $a(x)$.

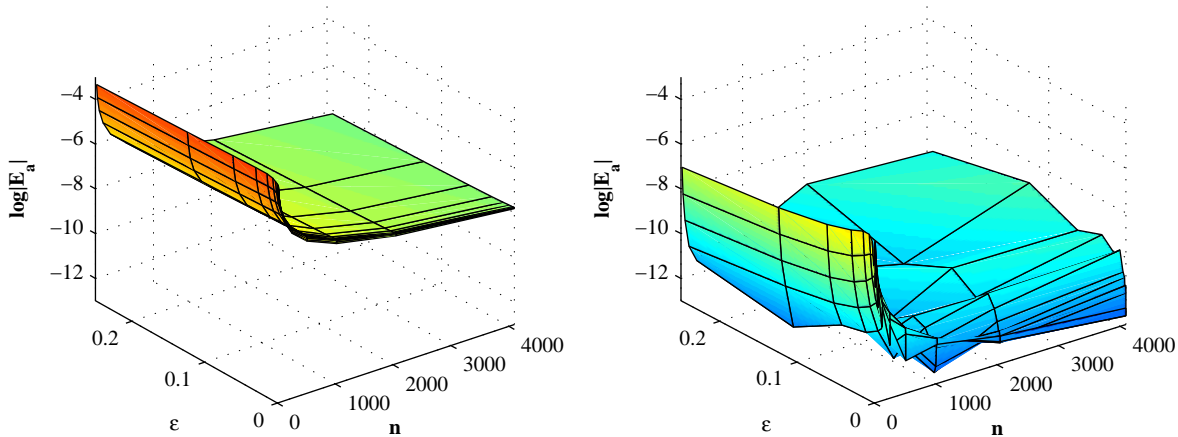


Figure 1: $\log |E_a|$ for problem (2), EMW method (left) and AD method (knot errors, right); ε and N are from (6).

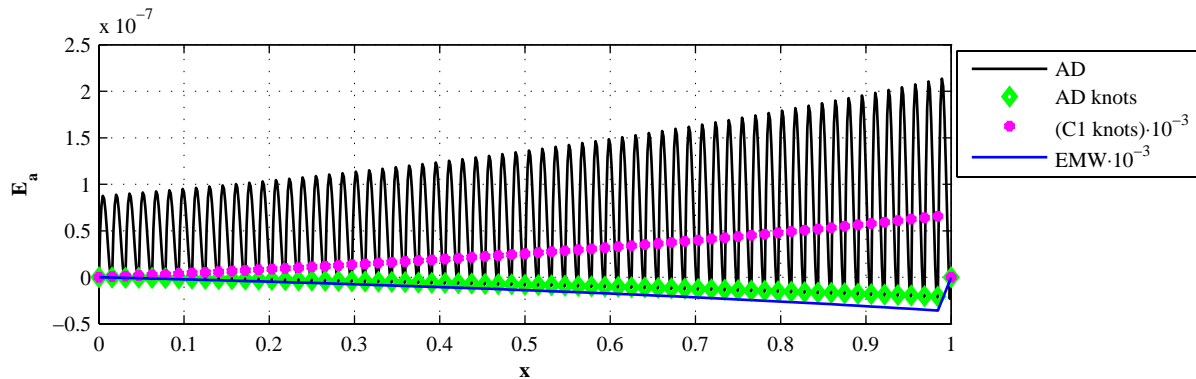


Figure 2: E_a for AD, C1 and EMW method. Here $\varepsilon = 2^{-13}$, $N = 64$, 10 points per interval are taken to plot the global AD error (black solid line). Knot errors for C1 and EMW errors are scaled by multiplication $\cdot 10^{-3}$.

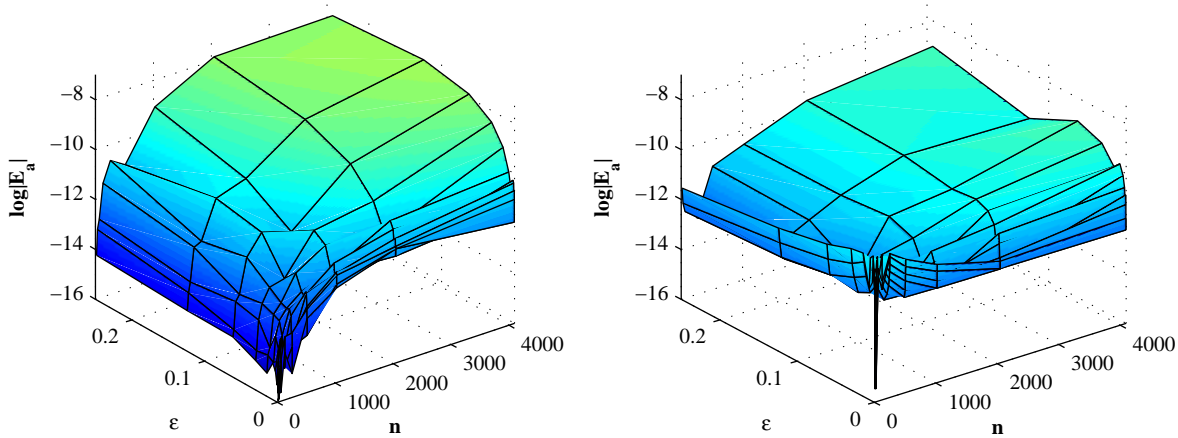


Figure 3: $\log |E_a|$ for problem (4), RAM method (left) and C1 method (knot errors, right). The rest as in Fig. 1.

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